PAPER NO. 6 Information from six experimental investigations for the side and bottom crack width of flexural members were analyzed statistically with the aid of a computer. A large number of equations and variables were examined. Two equations were proposed that best fit all the experimental data to predict the most probable maximum crack width in reinforced concrete flexural members.

Maximum Crack Width in Reinforced Concrete Flexural Members

By PETER GERGELY and LEROY A. LUTZ

■ A NUMBER OF INVESTIGATIONS in recent years were concerned with the cracking of reinforced concrete members. This interest has been stimulated by a trend toward the use of higher strength steels. In spite of these studies, the factors affecting the spacing and width of cracks are still not known for all conditions, and certainly not agreed on among different investigators.

Various semi-theoretical and experimental equations have been proposed by others that contain different variables. A short review of the most important studies is given in Ref. 1. Some of the features of these investigations will be discussed in this paper.

The width of cracks is subject to relatively large scatter, which makes it difficult to tell which equation is best for predicting these quantities. Also, the different characteristics of each investigator's specimens and the indirect effects of many of the variables result in differing conclusions. For these reasons, an extensive statistical evaluation of data from six different investigations has been made and is reported herein. The previously proposed equations are compared and new equations are proposed as the result of correlation and regression analyses.

Most investigators have reported only the maximum and average measured crack width at certain stress levels. For this reason, a complete statistical analysis based on, say, a 5 percent probability of a certain maximum crack width for a member, was not possible.

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The maximum reported crack width in a beam at a given stress level has been used. The average crack width might have been more convenient to study, but different investigators used diverse criteria in considering average crack width. Some included only primary cracks, while others (for example Broms[2]) considered all cracks observed. Furthermore, it is not the average but the maximum crack width that is of practical importance.

As a result of this study, two alternative pairs of simple formulas are proposed for predicting maximum crack width. The correlation of either pair with all data that have been studied appears to be significantly better than any of the previously published equations.

All dimensions are in inches, all areas in square inches and all stresses in ksi. The crack widths are given in thousandths of an inch throughout this paper. The cube concrete strength values, where used, have been converted to cylinder strength by using a factor of 0.85. All symbols are defined where they first appear and are listed in the Appendix.

SUMMARY OF CORNELL WORK ON MACROCRACKING

Investigations of macrocracking of concrete at Cornell University were started by B. B. Broms and results were published in a number of papers.[2, 3] The purpose of the study was to obtain detailed fundamental information on the cracking in concrete members reinforced with modern high strength bars.

The analytical and experimental investigations of tensile and flexural members showed that the primary crack spacing depended mainly on the maximum cover of the steel reinforcement. Since the strain in the cracked concrete is small, the crack width at a given steel stress is proportional to the crack spacing. Thus, the crack width was found to vary directly with the distance from the nearby bar. Watstein and Mathey[4] noticed that the crack width increased with distance from the reinforcement in tests of axially reinforced tensile specimens.

A number of crack width theories contain D/p as a primary variable. Tests by Broms showed that D/p and also D/p_e (p_e is the effective reinforcement ratio based on area of concrete located symmetrically about main reinforcement) may be very poor variables; tests with different diameter bars but with the same p_e and p resulted in similar crack widths. Also, T-beams with equal D/p and D/p_e ratios gave different crack widths, depending on the arrangement of the reinforcement in the flange.

In a test series by Hognestad[1] the crack width increased nearly in proportion to D/p for old type deformed bars and plain bars, while for the new type deformed bar the crack widths were appreciably lower and increased only slightly with increase in bar diameter.

The ratio D/p as a variable corresponds to the postulate that the total bar perimeter is a major factor in determining the crack width. This is not the case with modern deformed bars because of their excellent bond characteristics.

A large modern deformed bar is able to produce a crack spacing equal to about twice the maximum cover over the reinforcing bar, which is close to the limit that can be obtained. Broms concluded from his analytical work that the crack spacing should be about 1.5 times the maximum cover. Replacing the large bars with smaller bars and maintaining the same cover, it is found that the use of smaller bars does little to reduce the crack spacing. Thus, the bar diameter by itself is no longer a prime variable.

The other variable in the bond perimeter, the number of bars, was proven to be a significant factor in determining the crack width, inasmuch as it dictates the concrete cover indirectly. Research done at the PCA[1, 5, 6] concluded from beam tests with modern deformed bars that that the quantities $\sqrt{A}$ and/or $\sqrt[4]{A}$ were major variables, where $A = \frac{A_e}{m}$. (A_e is the area of concrete surrounding the main reinforcement and having the same centroid; m is the number of bars used; see Figure 6-2). However, $\sqrt{A}$ can also be interpreted as a distance which depends on the combined effects of the bottom and side covers, the bar spacings and the number of bars.

Broms' last series of tests consisted of tensile specimens which showed some of the shortcomings of using the variable A. Using 4 bars in various arrangements within a constant concrete area A_e, the crack width was found to vary with the steel arrangement. Also, when a No. 8 bar was replaced by 4 No. 4 bars an inch apart (thereby reducing $\sqrt{A}$ by one-half), there was very little difference in the cracking pattern and crack width.

The quantity $\sqrt{A}$ appears to be a significant variable if the steel is uniformly distributed throughout the effective area of concrete, making the crack width approximately uniform at all points on the surface of the concrete. This is also true for variables such as D/p_e, D/p, etc. since they do not consider the variation in the width of any one crack along the surface of a specimen.

Broms' investigation, based almost exclusively on testing tension members, resulted in the following expression for the maximum crack width:

$$w_{max} = 4\ t_e\ \epsilon_s \tag{1}$$

where ϵ_s is the strain in the steel, and $t_e = t$, the distance from the point in question on the surface (e.g., C in Fig. 6-1) to the nearest reinforcing bar, if that point is not between two reinforcing bars (between points A and B). Between two bars (e.g., at point D) $t_e = \sqrt{c^2 + e^2}$

$$\text{where } \frac{1}{e} = \frac{1}{e_1} + \frac{1}{e_2} \quad \text{or } e = \frac{e_1 e_2}{s} \quad \text{when } \frac{s}{c} > 1.$$

$$\text{If } \quad \frac{s}{c} \leqslant 1 \quad \text{then} \quad t_e = c \text{ between A and B.}$$

If the crack width is to be measured at the bottom face of the beam, the steel strain must be corrected to the "strain" at the bottom face:

$$w_b = 4\ t_e\ R\ \epsilon_s \tag{2}$$

where R is equal to h_2/h_1, the ratio of the distances from the neutral

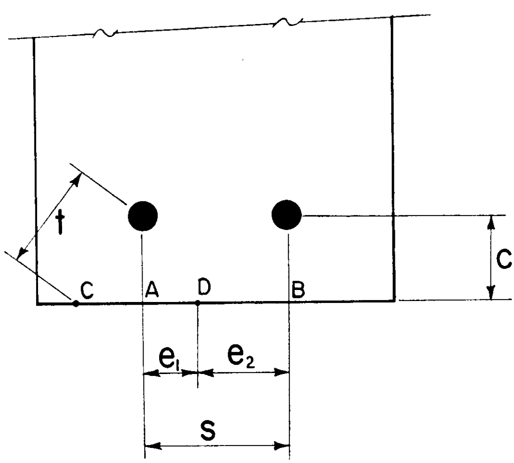

Figure 6-1 Reinforced concrete member showing terms used to define the effective cover, t_e

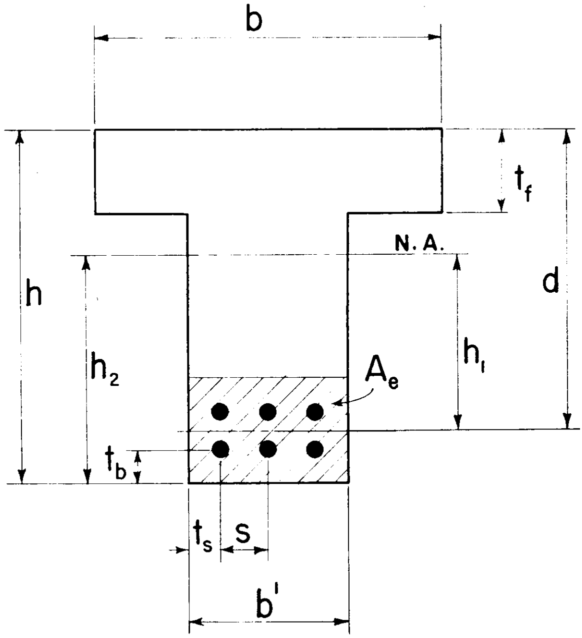

Figure 6-2 Dimensional notation

axis to the bottom face and to the centroid of the reinforcement (see Figure 6-2).

