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Effective Moment of Inertia for Calculating Deflections of Concrete Members Containing Steel Reinforcement and Fiber-Reinforced Polymer Reinforcement

by Peter H. Bischoff and Andrew Scanlon

The effective moment of inertia expression proposed by Branson in 1963 and incorporated into the ACI Code is reevaluated. It is found that Branson's expression is valid for members with steel reinforcement ratios greater than 1%. This expression, however, overestimates member stiffness at lower reinforcement ratios and gives a member deflection less than expected as demonstrated by comparison with test results. Branson's approach also underestimates deflection of slender walls with a central layer of reinforcement, as well as deflection of fiber-reinforced polymer (FRP)-reinforced concrete beams. An alternative expression is presented that is shown to be valid for all reinforcement ratios for both steel and FRP reinforcement.

Keywords: beam; deflection; reinforcement; slab.

INTRODUCTION

In 1966, ACI Committee 435 published "Deflections of Reinforced Concrete Flexural Members" (ACI Committee 435 1966). The report includes a comparison of several methods for computing immediate deflection including the effects of cracking on member response. The methods compared included the ACI Code method in use at the time (ACI Committee 318 1963) and the effective moment of inertia approach proposed by Branson (1963).

The ACI 318-63 approach considered two cases:

1. ρf_y < 500 psi (3.45 MPa), use the uncracked gross section moment of inertia I_g to compute immediate deflection at service load levels; and

2. ρf_y > 500 psi (3.45 MPa), use the cracked transformed section moment of inertia I_{cr} to compute immediate deflection at service load levels.

For Grade 60 (415 MPa) reinforcement, the transition occurs at ρ = 0.833%.

The effective moment of inertia I_e approach introduced by Branson allows for a gradual transition from uncracked to cracked transformed section as the ratio of service load moment M_a to cracking moment M_{cr} increases. This transition is given by the expression below, and a plot of I_e/I_g versus ρ is shown in Fig. 1 for both approaches.

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left(1 - \left(\frac{M_{cr}}{M_a}\right)^3\right) I_{cr} \le I_g \qquad (1)$$

The committee compared calculated deflections with measured deflections for several sets of laboratory tested beams. The test beams had ρ values ranging from 1 to 3.2%. In this range, Branson's I_e approaches I_{cr} as shown in Fig. 1. Based on the comparison with test results, the committee concluded that both the ACI 318-63 method and Branson's I_e method were adequate for practical purposes in predicting immediate deflections. ACI 318 subsequently adopted the Branson I_e expression for inclusion in the 1971 ACI Code (ACI Committee 318 1971), and this is currently the prescribed method in the ACI Code (ACI Committee 318 2005).

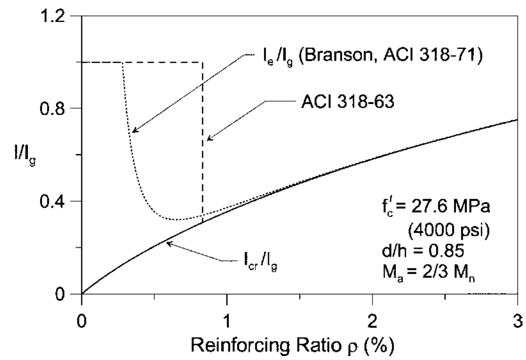

Fig. 1—Effective moment of inertia at service loads (M_a = 2/3M_n).

The comparison with test results did not include beams with reinforcement ratios in the lower range (ρ < 1%), which is more typical of slabs and lightly reinforced beams. Concerns have been raised that Branson's I_e equation is adequate for moderate to high reinforcement ratios but tends to underpredict immediate deflections at low reinforcement ratios (Bischoff 2005a,b). This problem is reflected by amendments to the Australian Standard AS3600 (1994) limiting I_e to a value of 0.6I_g for flexure members with a reinforcing ratio less than 0.5% (Gilbert 2001). In addition, past efforts to apply Branson's equation to members reinforced with fiber reinforced polymer (FRP) bars have found that a correction factor is necessary to correct for overprediction of member stiffness (ACI Committee 440 2006). In this paper, a formulation of the effective moment of inertia is presented that is applicable to all ranges of reinforcement ratio for steel reinforcement as well as FRP reinforcement.