MAJOR VARIABLES AND EQUATIONS

Various investigators proposed a somewhat bewildering variety of independent variables on which the crack width and crack spacing is supposed to depend. These are here briefly reviewed. The primary variable is, of course, the steel stress f_s. Most investigators use f_s to the first power. Broms has used the steel strain because he included crack widths at stresses larger than the proportional limit of the reinforcement.

The second major variable, the multiplier of the steel stress, takes a variety of forms. One of the earliest and most often mentioned is the ratio of the bar diameter to the reinforcement ratio, D/p, which is equivalent to $\frac{4bd}{\Sigma o}$, b being the width of the beam in the compression zone, Σo the total bar perimeter. A similar ratio $\frac{D}{p_t} = \frac{4A_t}{\Sigma o}$ has also been considered, as well as the ratio $\frac{D}{p_e} = \frac{4A_e}{\Sigma o}$. A_t is the area of concrete in tension (below the neutral axis). These expressions differ in the definition of the participating area of concrete surrounding the reinforcement.

Based on experimental evidence, formulas were proposed in which the influence of the above expressions was variously modified in the following ways:

$$\frac{D}{\sqrt{p_e}} \quad \text{(Ref. 5)}, \qquad (C_1 + \frac{C_2}{p_e})D \qquad \text{(Ref. 7)}$$

$$\frac{h - d}{d} \cdot \frac{D}{p} \text{ (Ref. 8) or } (C_1 + C_2 \frac{D}{p}) \qquad \text{(Ref. 9)}$$

In all of the above expressions, for a given percentage of steel (p, p_t or p_e), the crack width increases with the bar diameter. Recent investigations in the United States[1, 2, 3, 5] however, have indicated that the diameter is not a pertinent variable. Actually, one of the expressions in the above list does not depend on D, namely $\frac{D}{\sqrt{p_e}}$ which is equal to $\frac{2}{\sqrt{\pi}}\sqrt{\frac{A_e}{m}}$, and is independent of D. The term $\sqrt{\frac{A_e}{m}} = \sqrt{A}$ is one of the variables recommended by Kaar and Mattock, the alternative being $\sqrt[4]{A}$. Of course, A is usually larger when D is large, but this need not be the case.

Broms has recommended the use of the variable t_e, which is an effective distance from any point on the beam surface to the centroid of the nearest reinforcing bar. This is the only variable that considers the variation of the width of the crack along the surface of the member.

In this investigation, these and a number of other variables were examined statistically and it was determined which of them do a relatively better job in predicting the maximum crack width for most situations.

Some frequently used equations for the determination of the maximum (in one case average) crack width in thousandths of an inch (where numbers are given), are:

Broms:[2, 3]

$$w_s = \frac{w_b}{R} = 4\, t_e\, \epsilon_s = 0.133\, t_e\, f_s \qquad (3)$$

(ϵ_s is in milli-in/in)

Kaar-Mattock:[5]

$$w_s = \frac{w_b}{R} = 0.067\, \sqrt{A}\, f_s \qquad (4)$$

$$w_s = \frac{w_b}{R} = 0.115\, \sqrt[4]{A}\, f_s \qquad (5)$$

Comité Européen du Béton (CEB):[7]

$$w_{max} = (4.5 + \frac{0.40}{p_e}) \frac{Df_s}{47.5} \quad (6)$$

Clark:[8]

$$w_{ave} = 0.0227(\frac{h-d}{d}) \frac{D}{p}(f_s - 0.0566(\frac{1}{p} + n)) \quad (7)$$

$$w_{max} = 1.64\ w_{ave}$$

Borges Lima:[9]

$$w_{max} = (C_1 + C_2\frac{D}{p})f_s \quad (8)$$

Rüsch (Presented in lecture at Cornell University)

$$w_{max} = \frac{1}{\alpha}\frac{D}{p}f_s^2 \quad (9)$$

$$w_{max} = (C_1 + C_2\frac{D}{p_t})f_s \quad (10)$$

Brice:[10]

$$w_{max} = C_1 \frac{p_e + 0.1}{p_e} D\ f_s \quad (11)$$

Swedish:[11]

$$w_{max} = C_1\ D \left[\frac{f_s}{p}\right]^{2/3} \quad (12)$$

DESCRIPTION OF DATA

The data used in this study were obtained from the investigations by Broms (taken from original data of the tests described in Ref. 2), Hognestad,[1] Kaar and Mattock,[5] Kaar and Hognestad,[6] Clark,[8] and Rüsch and Rehm.[12] All these investigations used only reinforcing bars that satisfied the ASTM A-305 standard on bar deformations, with two exceptions: 1) 7 of 36 beams tested by Hognestad were not considered because they had bars that did not conform with the A-305; 2) only beams having bars with transverse lugs similar to American bars were considered from the Rüsch-Rehm investigations (Querrippenstahl, Nori-Stahl, Noreck-Stahl, old Hi-Bond-Stahl). However, a few beams with bars having a deformation spacing somewhat greater than 0.7D (up to 0.85D) were included, even though, for this reason, they did not precisely satisfy A-305.

The maximum side crack width at the level of the centroid of the tensile reinforcement taken from references 1, 5, 12, and from Broms' work, and the maximum crack width on the bottom face of the beam

TABLE 6-1 PROPERTIES OF BOTTOM CRACK SPECIMENS

Investigator	No. of Specs.	No. of Obs.	h in	b_o in	t_s in	t_b in	s in	D in	R	$\sqrt{A}$ in	p %	p_t %	p_e %
Hognestad	8	32	all 16.0	all 8.00	.81† 3.31 (1.93)	.81 3.31 (2.31)	1.38 6.38 (4.13)	all .875	1.07 1.68 (1.27)	2.55 6.20 (4.11)	1.06 1.34 (1.11)	1.27 1.36 (1.33)	1.56 9.26 (4.63)
Kaar-Mattock	13	65	8.0 17.5	4.75 32.0	1.62 6.00 (2.51)	1.62 2.00 (1.74)	1.50 12.0 (5.48)	.50 1.27	1.14 1.60 (1.33)	2.14 6.93 (3.88)	.37 2.64 (.98)	0.38 3.24 (1.23)	.62 4.35 (2.61)
Rüsch-Rehm	23	163	5.90 24.6	7.88 11.8	0.98 2.95 (1.71)	0.75 2.20 (1.48)	1.73 8.85 (4.60)	.314 1.26 (.746)	1.08 1.23 (1.13)	2.26 4.67 (3.27)	.23 .96 (.53)	.30 2.79 (.92)	.94 13.1 (4.53)
Kaar-Hognestad	8	46	22.0 *	48.0 *	1.75 4.00 (3.14)	all 1.50	3.50 8.00 (6.27)	.500 *	1.14 1.17 (1.15)	2.65 4.90 (4.21)	1.12 4.37 (4.21)	0.44 1.56 (0.70)	0.82 2.14 (1.19)
Clark	54	326	6.0 23.0 (11.4)	6.0 15.0 (7.31)	1.25 4.50 (2.26)	.69 2.70 (1.37)	2.50 7.50	.375 1.375 (.81)	1.11 1.35 (1.24)	1.86 5.69 (3.40)	.35 2.58 (1.49)	.375 3.78 (2.01)	1.33 11.4 (5.30)
Total	106	632											

* All but one.

† The first two lines give the range, numbers in parentheses are the averages.

TABLE 6-2 PROPERTIES OF SIDE CRACK SPECIMENS

Investigator	No. of Specs.	No. of Obs.	h	b_o	t_s	t_b	s	D	$\sqrt{A}$	p	p_t	p_e
Broms	7	20	14.25 (all but one)	3.50 12.0	1.75* 5.50 (2.29)	1.25 3.50 (1.86)	1.0 3.50	.50 1.0	1.75 3.50	1.81	1.29 2.51	1.88 6.45
Hognestad	28	109	8.0 24.0	4.0 12.0	0.81 4.00 (1.76)	0.81 4.81 (1.88)	1.22 6.38	.375 1.00	1.76 6.20 (3.19)	.70 3.28 (1.44)	0.81 4.73 (1.81)	2.04 8.49 (4.67)
Kaar-Mattock	13	66	8.0 17.5	4.75 32.0	1.62 6.00 (2.51)	1.62 2.00 (1.74)	1.50 12.0 (5.48)	.50 1.27	2.14 6.93 (3.88)	.37 2.64 (.98)	.38 3.24 (1.23)	.62 4.35 (2.61)
Rüsch-Rehm	21	160	All 24.6	7.88 11.8 (10.3)	1.30 1.97 (1.55)	1.30 2.20 (1.58)	1.73 8.85 (3.78)	.394 1.26 (.79)	2.26 4.67 (3.14)	.23 .96 (.55)	.30 2.97 (1.05)	.94 13.1 (5.28)
Total	69	355										

* The first two lines give the range, numbers in parentheses give the averages.

taken from references 1, 5, 6, 8, 12, were analyzed separately. In fact, the results from each investigation were studied separately, because each investigator used somewhat different methods. Some of these differences are listed below:

a) Kaar and Hognestad[6] used measured instead of calculated stress values.

b) Kaar and Mattock[5] used a measured stress value at a place where a "crack former" was used and presumably measured the maximum crack width at this location.

c) Clark[8] measured the bottom crack width with Tuckerman gages; their precise location was not reported.

d) Rüsch and Rehm[12] measured the bottom crack widths directly below a reinforcing bar, while Hognestad determined the crack widths between two bars.

The maximum crack width measured by an investigator at a certain stress level is considered statistically as an observation. If the maximum crack width is measured at 6 stress levels, there will be 6 observations resulting from this one beam test. Since an investigator measures the cracks at about the same number of stress levels for every beam, there is little weighting of one beam over the others in an investigator's sample.

A number of observations were removed from the total population because of unusual characteristics. If the stress in the reinforcement exceeded 80 ksi, or it appeared obvious that the steel had yielded (evidenced by an exceptionally large increase in the maximum crack width) the observation was omitted from the analysis. Readings at steel stresses below 14 ksi were omitted, except in the Kaar-Mattock data where a crack former was used.

Two beams, 11.8 in. wide and 23.6 in. deep, reinforced with a single bar, were removed from Rüsch-Rehm's sample because of the impractical nature of the information and of the distortion caused by this erratic data.

A total of 33 observations were thus removed for various reasons; the total number of observations used in this study was 612 for bottom cracks and 355 for side cracks. In order to give an idea of the number of specimens used and their properties, the principal characteristics of the total population are summarized in Tables 6-1 and 6-2. Averages are given only where there is enough variation to make it meaningful.

DESCRIPTION OF STATISTICAL ANALYSIS

The large number of data and the contribution of the various variables can best be studied by statistical analysis. A Multiple Regression Analysis computer program was used for this purpose. The output consisted of the totals and means, the uncorrected sums of squares, the standard errors, the correlation matrix, the regression coefficients and their standard errors, etc. In addition, with the help of separate Fortran programs, the distribution of the input data and the percentage deviations of the measured values from the experimental values were determined.

A regression analysis determines the coefficients of a linear expression that best fits a given set of experimental data by a least squares of deviations criterion. By suitable transformations of the variables, other than linear expressions can be analyzed. For example, by taking the logarithms of the variables, equations containing exponential terms are created. The goodness of fit is measured by the correlation coefficient, which is unity if a perfect linear relationship exists between the variables.

In some cases, stepwise regression was employed, in which case first one, then two, then three or more independent variables were included by the program. The criterion for selection of the first independent variable is that the best correlation with the dependent variable is obtained; the other independent variables are then combined linearly to account as much as possible for the variation in the dependent variable. The contribution of a variable or of a combination of variables can be studied in this manner.

From the statistical analysis, various forms and combinations of variables were obtained; for example, $\sqrt{t\sqrt{A}}$. The numerical constants in the various crack width equations containing these composite variables were obtained from the aforementioned Fortran program.

This constant, C, for the equation $w_c = CX$ is given by

$$C = \frac{\Sigma w_0 X}{\Sigma X^2} \quad (13)$$

where X is the composite independent variable, e.g. $\sqrt{A}\, f_s$. The measure of goodness of the correlation is expressed by the standard error, σ

$$\sigma = \sqrt{\frac{\Sigma(w_0 - w_c)^2}{N - 1}} \quad (14)$$

which is analogous to the standard deviation and has similar properties. Since it was observed that the error $w_0 - w_c$ generally increased with stress, the relative standard deviation (in percent)

$$100\sqrt{\frac{\Sigma\left(\frac{w_0 - w_c}{w_c}\right)^2}{N - 1}} \quad (15)$$

was also calculated.

RESULTS OF STATISTICAL ANALYSIS

The statistical analysis was used to examine the scatter of the data from the various investigations, the peculiarities of some variables and equations, the relative merits of the various crack width formulas, and the practical value of the different equations. Before discussing the various equations and variables, some general comments will be made about the accuracy possible in predicting maximum crack widths.

Scatter of crack width data

With the observations consisting of the maximum observed crack widths at various loads, the equations resulting from the statistical analysis will predict the most probable maximum crack width of the

TABLE 6-3 MAXIMUM BOTTOM CRACK WIDTH EQUATIONS

Composite Independent Variable, X	Hognestad		Rüsch-Rehm		Kaar-Mattock*		Clark		Kaar-Hognestad*	
	C†	σ‡	C†	σ‡	C†	σ‡	C†	σ‡	C†	σ‡
1 Best Fit with f_s	—	1.92	—	1.49	—	1.25	—	0.78	—	1.27
2 Rf_s	.268	5.66	.218	3.31	.293	4.17	.1890	2.42	.1762	2.05
3 $t_b Rf_s$	.0934	3.35	.1382	4.16	.1732	3.53	.1301	2.38	.1439	2.05
4 $\sqrt{t_b} Rf_s$	.1727	3.41	.1772	3.60	.226	3.81	.1690	1.83	.1762	2.05
5 $\sqrt{A} Rf_s$	.0612	3.41	.0676	2.85	.0776	3.35	.0564	1.68	.0494	2.53
6 $\sqrt[4]{A} Rf_s$	.1319	4.31	.1229	2.96	.1597	2.23	.1060	1.85	.1043	2.21
7 $\sqrt[3]{t_b A} Rf_s$	.0714	3.15	.0880	3.11	.1047	2.54	.0767	1.65	.0711	2.30
8 $t_{e_m} Rf_s$	.0890	2.61	.1109	3.07	.1320	3.07	.1113	2.12	.0964	2.36
9 $\sqrt{t_b \sqrt{A}} Rf_s$	.0767	3.12	.0994	3.33	.1203	2.42	.0885	1.76	.0851	2.21
10 $\sqrt{s_1 t_b} Rf_s$	.0885	3.31	.0869	3.29	.0999	3.62	.0798	1.68	.0695	2.41
11 $\sqrt[3]{s_1 t_b^2} Rf_s$	.0930	2.67	.1043	3.37	.1238	2.82	.0967	1.65	.0892	2.23
12 $\sqrt[3]{s_1 t_b (h-d)} Rf_s$	.0930	2.67	.0993	3.58	.1138	2.65	.0967	1.65	.0888	2.20
13 $\frac{D}{p_e} Rf_s$	.00805	3.36	.00968	4.13	.00787	6.39	.01018	2.52	.00416	3.23
14 $(11.25 + \frac{1}{p_e}) DRf_s$	.00646	2.95	.00745	3.24	.00691	5.29	.00691	1.92	.00378	3.06
15 $(50 + \frac{1}{p_e}) DRf_s$	.00369	3.56	.00376	3.42	.00462	3.61	.00317	1.64	.00287	2.69
16 $(150 + \frac{1}{p_e}) DRf_s$	.00169	4.59	.00157	4.14	.00237	3.39	.00130	1.69	.00174	2.35
17 $\frac{D}{p_t} Rf_s$	.00408	5.40	.00239	2.61	.00365	6.43	.00358	3.07	.00225	3.42
18 $\frac{D}{\sqrt{p_t}} Rf_s$	.0354	5.53	.0278	2.84	.0485	2.73	.0306	1.74	.0323	2.65
19 $\sqrt{\frac{D}{p_t}} Rf_s$	.0331	5.53	.0236	2.53	.0364	3.76	0.274	2.42	.0228	2.61
20 $(30 + \frac{1}{p_t}) DRf_s$	.00292	5.47	.00196	2.43	.00315	4.79	.00252	2.22	.00197	3.16
21 $(100 + \frac{1}{p_t}) DRf_s$	.00175	5.54	.00134	2.80	.00227	2.87	.00143	1.71	.00152	2.78
22 $(300 + \frac{1}{p_t}) DRf_s$	.00082	5.60	.00068	3.68	.00118	2.61	.00063	1.65	.00090	2.38
23 $\frac{D}{p_e}(f_s - .0566 (\frac{1}{p} + n)) R$	.00927	3.11	.01387	3.86	—x	—x	.0125	2.08	—x	—x
24 $\sqrt[3]{t_b A} (f_s - 5) R$	.0824	2.86	.0993	3.05	—x	—x	.0911	1.59	—x	—x

* f_{sm} used in place of f_s for the variables listed.
† Numerical Constant in crack width equation (see Eq. 13).
‡ Standard Error of crack width equation (see Eq. 14).
x Since it was found that f_{sm} is essentially proportional to the crack width (see section 6—"Steel stress as a variable"), terms of the form $f_{sm} - K$ were not considered for these investigations.