RESEARCH SIGNIFICANCE

The results presented in this paper are directly applicable to design practice related to deflection control of structural concrete members. This paper deals with computation of short-term (immediate) deflections only. Proposed changes to ACI 318 are presented.

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ACI member **Peter H. Bischoff** *is a Professor in the Civil Engineering Department at the University of New Brunswick, Fredericton, New Brunswick, Canada. He is a member of ACI Committees 224, Cracking; 360, Design of Slabs on Ground; 408, Bond and Development of Reinforcement; 435, Deflection of Concrete Building Structures; and 544, Fiber Reinforced Concrete. His research interests include serviceability behavior of concrete structures.*

Andrew Scanlon, *FACI, is a Professor of civil engineering at Pennsylvania State University. He is a member of ACI Committees 224, Cracking; 318, Structural Concrete Building Code; 318-C, Safety, Serviceability, and Analysis (Structural Concrete Building Code); 342, Evaluation of Concrete Bridges and Bridge Elements; 348, Structural Safety; 435, Deflection of Concrete Building Structures; and E 803, Faculty Network Coordinating Committee.*

FLEXURAL BEHAVIOR AT SERVICE LOAD LEVELS

The flexural stiffness of a concrete beam varies along its length due to the presence of cracking that can occur from the applied loading and possibly tension stress caused by shrinkage restraint. At crack locations, the concrete carries essentially zero tension. Between cracks, however, the concrete participates in resisting tensile stress because of bond between the reinforcement and concrete. This effect is often referred to as tension stiffening and is taken into account with the effective moment of inertia I_e.

Simple spring models are used to demonstrate the effect of stiffness variation along a member. Examples are given in Fig. 2 for linear and rotational springs arranged in both series and parallel. The value P represents the axial load for linear springs and moment for rotational springs. The term Δ represents displacement for linear springs and rotation for rotational springs, whereas the term k represents stiffness for both linear and rotational springs. Applying equilibrium and compatibility to the linear elastic systems gives the following expressions for effective stiffness of the two spring models considered:

1) Springs in series

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \tag{2}$$

2) Springs in parallel

$$k_e = k_1 + k_2 \tag{3}$$

From examination of the deformed shapes, it is clear that the springs-in-series model is more appropriate for members with discrete cracks along the member. This suggests that the interpolation formula to model tension stiffening should be based on a weighted average of flexibility rather than stiffness (Bischoff 2005a). This then leads to a subtle change in Branson's original expression, giving

$$\frac{1}{I_e} = \left(\frac{M_{cr}}{M_a}\right)^m \frac{1}{I_g} + \left(1 - \left(\frac{M_{cr}}{M_a}\right)^m\right)\frac{1}{I_{cr}} \geq \frac{1}{I_g} \tag{4a}$$

By rearranging terms, Eq. (4a) can be rewritten as

$$I_e = \frac{I_{cr}}{1 - \left(\frac{M_{cr}}{M_a}\right)^m \left[1 - \frac{I_{cr}}{I_g}\right]} \leq I_g \tag{4b}$$

It is found that a value of $m = 2$ in Eq. (4) correlates well with Branson's original formulation where the power $m = 3$. This correlation is carried out for a beam cross section with a ratio of $I_g/I_{cr} = 2.2$ ($\rho \approx 1.5\%$), and is representative of the beams used by ACI 435 to verify Branson's equation.

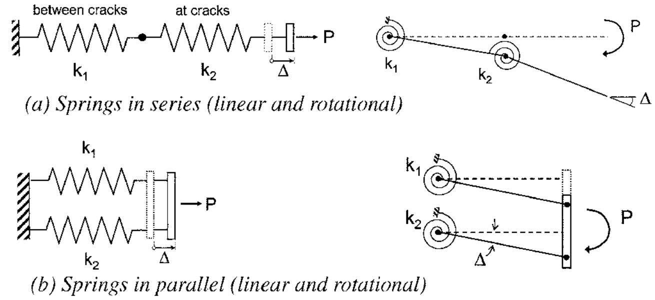

Fig. 2—Simple spring models.

COMPARISON OF EFFECTIVE MOMENT CURVATURE RELATIONSHIPS

Moment-curvature relationships based on the effective moment of inertia concept are plotted in Fig. 3 to compare the original Branson formulation with the approach proposed by Bischoff (2005a). Reinforcement ratios of 1.5, 1.0, 0.5, and 0.3% are considered. The applied moment M_a is assumed to be 2/3 of the nominal flexural strength M_n, based on Grade 60 reinforcement (f_y = 60 ksi [415 MPa]). Calculations are carried out for a 300 mm (12 in.) wide by 200 mm (8 in.) deep section representative of slabs, and f_c' is taken as 27.6 MPa (4000 psi).