TABLE 6-4 MAXIMUM SIDE CRACK WIDTH EQUATIONS

Composite Independent Variable, X	Hognestad and Broms C†	Hognestad and Broms σ‡	Rüsch-Rehm C	Rüsch-Rehm σ	Kaar-Mattock* C	Kaar-Mattock* σ
1 Best fit with f_s	—	1.09	—	1.36	—	1.02
2 f_s	.1829	2.54	.1986	2.74	.269	2.71
3 $t_s\ f_s$	.0858	2.33	.1295	2.48	.1079	4.48
4 $\sqrt{t_s}\ f_s$	.1364	1.78	.1612	2.57	.1875	2.58
5 $\sqrt{A}\ f_s$	.0521	2.96	.0651	2.29	.0735	2.73
6 $\sqrt[4]{A}\ f_s$	.1018	2.50	.1157	2.35	.1482	1.83
7 $\sqrt{t_s \sqrt{A}}\ f_s$	.0725	2.13	.0927	2.30	.0911	3.52
8 $\sqrt[4]{t_s \sqrt{A}}\ f_s$	.1198	2.06	.1369	2.44	.1677	2.13
9 $\sqrt[3]{t_s\ A}\ f_s$	.0660	2.35	.0826	2.27	.0853	3.22
10 $\sqrt[6]{t_s\ A}\ f_s$	.1138	2.19	.1295	2.41	.1611	2.01
11 $\frac{D}{p_e} f_s$	.00775	3.53	.01188	2.36	.00751	4.38
12 $(11.25 + \frac{1}{p_e}) D f_s$	.00589	3.14	.00798	2.15	.00658	3.73
13 $(50 + \frac{1}{p_e}) D f_s$	.00306	2.95	.00356	2.71	.00440	2.84
14 $(150 + \frac{1}{p_e}) D f_s$	.00133	3.06	.00143	3.15	.00226	2.82
15 $\frac{D}{p_t} f_s$	.00330	3.32	.00242	2.26	.00328	4.37
16 $\frac{D}{\sqrt{p_t}} f_s$	.02925	3.02	.0261	2.22	.0443	2.31
17 $\sqrt{\frac{D}{p_t}}\ f_s$	.0263	2.56	.0232	1.87	.0329	2.49
18 $(30 + \frac{1}{p_t}) D f_s$	.00239	3.05	.00193	2.03	.00284	3.46
19 $(100 + \frac{1}{p_t}) D f_s$	.00141	2.98	.00127	2.22	.00207	2.49
20 $(300 + \frac{1}{p_t}) D f_s$	.00064	3.07	.00062	2.77	.00111	2.43

* f_{sm} used in place of f_s for the variables listed.
† Numerical constant in crack width equation (see Eq. 13).
‡ Standard error of crack width equation (see Eq. 14).

TABLE 6-5 MAXIMUM SIDE CRACK WIDTH EQUATIONS CONSIDERING THE INFLUENCE OF t_s/h_1

Composite Independent Variable, X	Hognestad and Broms C†	Hognestad and Broms σ‡	Rüsch-Rehm C	Rüsch-Rehm σ	Kaar-Mattock* C	Kaar-Mattock* σ
1 $t_s f_s$	.0935	2.05	.1295	2.48	.1079	4.48
2 $\frac{t_s f_s}{1 + 2/3\ t_s/h_1}$	.1107	1.84	.1379	2.48	.1608	2.61
3 $\frac{t_s f_s}{1 + t_s/h_1}$	.1189	1.77	.1421	2.49	.1780	2.34
4 $\frac{t_s f_s}{1 + 3/2\ t_s/h_1}$	.1309	1.69	.1483	2.49	.1992	2.30
5 $\sqrt{A} f_s$	.0496	2.26	.0651	2.29	.0735	2.73
6 $\frac{\sqrt{A}}{1 + 2/3\ t_s/h_1} f_s$	.0577	2.21	.0693	2.28	.0960	1.93
7 $\frac{\sqrt{A}}{1 + t_s/h_1} f_s$	.0615	2.20	.0715	2.28	.1032	2.15
8 $\frac{\sqrt{A}}{1 + t_s/h_1} f_s$	.0671	2.21	.0747	2.27	.1123	2.52
9 $\sqrt{t_s \sqrt{A}}\ fs$	.0709	1.87	.0927	2.30	.0911	3.52
10 $\frac{t_s}{1 + 2/3\ t_s/h_1}$	.0832	1.71	.0988	2.29	.1273	1.84
11 $\frac{\sqrt{t_s\sqrt{A}}}{1 + t_s/h_1} f_s$	.0891	1.66	.1018	2.29	.1389	1.78
12 $\frac{\sqrt{t_s \sqrt{A}}}{1 + 3/2\ t_s/h_1}$	.0976	1.62	.1063	2.29	.153	2.00
13 $\sqrt[3]{t_s A}\ f_s$	.0636	1.94	.0826	2.27	.0853	3.22
14 $\frac{\sqrt[3]{t_s A}}{1 + 2/3\ t_s/h_1} f_s$	.0743	1.81	.0880	2.26	.1165	1.75
15 $\frac{\sqrt[3]{t_s A}}{1 + t_s/h_1} f_s$	.0795	1.78	.0907	2.26	.1265	1.80
16 $\frac{\sqrt[3]{t_s A}}{1 + 3/2\ t_s/h_1} f_s$	.0870	1.75	.0947	2.26	.1388	2.09
17 $\sqrt{\frac{t_s\sqrt{A}}{1 + 4(t_s/h_1)^2} f_s}$	.08045	1.68	.0945	2.29	.1308	2.21
18 $\frac{\sqrt[3]{t_s A}}{1 + 2/3\ t_s/h_1} (f_s - 5)$	.0847	1.82	.0989	2.26	—x	—x
19 $\frac{\sqrt[3]{t_s A}}{1 + t_s/h_1} (f_s - 5)$	.0906	1.79	.1020	2.26	—x	—x
20 $\frac{\sqrt{t_s \sqrt{A}}}{1 + t_s/h_1} (f_s - 5)$	.1015	1.67	.1144	2.30	—x	—x

* f_{sm} used in place of f_s for the variables listed.

† Numerical constant in crack width equation (see Eq. 13).

‡ Standard error of crack width equation (see Eq. 14).

x Since it was found that f_{sm} is essentially proportional to the crack width (see section 6—"Steel stress as a variable"), terms of the form $f_{sm} - K$ were not considered for these investigations.

sample. Even using the best equations, the scatter in the data was found to produce maximum crack widths that range from less than ½ to more than 1½ times the most probable maximum crack width. Approximately ⅔ of the cracks will be within 25 percent of the most probable value. To evaluate this scatter, the standard error σ is used.

In comparing the standard errors for the various expressions (Tables 6-3, 6-4, and 6-5), it appears that the numerical differences are not great. The following considerations should be noted:

a) Using f_s or any other similar variable, perfect correlation cannot be obtained even for a single beam. Using the best fit line for every beam separately with f_s as the variable, the standard error for each investigation (i.e., considering all beams), was found and is given in the first row of Tables 6-3 and 6-4. This is the absolute minimum error possible with f_s as the variable.

b) In most of the investigations, the concrete cover was kept nearly constant and the steel was well distributed. This makes the standard error for f_s (or Rf_s) relatively low and makes possible improvement small. For instance, for Hognestad's bottom crack data, representing extensive variation in the distribution of the steel, the modifying variable can produce a 0.003 in. reduction in the standard error, but generally the improvement in the standard error is less than 0.001 in.

c) Due to the random nature of cracking, variations occur in the maximum crack widths which cannot be accounted for by the properties of the beams.

Thus, one portion of the error represents the irregularities possible in any one beam, the second portion is caused by the scatter from beam to beam, and the third portion can be accounted for by the variable that modifies f_s (or Rf_s).

Steel stress as a variable

The steel stress is the most important variable to be considered in evaluating the crack width. Since the width of a crack is usually desired at working loads, the elastic cracked section theory was used to evaluate the steel stress, f_s. This was done by Hognestad, Rüsch-Rehm, Clark, and Broms in their investigations. The other investigations used the measured steel stress, f_{sm}.

Regression analyses made on the proposed equations in logarithmic form, give f_s to a certain power. Clark's and Rüsch-Rehm's bottom crack data fit $f_s^{1.25}$ best, and Hognestad's few bottom crack data fit $f_s^{1.43}$. Similar analyses for the two investigations using measured stress[5, 6] f_{sm}, showed that $f_{sm}^{.90}$ and $f_{sm}^{.81}$ provide the best fit for these sets of data, respectively. Assuming that f_s is approximately equal to f_{sm} at high stresses, it follows that f_s is larger than f_{sm} at low stresses. This agrees with the fact that a beam is not as fully cracked at lower stresses as is assumed in computing f_s.

An examination of the side crack information reveals Hognestad's data to correlate best with $f_s^{1.37}$, while the exponent for the Rüsch-Rehm data was found to be 1.05 and for Kaar-Mattock's data 1.08. No adequate reason was found for the differences between the exponents for side and bottom crack widths of a given investigation.

The crack width appears to be essentially proportional to the measured stress, f_{sm}; considering the calculated stress, f_s, this is not as true. Between the stress range of 15 to 70 ksi, the variable f_s^n with $1<n<1.4$ can be approximated by a straight line of the form $f_s - K$. Regression analyses using f_s^n and a corresponding $f_s - K$ show approximately the same accuracy.

The average K values found from the individual beams of the Rüsch-Rehm, Clark, and Hognestad bottom crack investigations are 7.3 ksi, 5.6 ksi, and 7.3 ksi, respectively. For the side crack investigations, the average K values were 4.1 ksi for the Rüsch-Rehm beams, and 8.3 for the Hognestad beams. Regression analyses found generally similar values of K.

Clark's equation, for example, considered K to be a function of the reinforcement ratio, p. Examination of equations containing the term $f_s - C_l/p_i$ (where p_i presents p, p_e, or p_t) show improvement in correlation though no consistent value of C_l could be found. C_l was a function of the investigation as well as the variable modifying the term $f_s - C_l/p_i$. An examination of the individual K values for an investigation revealed that there was little correlation between K and $1/p_i$.

A plot of the equations using f_s, f_s^n, and $f_s - K$ for any set of data, would show that the equations using f_s overestimates the crack width at low stresses and underestimates them somewhat at high stresses. This is due to the fact that the variable is forced to pass through the origin of a $w - f_s$ plot. The variable $f_s - K$ is more correct yet no more difficult than f_s to use.

The influence of the second variable, the multiplier to the stress term is much more important than the differences between the stress terms. Because of this, the values of the best modifying variables were affected little by the choice of the form of the stress term. Therefore, the analysis to find the best modifying variable was done using f_s. The selection of the best value of K was then made using the best modifying variables found for f_s.

Bottom crack width analysis

The crack width at the bottom of the beam is generally larger than on the side of the beam. This increase is mainly due to larger extension of the bottom face than at the level of the steel. This strain gradient is recognized by a correction factor $R = h_2/h_1$. This has been proposed by Broms and used by Kaar and Hognestad. If the tensile reinforcement is well distributed and the side crack width is little affected by the compression zone of the beam, the width of the side and bottom cracks equations should be the same except for the ratio R in the bottom crack equation. The suitability of such a multiplier is illustrated by the best fit equations for the Rüsch-Rehm data:

$$w_s = 0.0232 \sqrt{\frac{D}{P_t}}\, f_s, \qquad w_b = 0.0236 \sqrt{\frac{D}{P_t}} \cdot R\, f_s \qquad (16)$$

A large sample of the variables examined for the bottom crack width is presented in Table 6-3.

Hognestad's bottom crack data consists of 8 beams (32 observations) with 2 No. 7 bars having a wide variation of side and bottom covers. The crack width was measured at the beam centerline, midway between the two bars.

Variables that considered the combined effects of the bar spacing and the bottom cover produced the best correlation. The Broms-Lutz[3] variable for crack width midway between two bars,

$$t_{e_m} = \sqrt{t_b^2 + \left(\frac{s}{4}\right)^2} \qquad (17)$$

correlated best, while $\sqrt[3]{s_1 t_b^2}$ was next best. Any stronger or weaker influence of s with respect to t_b significantly increased the standard error. $\sqrt[3]{t_b A}$ was a reasonably good variable; in this case it is equivalent to $t_b^{2/3}$, since, for 2 bars in an 8 in. beam width, $A = 8t_b$. The CEB variable proved to be better than the equation involving $\sqrt[3]{t_b A}$, but since the bar diameter was the same in all specimens, the CEB variable $(4.5 + \frac{0.4}{p_e})D$ is equivalent to $(4.5 + .665A)$ or $(4.5 + 5.32t_b)$. These variables are significantly better than $\sqrt{A}$ or $\sqrt[4]{A}$ or $\frac{D}{p_e}$ (which is equivalent to A for this investigation).

Clark measured the bottom crack width with Tuckerman strain gages along some lines on the bottom face of 54 beams, 28 of which had 6 in. overall depth; 19 had 15 in. depth, and 7 had 23 in. depth. In all of the 6 in. deep beams, the bar spacing was twice the side cover. This was presumed to be the case for the deeper beams, too. All specimens had only one row of steel. For these reasons, A was equal to $2s_1t_b$. The three best variables were found to be $\sqrt[3]{s_1 t_b^2}$ [or $\sqrt[3]{t_b A}$], $(50 + \frac{1}{p_e})D$, and $(300 + \frac{1}{p_t})D$. The variables $\sqrt{A}$ (or $\sqrt{2s_1t_b}$ or $\sqrt{t_s t_b}$),

$\frac{D}{\sqrt{p_t}}$, $\sqrt{t_b}$, $\sqrt{t_b\sqrt{A}}$, and the CEB variable were also quite good.

The Rüsch-Rehm bottom crack data consists of readings from 23 specimens; of these 13 were 24.6 in. deep rectangular beams, 6 were 24.6 in. deep T-beams, and 4 were slabs.

The use of the ratio D/p_t worked best for the Rüsch-Rehm data, the best variables being $(30 + \frac{1}{p_t})D$, $\sqrt{\frac{D}{p_t}}$, $\frac{D}{p_t}$, in that order. Variables independent of D, such as $\frac{D}{\sqrt{p_t}}$, $\sqrt{A}$ and $\sqrt[4]{A}$ were next best, followed by t_{em}, $\sqrt{t_b\sqrt{A}}$, and $\sqrt[3]{t_bA}$. It should be noted that t_{em} should be used for crack widths measured between two bars according to Broms-Lutz,[3] although the use of t_{em} instead of t_b did produce a marked improvement for the Rüsch-Rehm data where crack widths were measured below the bar positions. This indicates that the crack width is dependent on the bar spacing even below the reinforcement.

The following characteristics of the Rüsch-Rehm investigation should be noted: a) The crack widths were measured below the bars; b) The percentage of the steel was low, all beams having less than one percent reinforcement; the average was 0.53 percent; c) The deformation spacing of the bars is significantly larger than that found on American modern deformed bars (Average spacing, c'D, is 0.69D as compared with 0.46D in Ref. 1); d) The concrete strength was low (less than 2000 psi) for the majority of the beams; e) A considerable number of the beams had more than one row of bars. These attributes are to be remembered when comparing with other sets of data.

Kaar and Mattock determined the crack widths at measured steel stress levels. The measurement was made by electrical strain gages placed at midspan where a crack was forced to form. They tested 9 beams with the same effective depth, reinforced with 8 No. 4 bars and having constant side and bottom cover; p, p_e and A varied. They also tested 4 slabs in which the size and number of bars were varied. The maximum crack width was apparently measured wherever it happened to occur on the tension face, regardless of its position relative to the bars.

The variable $\sqrt[4]{A}$ was found to correlate best; this was suggested by the investigators. The expression $\sqrt{t_b\sqrt{A}}$ was second with $\sqrt[3]{t_bA}$ third best (t_b was 2 in. for the slabs and 1.62 in. for the beams). The quantities $(300 + \frac{1}{p_t})D$, $\frac{D}{\sqrt{p_t}}$, as well as $\sqrt[3]{s_1t_b^2}$ and t_{em} are also good variables.

Equations containing s and t_b are good; however, equations using A are better. This may well be due to the fact that most of the above specimens had more than one row of reinforcement, in which case s and t_b alone cannot present the complete picture.

Kaar and Hognestad's investigation was concerned with the crack width on the flange of T-beams over a support. Strain gages were placed on the steel at the edge of the support diaphragm where the largest cracks were likely to occur.