The plots shown in Fig. 3 demonstrate that the effective moment of inertia corresponding to the service load moment M_a is insensitive to the formulation of I_e at the higher reinforcement ratios above 1%. Differences between the Branson expression and proposed approach are less than a few percent in this case. At the lower reinforcement ratios (0.3 and 0.5%), there is a significant difference in I_e, with Branson's original expression displaying a much stiffer response than the proposed alternative form. Deflections calculated with Branson's expression for I_e can be as much as 50% less than deflection calculations using the alternative approach.

Branson's Eq. (1) only works well for flexure members with an I_g/I_{cr} ratio less than approximately 3, and this corresponds to beams and slabs with a steel reinforcing ratio greater than 1% (refer to Fig. 4). This expression for I_e essentially represents a weighted average of two springs in parallel (Fig. 2(b)), where the equivalent stiffness approaches the stiffness of the stiffer spring as one spring becomes much stiffer than the other. That is, $k_e = k_1(1 + k_2/k_1) \approx k_1$ when $k_1 >> k_2$. Hence, a beam response modeled with Branson's expression for I_e is pulled toward the uncracked I_g response for beams with I_g/I_{cr} greater than 3. This trend is clearly demonstrated in Fig. 3. The proposed approach using Eq. (4), on the other hand, represents a weighted average of two springs in series (Fig. 2(a)), and the beam response with this model is now pulled toward the cracked I_{cr} response as I_g/I_{cr} increases ($k_e = k_2/(1 + k_2/k_1) \approx k_2$ when $k_1 >> k_2$). Other factors such as the assumed value of modulus of rupture also affect the beam response at lower reinforcement ratios.

COMPARISON WITH EXPERIMENTAL DATA

As noted previously, the comparisons reported by ACI Committee 435 (1966) were restricted to beams with reinforcement ratios greater than 1%. A comparison with slab tests reported by Gilbert (2006) is presented in Fig. 5. Simply supported one way slabs of rectangular section with a thickness