No variable was found that would improve the correlation for the 8 beams examined beyond that of using Rf_s. Since t_b was 1.50 in. for all specimens, $\sqrt{t_b}\ Rf_s$, t_bRf_s etc. provided the same standard error as Rf_s. Other good variables were $(20 + \frac{1}{\sqrt{p_e}})D$, $\sqrt[4]{A}$, $\sqrt{t_b\sqrt{A}}$, $\sqrt[3]{st_b^2}$, $\sqrt[3]{t_bA}$, $(150 + \frac{1}{p_e})D$ in that order. From the analysis it appears that the bar spacing had little or no effect on the crack width. This was not indicated by the other investigations. The different nature of the specimens and the limited number of tests prevent more precise evaluation.

Side crack width analysis

The maximum side crack width is analyzed independently of the bottom crack width. Though it has been customary to represent the two cases by one equation it is believed that there are differences to warrant this separate treatment.

Broms' investigations have indicated that the crack width is dependent on the effective concrete cover, that is, the width of the same crack changes along the surface of the member. Obviously, there may be differences in the side and bottom cover. In beams, however, not only

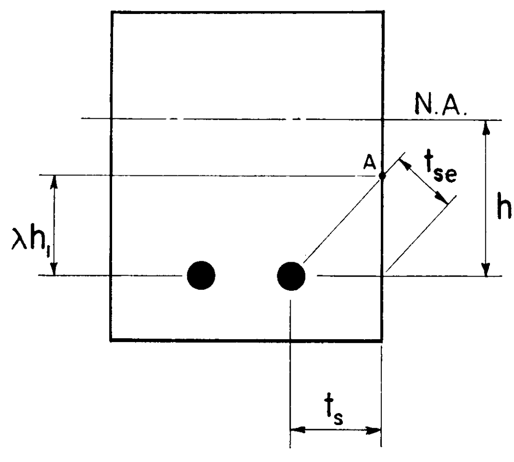

Figure 6-3 Definition of effective side cover, t_{se}

the location of the reinforcement, but also the location of the compression zone affects the size of the side crack width at the level of the reinforcement. It appeared during this investigation that an effective cover based on the actual side cover, t_s, and the distance from the reinforcement to the neutral axis, h_1, might account for this influence. The result was an effective side cover, t_{se}, as shown in Figure 6-3. This cover

$$t_{se} = \frac{t_s}{\sqrt{1 + (\frac{t_s}{\lambda h_1})^2}} \qquad (18)$$

is the distance from where the crack is measured to the line connecting the two points of restraint. At first, it would seem that the distance h_1 itself should be used, but available data seem to indicate that λ is less than one. The side crack width reduction factor, $1/\sqrt{1 + k_1(\frac{t_s}{h_1})^2}$ was later simplified to an equally good expression, $\frac{1}{1 + k_2 \frac{t_s}{h_1}}$ although the physical significance of t_{se}, as defined, no longer holds. The influence of this factor is investigated in Table 6-5.

In the investigations by Hognestad and Kaar-Mattock, where several beams were tested with relatively large t_s/h_1 ratios, the improvement in the standard error was clearly evidenced. For example, the term $\sqrt{t_s\sqrt{A}}\ f_s$ has a standard error of 1.87 for the Hognestad and Broms data, while the term $\frac{\sqrt{t_s\sqrt{A}}\ f_s}{1 + \frac{3}{2}\ t_s/h_1}$ has a standard error of 1.62. Similarly, the standard error decreases from 3.52 to 1.78 by the use of $\frac{\sqrt{t_s\sqrt{A}}}{1 + t_s/h_1} f_s$ instead of $\sqrt{t_s\sqrt{A}}\ f_s$ for the Kaar-Mattock data. Since t_s/h_1 is small and has little variation for the Rüsch-Rehm data (h = 24.6 in. for all beams), negligible improvement is noticed using $1 + k_2t_s/h_1$ in the denominator of any variable.

The Hognestad and Broms data consisted of 36 beams, most of which had an overall depth of 14¼ in. or 16 in., with a wide variation in p, p_e, A and cover thicknesses. For these investigations, the variable $\sqrt{t_s}$ provided the best correlation. The variables $\sqrt{A}$ or $\sqrt[4]{A}$ did little or nothing to improve the standard error. Equations containing D/p_e or D/p_t were also quite poor (Table 6-4). Considering the reduction factor $\frac{1}{1 + k_2t_s/h_1}$, the terms $\frac{t_sf_s}{1 + \frac{3}{2}\ t_s/h_1}$, $\frac{\sqrt{t_s\sqrt{A}}}{1 + \frac{3}{2}\ t_s/h_1} f_s$ and

$\sqrt[3]{t_sA}\, f_s/(1 + \frac{3}{2} t_s/h_1)$ were found to be best (see Table 6-5). Since two of Broms' beams were inverted T-sections, where crack widths were measured on the ends of the flanges, these reduction factors do not apply, hence six observations were omitted in preparing Table 6-5.

Rüsch-Rehm tested beams with equal overall depth, low percentages of steel, and low concrete strength. The variation of cover thickness was moderate, the variation of A and p_e was considerable. For this set of data $\sqrt{\frac{D}{p_t}}$ was by far the best variable, with $(30 + \frac{1}{p_t})D$ second. The CEB expression ranked third, followed by $\sqrt{A}$, $\sqrt[3]{t_sA}$, and $\sqrt{t_s\sqrt{A}}$. The use of the variable t_s alone in any form showed up poorly. As mentioned before, the use of the reduction factor showed negligible improvement.

For the Kaar-Mattock side crack data $\frac{\sqrt{t_s\sqrt{A}}}{1 + t_s/h_1}$ and $\frac{\sqrt[3]{t_sA}}{1 + t_s/h_1}$ were the best variables, followed closely by $\sqrt[4]{A}$. Somewhat poorer correlation was achieved using $\sqrt[6]{t_sA}$, $\sqrt[4]{t_s\sqrt{A}}$, and $\frac{t_s}{1 + 1.5\frac{t_s}{h_1}}$.

General discussion of results

Comparison of Crack Widths from Different Sources

In considering the variables for each investigation individually, the standard error was the basis for evaluation, while in comparing the results of the various investigations, the magnitude of the regression coefficient becomes of importance.

For the bottom crack width study, the regression coefficient applying to Clark's data will be used as a basis for comparison. In order to examine the differences between the crack width data from different investigations, a variable is taken that is good for all five sets of data; such a variable is $\sqrt[3]{t_bA}\, Rf_s$. Thus, one finds that the maximum crack width from the Rüsch-Rehm data is 15 percent larger, Hognestad's and Kaar-Mattock's 7 percent smaller and Kaar-Mattock's 37 percent larger than Clark's values. These percentages, of course, are somewhat different for other variables, depending on how good the variables are.

A similar comparison with the side crack data taking Hognestad's and Broms' data as a basis with $\frac{\sqrt{t_s\sqrt{A}}}{1 + t_s/h_1} f_s$ as the variable shows that Rüsch-Rehm's side crack widths are 14 percent larger, while Kaar-Hognestad's are 56 percent larger.

In general, Rüsch-Rehm's crack values are about 10 percent to 15 percent larger than the American data (excluding those where the measured stress was used[5, 6]). It is believed that this difference is due to the difference in bar deformation spacing, and to the fact that they unloaded the specimens 14 times from a steel stress of 43 ksi before proceeding to higher stresses.

The crack values of Kaar-Mattock's tests are on the order of 40 to 50 percent higher than that of the other American tests (including Kaar-Hognestad's work where measured stresses were also used). This difference seems to indicate that the maximum crack widths may have been observed at midspan where the "crack former" was located. The reported maximum crack widths at or below 5 ksi steel stress must have been measured there, since the concrete would not have cracked at a moment calculated from these steel stresses.

For the above reasons, and since the calculated stress has to be used in practice, the results of Hognestad's and Clark's investigations were used to determine the numerical constants in the recommended equations.

Evaluation of Major Variables

The ratio D/p was not a very good variable in any form, especially for T-beams, where bd greatly overestimates the actual concrete area. The use of b′ instead of b in determining the reinforcement ratio would be better.

The reinforcement ratio p_t based on the area of concrete in tension, was examined in some detail. The ratio D/p_t in itself was a very poor variable for all except the Rüsch-Rehm data; even in this case $\sqrt{\frac{D}{p_t}}$ or $(30 + \frac{1}{p_t})D$ were better. $\frac{D}{\sqrt{p_t}}$, which is independent of D, proved to be much better than $\sqrt{\frac{D}{p_t}}$ for the data of Clark and Kaar-Mattock. None of these terms worked well for the data of Hognestad and of Broms; but if a variable involving p_t is desired, the form $\frac{D}{\sqrt{p_t}}$ would probably be best, considering all investigations.

The CEB variable $(4.5 + \frac{0.4}{p_e})D$ greatly overestimates the crack width when p_e is low. This is shown by the fact that the regression coefficient of this variable for Kaar-Hognestad's data is about half of that for Rüsch-Rehm's data. Individual beams show the same trend. To be sure, the CEB equation is to be used only for p_e between 2 and 20 percent; however, actually p_e is often below this range; 5 of 13 of Kaar-Mattock's beams, 7 of 8 of Kaar-Hognestad's beams, and 11 specimens from the other investigations had effective percentages less than 2 percent.

The effect of p_e can be reduced in two ways: by having an additive constant (as above), which cannot work for the whole range of p_e, or by raising p_e to a power less than one. Kaar-Mattock have shown that $\sqrt{A}$ (which is equivalent to $\frac{D}{\sqrt{p_e}}$) is similar to $(4.5 + \frac{0.4}{p_e})D$, the deviation being less than 4 percent for p_e values between 4 and 20 percent. Below $p_e = 4$ percent, the equations begin to deviate. Considering the bottom crack data, $\sqrt{A}$ is significantly better for all but Hognestad's 8 beams. Nevertheless, also with $\sqrt{A}$ there is some tendency to overestimate the crack width for low p_e, though much less than with the CEB variable. This is why $\sqrt[4]{A}$ works well for the Kaar-Mattock and Kaar-Hognestad data in which p_e was notably small. This smaller dependence on A (i.e., on $\frac{D^2}{p_e}$) is evidenced in these two investigations by the fact that the variable $(150 + \frac{1}{p_e})D$ is better than the CEB variable $(11.25 + \frac{1}{p_e})D$.

The simplest approach to the determination of the crack width is to use the cover thickness t_s or t_b, whatever the case may be, as suggested by Broms. The cover thickness or its square root were shown to be good in several of the investigations, especially for the side crack data of Hognestad and Broms where $\sqrt{t_s}$ was very good, but in other investigations it was poor. Though the cover thickness is an important variable, it is apparent that the crack width depends on other factors in addition to t_s or t_b. This is substantiated by the improvement observed in the correlation for variables that included the cover thickness, plus the bar spacing or A. The variable A (or s_1t_b for a single row of bars) indicates the crack spacing, and t_b (or t_s) shows how this crack spacing and thus the crack width is influenced by the actual location of the reinforcement. In the case of the side crack width, the effect of the location of the neutral axis on reducing the width is considered by the $1 + k_2 \frac{t_s}{h_1}$ term in the denominator.

RECOMMENDED EQUATIONS

It is difficult to select an equation that fits well all sets of data. An equation that is very good for one investigation and poor for other data cannot be considered acceptable. Such an equation is usually based on data of limited scope and often reflects the characteristics of that particular investigation.

One important consideration is that the equation be dimensionally correct. Since the stress term is used in lieu of the strain which is dimensionless, the modifying variable should have the dimension of length. If the equation is not dimensionally correct (e.g., if $\sqrt[4]{A}$, $\sqrt{t_s}$

or $\frac{D}{p_t}$ are the modifying variables) the size of the beam is a factor in choosing the coefficient, and scaling of models becomes impossible.

Bottom Crack Width. Considering all of the data, with f_s as the stress variable, the best general variable appears to be $\sqrt[3]{t_b A}\, Rf_s$. Other good variables are $\sqrt{A}$, $\sqrt[4]{A}$, $\frac{D}{\sqrt{p_t}}$ and $\sqrt{t_b\sqrt{A}}$. All these variables are independent of the bar diameter D, and all consider the area of concrete around a bar in some manner.

The variable $\sqrt[3]{t_b A}$ reflects the variation in the bottom concrete cover as well as the average effective area of concrete around a reinforcing bar. Realizing that the magnitude of the crack widths for the Rüsch-Rehm data and the Kaar-Mattock data are out of line with the other investigations, and that Clark's test program was more extensive than the other American studies, the following equation is recommended:

$$w_b = 0.076\ \sqrt[3]{t_b A}\ R\ f_s \qquad (19)$$

This equation represents the most probable maximum crack width on the bottom face of a beam, with f_s as the variable.

Examining possible improvements in the equation using a stress term of the form $f_s - K$ with the modifying variable $\sqrt[3]{t_b A}$, it was found that $f_s - 5$ produced the lowest standard errors. The improvement in the absolute error by using $f_s - 5$ instead of f_s is reflected by the reduction in σ as seen by comparing lines 7 and 24 in Table 6-3. Correspondingly, 7 percent more of the data falls within 25 percent of the most probable values (Table 6-6). The improved equation for the most probable maximum bottom crack width is

$$w_b = 0.91\ \sqrt[3]{t_b A}(f_s - 5)R \qquad (20)$$

As mentioned before, the scatter in the data is appreciable even for the best probable maximum crack width equations, such as Eqs. 19 and 20. To give some information about the distribution of the measured maximum bottom crack widths for each individual investigation, Table 6-6-A is presented. For example, using the optimum C values for each investigation, only about 15 percent of the data exceed $1.25w_b$. Using the recommended equations with C values 0.076 and 0.091 obtained by considering all investigations, Table 6-6-B gives similar information on crack width distribution. The standard errors in Table 6-6-B are, of course, larger than some in Tables 6-3, 6-4 and 6-5, since they relate to the indicated overall C values rather than to those derived for each individual investigation.

Side Crack Width. Each of the variables $\sqrt{t_s}$, $\sqrt[4]{A}$, $\sqrt{\frac{D}{p_t}}$ are very good for just one of the investigations—apparently by taking account

TABLE 6-6-A PERCENTAGE DISTRIBUTION OF MAXIMUM BOTTOM CRACK WIDTH DATA

Equation—$w_b = C\sqrt[3]{t_b A}\ Rf_s$

Investigation	Hognestad	Rüsch-Rehm	Kaar-Mattock*	Clark	Kaar-Hognestad*
No. Beams	8	23	13	54	8
C	.0714	.0880	.1047	.0767	.0711
$1.5w_b$	12‡	6	5	3	12
$4/3w_b$	12	10	9	10	24
$1.25w_b$	16	15	17	15	33
w_b	38	37	60	42	61
$.75w_b$	75	77	95	84	78
$.5w_b$	97	95	100	98	98

* f_{sm} used in place of f_s.

Equation—$w_b = C\sqrt[3]{t_b A}\ R(f_s - 5)$

Investigation	Hognestad	Rüsch-Rehm	Kaar-Mattock†	Clark	Kaar-Hognestad†
No. Beams	8	23		54	
C	.0824	.0993		.0911	
$1.5w_b$	12	6		4	
$4/3w_b$	12	14		12	
$1.25w_b$	12	18		17	
w_b	41	44		54	
$.75w_b$	78	87		92	
$.5w_b$	97	98		98	

† Equation not applicable using f_{sm} (see note in Table 6-III).
‡ The number 12 indicates that 12% of the crack width observations were larger than $1.5w_b$.

of the characteristics of the particular beams tested (see Table 6-4). The only measure that really worked in improving the correlation with all of the side crack data was to consider the effect of the location of the neutral axis in reducing the crack width, especially in shallow beams with large side cover. This reduction can be most simply expressed by the factor $\dfrac{1}{1 + k_2 \dfrac{t_s}{h_1}}$. The variables $\sqrt{t_s\sqrt{A}}$ and $\sqrt[3]{t_s A}$ were found to work best in conjunction with this reduction factor.

It is difficult to determine whether $\dfrac{\sqrt{t_s\sqrt{A}}}{1 + t_s/h_1}$, $\dfrac{\sqrt[3]{t_s A}}{1 + \frac{2}{3}\, t_s/h_1}$, or $\dfrac{\sqrt[3]{t_s A}}{1 + t_s/h_1}$ is the best variable (see Table 6-5). The first of these variables was slightly better than the other two, but the difference was not significant. Hence, the following equation is recommended because it resembles Equation 19 for bottom cracks:

$$w_s = 0.076 \frac{\sqrt[3]{t_s A}}{1 + \frac{2}{3} t_s/h_1} f_s \qquad (21)$$

This is the proposed equation for the probable maximum side crack width with f_s as the variable.

Considering equations with an $f_s - K$ stress term, the recommended equation is

$$w_s = 0.091 \frac{\sqrt[3]{t_s A}}{1 + t_s/h_1} (f_s - 5) \qquad (22)$$

The average K values for the Hognestad and Rüsch-Rehm beams were 8.3 and 4.1, respectively, as mentioned before. Examination of Table 6-7 shows that the improvement of Eq. 22 over Eq. 21 is negligible. The value of $K = 5$ was selected because it is also used in the recommended side crack equation. Tables 6-7-A and 6-7-B show the measured maximum bottom crack widths in the same manner as Tables 6-6-A and 6-6-B do for the maximum bottom crack width.