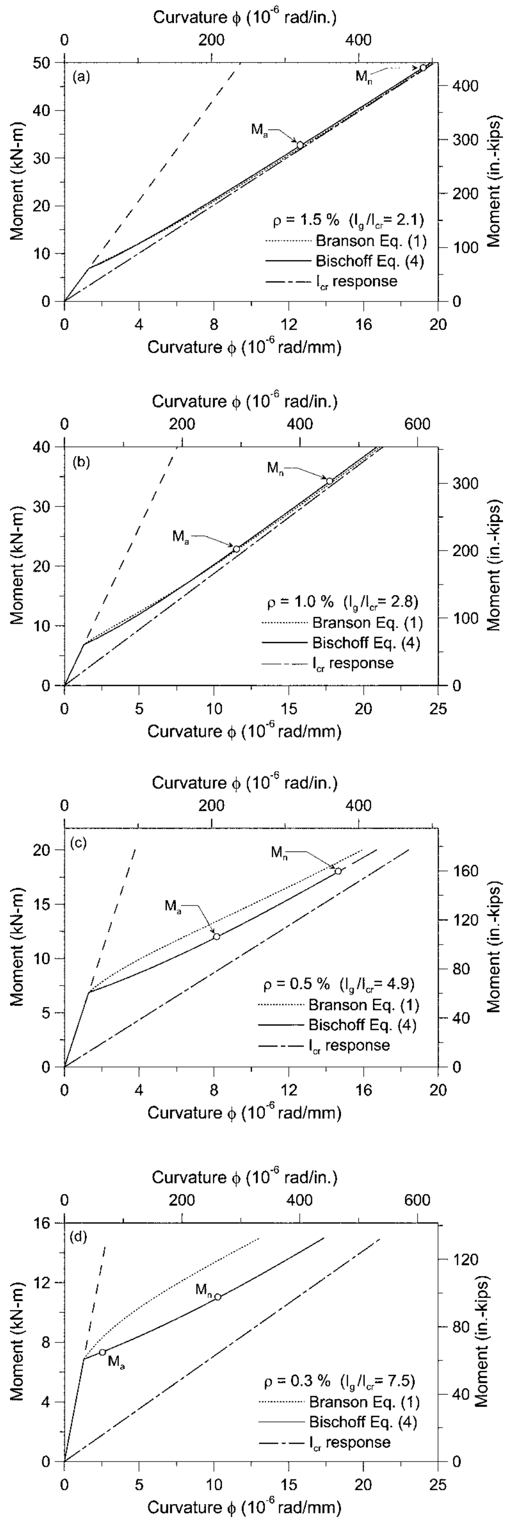

Fig. 3—Computed moment curvature response for: (a) ρ = 1.5%; (b) ρ = 1.0%; (c) ρ = 0.5%; and (d) ρ = 0.3%.

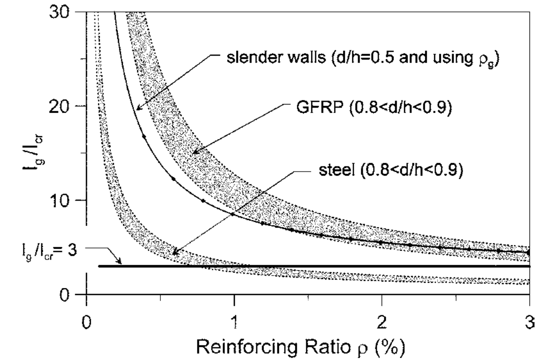

Fig. 4—Variation of I_g/I_{cr} ratio with reinforcing ratio.

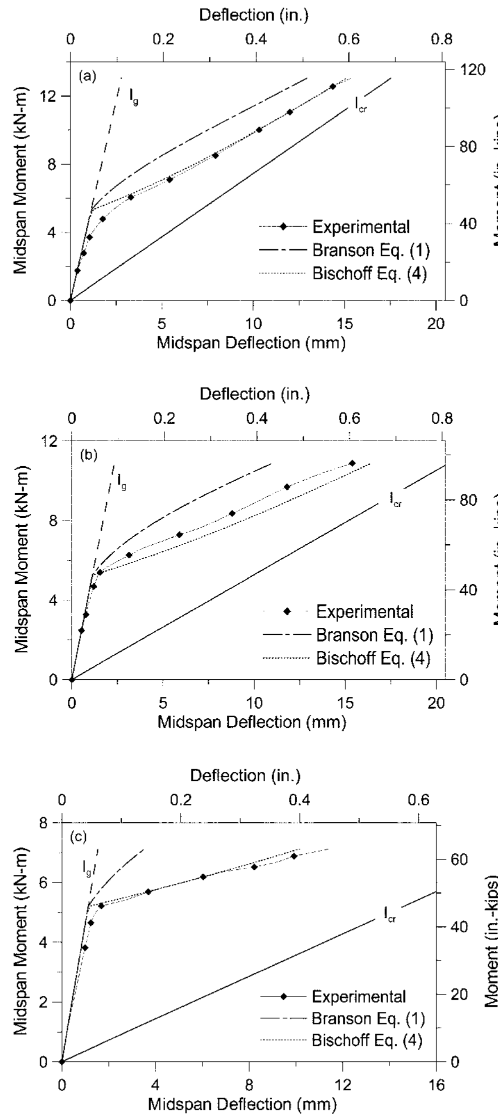

Fig. 5—Slab response for: (a) ρ = 0.52% and I_g/I_{cr} = 6.3; (b) ρ = 0.33% and I_g/I_{cr} = 8.8; and (c) ρ = 0.20% and I_g/I_{cr} = 13.0 (after Gilbert 2006).

of 100 mm (4 in.) and span of 2 m (6 ft 7 in.) were subjected to third-point loading. The results are quite conclusive in showing that the original Branson formulation produces a load-deflection response at service load levels that is too stiff for steel reinforced members at low reinforcement ratios, whereas the proposed formulation provides a better correlation with the test results. Both formulations were satisfactory at higher reinforcement ratios.

Figure 6 compares the two approaches for beams with a 0.31% reinforcing steel ratio and having a cross section 250 mm (10 in.) wide by 300 mm (12 in.) high. The test response is plotted for two identical beams with a simply supported span of 3 m (9 ft 10 in.) and loaded at the third points. Branson's Eq. (1) provides a response that is too stiff, whereas Eq. (4) slightly overestimates member deflection.

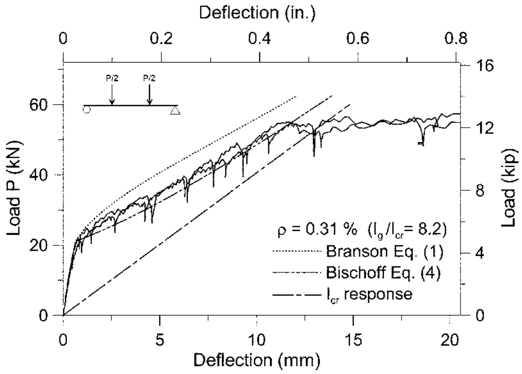

Fig. 6—Steel reinforcement beam response for ρ = 0.31% (I_g/I_{cr} = 8.2).

APPLICATION TO BEAMS WITH FRP REINFORCEMENT

A number of researchers (ACI Committee 440 2006) have shown that Branson's original formulation produces a response that is overly stiff for beams reinforced with FRP bars for which the modulus of elasticity is considerably lower than for steel reinforcement. A modified form of Branson's equation has been recommended for FRP reinforced members (ACI Committee 440 2006) as follows

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 \beta_d I_g + \left(1 - \left(\frac{M_{cr}}{M_a}\right)^3\right) I_{cr} \le I_g \qquad (5)$$

where the correction factor $\beta_d = 0.2\rho/\rho_b \le 1.0$ was empirically derived using a statistical fit of available data. The term ρ_b is the balanced reinforcing bar ratio.

A comparison between the ACI Committee 440 expression in Eq. (5) and the proposed alternative expression with no correction factors (Eq. (4)) is shown in Fig. 7 for beams reinforced with glass FRP (GFRP) bars having an elastic modulus E_b of 40 GPa (5800 ksi) and ultimate strength f_u of 690 MPa (100 ksi). f_c' is taken as 27.6 MPa (4000 psi), and results are normalized with respect to the cracking moment M_{cr} and corresponding uncracked curvature. These plots demonstrate that the proposed expression produces close agreement with the ACI Committee 440 recommended equation without the need to introduce correction factors for FRP. Similar agreement is obtained for carbon FRP (CFRP) reinforced beams. Once again, the reason that Branson's original expression under-predicts deflection is because the I_g/I_{cr} ratio for FRP beams is typically much greater than 3 (refer to Fig. 4).

While the ACI Committee 440 expression for I_e works well for concrete reinforced with either GFRP or CFRP bars, deflection is underestimated with aramid FRP (AFRP) reinforcement (Bischoff 2007). Comparison with the measured load-deflection response of an AFRP reinforced concrete beam tested by Rashid et al. (2005) is presented in Fig. 8, and clearly shows that Bischoff's alternative expression for I_e computes deflection reasonably well for beams with this type of reinforcement. The beam evaluated had a simply supported span of 2.4 m (7ft 10.5 in.), was loaded at the third points, and had a rectangular 150 x 300 mm (6 x 12 in.) cross section with a reinforcing ratio of 0.4%.

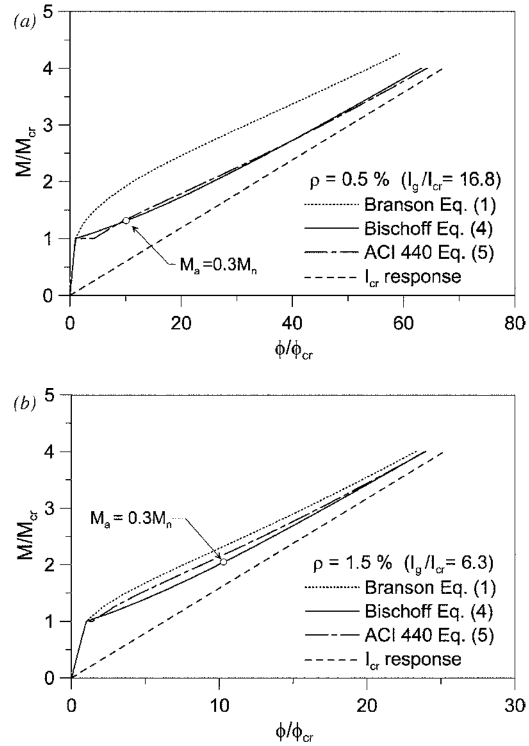

Fig. 7—GFRP beam response for: (a) ρ = 0.5% (I_g/I_{cr} = 16.8); and (b) ρ = 1.5% (I_g/I_{cr} = 6.3).