TABLE 6-6-B PERCENTAGE DISTRIBUTION OF MAXIMUM BOTTOM CRACK WIDTH DATA USING RECOMMENDED EQUATIONS

Equation—$w_b = 0.076 \sqrt[3]{t_b A}\ R f_s$

Investigation	Hognestad	Rüsch-Rehm	Kaar-Mattock*	Clark	Kaar-Hognestad*
σ	3.27	3.41	4.97	1.65	2.37
$1.5w_b$	12	12	38	4	4
$4/3w_b$	12	20	74	10	17
$1.25w_b$	12	28	82	16	24
w_b	31	58	98	45	46
$.75w_b$	66	85	100	85	74
$.50w_b$	97	98	100	98	96

* f_{sm} used in place of f_s.

Equation—$w_b = 0.091 \sqrt[3]{t_b A}\ R(f_s - 5)$

Investigation	Hognestad	Rüsch-Rehm	Kaar-Mattock*	Clark	Kaar-Hognestad*
σ	3.20	3.17	4.50	1.59	2.60
$1.5w_b$	9	12	31	4	11
$4/3w_b$	12	20	58	12	26
$1.25w_b$	12	26	71	17	33
w_b	25	58	89	52	50
$.75w_b$	69	90	91	90	76
$.50w_b$	97	98	91	98	100

TABLE 6-7-A PERCENTAGE DISTRIBUTION OF MAXIMUM SIDE CRACK DATA

Equation—$w_s = C \frac{\sqrt[3]{t_s A}}{1 + \frac{2}{3} t_s/h_1} f_s$

Investigation No. Beams C	Hognestad-Broms 34 .0743	Rüsch-Rehm 21 .0880	Kaar-Mattock* 13 .1165
$1.5w_s$	6‡	4	0
$4/3w_s$	14	7	3
$1.25w_s$	28	14	3
w_s	54	41	42
$.75w_s$	88	79	89
$.5w_s$	98	100	98

* f_{sm} used instead of f_s.

Equation—$w_s = C \frac{\sqrt[3]{t_s A}}{1 + t_s/h_1} (f_s - 5)$

Investigation No. Beams C	Hognestad-Broms 34 .0906	Rüsch-Rehm 21 .1020	Kaar-Mattock†
$1.5w_s$	8	7	
$4/3w_s$	19	12	
$1.25w_s$	31	16	
w_s	60	53	
$.75w_s$	89	87	
$.5w_s$	98	100	

† Equation not applicable using f_{sm} (see note in Table 6-III).
‡ The number 6 means that 6% of the observations are greater than 1.5 w_s where w_s is as given above the table.

Comparison of Side and Bottom Crack Equations. All these equations contain the concrete cover t_s or t_b, the effective concrete area A_e, and the number of bars m, as well as the steel stress. In general, the effect of the factors R and the denominator in the side crack equations is to produce smaller side crack widths than bottom crack widths.

It is interesting to note that both Equations 20 and 22 reduce to

$$w_{max} = 0.133\ t_s(f_s - 5)$$

for a cylindrical tensile specimen for which $A = \pi t_s^2$ and $t_s = t_b$. In a similar manner, Eq. 19 and 21 reduce to

$$w_{max} = 0.1113\ t_s f_s$$

These equations are similar to Broms' expression (Eq. 3) that was derived from tensile tests for high stresses.

TABLE 6-7-B PERCENTAGE DISTRIBUTION OF MAXIMUM SIDE CRACK DATA USING RECOMMENDED EQUATIONS

$$\text{Equation—}w_s = \frac{0.076\sqrt[3]{t_sA}}{1 + \frac{2}{3}\, t_s/h_1} f_s$$

Investigation σ	Hognestad-Broms 1.82	Rüsch-Rehm 2.54	Kaar-Mattock* 4.15
$1.5w_s$	6	11	50
$4/3w_s$	12	19	67
$1.25w_s$	25	28	80
w_s	54	66	97
$.75w_s$	88	91	98
$.50w_s$	98	100	98

* f_{sm} used instead of f_s.

$$\text{Equation—}w_s = 0.091 \frac{\sqrt[3]{t_sA}}{1 + t_s/h_1} (f_s - 5)$$

Investigation σ	Hognestad-Broms 1.79	Rüsch-Rehm 2.44	Kaar-Mattock* 4.29
$1.5w_s$	8	12	59
$4/3w_s$	17	20	71
$1.25w_s$	32	28	80
w_s	59	73	89
$.75w_s$	89	93	89
$.50w_s$	98	100	89

SUMMARY AND CONCLUSIONS

The maximum side and bottom flexural crack data taken from 6 investigations were analyzed statistically with the aid of a computer. Many variables and equations, both old and new, were examined.

The following major conclusions were reached regarding the factors affecting the crack width:

1. The steel stress is the most important variable.

2. The cover thickness is an important variable but is not the only consideration.

3. The bar diameter is not a major variable.

4. The size of the side crack width is reduced by the proximity of the compression zone in flexural members.

5. The bottom crack width increases with the strain gradient.

6. The major variables are the effective area of concrete A_e, the number of bars m, the side or bottom cover, and the steel stress.

In agreement with points 2, 4 and 5 above, different equations were necessary to determine the side and bottom crack widths. The equations recommended as best (and yet practical) are:

1. For the most probable maximum crack width at the level of the reinforcement:

$$w_s = .091 \frac{\sqrt[3]{t_s A}}{1 + t_s/h_1} (f_s - 5)$$

2. For the most probable maximum bottom crack width on the bottom (or tension) face of the beam:

$$w_b = .091 \sqrt[3]{t_b A}\ R(f_s - 5)$$

Two other equations, slightly simpler but not quite as good as the above were also suggested:

3. For the side crack width

$$w_s = 0.076 \frac{\sqrt[3]{t_s A}}{1 + \frac{2}{3} t_s/h_1} f_s$$

4. For the bottom crack width

$$w_b = 0.076 \sqrt[3]{t_b A}\ Rf_s$$

All proposed equations are dimensionally correct.

Since the recommended equations predict the probable maximum crack width, using appropriately larger numerical coefficients in design practice would reduce the probability of underestimating the maximum crack width. Such modifications for design purposes can easily be done with the help of Tables 6-6 and 6-7.

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APPENDIX

Notation

$A = A_e/m$	=	average effective concrete area around a reinforcing bar, in^2
$A_e = 2b'(h - d)$	=	effective area of concrete, in^2
A_s	=	area of tension reinforcement, in^2
A_t	=	area of concrete in tension, $in^2 = b'h_2$ for rectangular section
a	=	ratio of bar deformation height to nominal bar diameter
b	=	width of beam at compression side of beam, in
b'	=	width of beam at centroid of tensile reinforcement, in
c'	=	ratio of bar deformation spacing to nominal bar diameter
C	=	numerical constant in crack width equation (see Eq. 13)
D	=	nominal bar diameter, in
d	=	effective depth of beam, in
f_c'	=	compressive cylinder of concrete, ksi
f_s	=	steel stress calculated by elastic cracked section theory, ksi
f_{sm}	=	measured steel stress, ksi
h	=	overall depth of beam, in
h_1	=	$(1 - k)d$ $\quad h_2 = h - kd$
k	=	distance from neutral axis to compression face divided by effective depth of beam
m	=	number of tensile reinforcing bars
$n = 15.95/f_c'$	=	ratio of modulus of elasticity of steel to that of concrete
N	=	number of observations
p	=	A_s/bd
p_e	=	A_s/A_e
p_t	=	A_s/A_t
R	=	h_2/h_1
σ	=	standard error (see Eq. 14)
s	=	spacing of bars in outer row of reinforcement $s = 0$ when only one bar is used
s_1	=	s when $m > 1$; $s_1 = b'$ when $m = 1$
t_f	=	thickness of T – beam flange, in
t_e	=	the effective cover thickness as defined by Broms[3]

t_{em} = $\sqrt{t_b^2 + (s/4)^2}$
t_s = side cover measured from the center of outer bar, in
t_b = bottom cover measured from the center of lowest bar, in
t_{se} = effective side cover as defined in Eq. 18 and Figure 6-3
w_s = maximum (measured or calculated) side crack width at the level of steel centroid in constant moment region, .001 in
w_b = maximum (measured or calculated) bottom crack width in constant moment region, .001 in
w_o = observed maximum crack width
w_c = calculated maximum crack width
X = composite independent variable

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