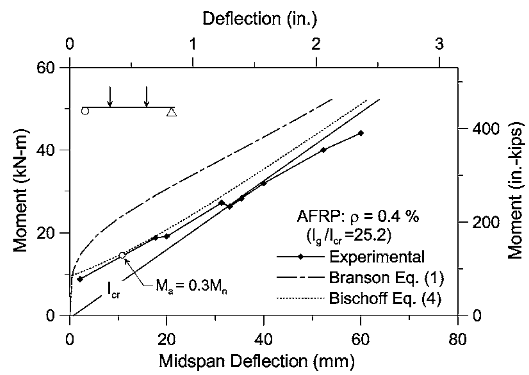

Fig. 8—AFRP beam response comparison (I_g/I_{cr} = 25.2).

DEFLECTION OF SLENDER WALLS

ACI 318-05 (ACI Committee 318 2005) includes an approach for design of slender walls (Section 14.8) that takes account of $P\Delta$ effects when computing the maximum service load deflection Δ_s at mid-height of the wall. This is done by setting

$$M = \frac{M_{sa}}{1 - 5P_s l_c^2 / 48 E_c I_e} \qquad (6)$$

to compute

$$\Delta_s = \frac{5Ml_c^2}{48E_cI_e} \le l_c/150 \qquad (7)$$

where M_{sa} is the maximum unfactored moment arising from lateral loads and the effect of eccentric axial loads (not

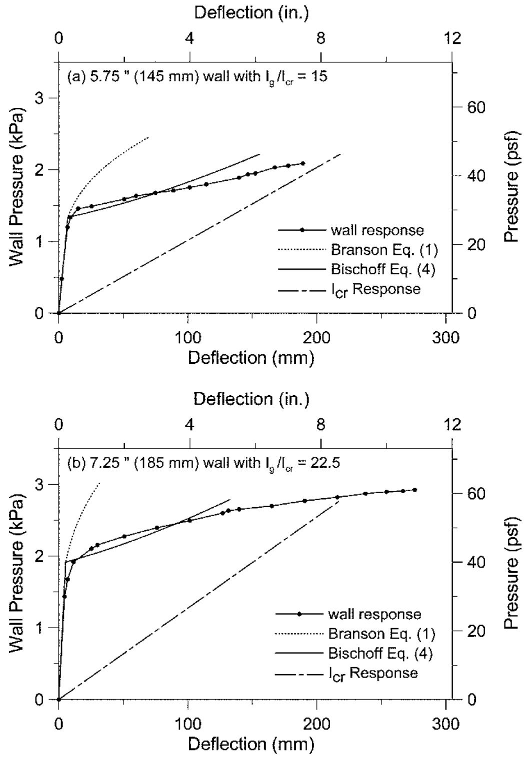

Fig. 9—Response for: (a) 145 mm (5.75 in.) thick wall (ρ_g = 0.28%); and (b) 185 mm (7.25 in.) thick wall (ρ_g = 0.23%).

Table 1—Summary of short-term deflections for cantilevered slab examples

Reinforcement	Cantilever slab	Approach	Full service load ($D + L$)	
			Δ, mm (in.)	L/Δ
Steel	2 m (6.5 ft) span (ρ = 0.29%)	Branson Bischoff	2.2 (0.09) 3.8 (0.15)	895 520
	2.4 m (8 ft) span (ρ = 0.42%)	Branson Bischoff	8.8 (0.35) 12.2 (0.48)	275 200
GFRP	2 m (6.5 ft) span (ρ = 0.5%)	Branson* Bischoff	1.1 (0.043)† 1.1 (0.043)†	1810 1810
	2.4 m (8 ft) span (ρ = 0.5%)	Branson* Bischoff	4.6 (0.18) 17.1 (0.673)	530 145
	2.4 m (8 ft) span (ρ = 1%)	Branson* Bischoff	4.4 (0.173) 10.2 (0.40)	555 240

*Deflection calculations for GFRP slabs are based on Branson's original Eq. (1).
†Deflection values based on gross (uncracked) moment of inertia.

including $P\Delta$ effects), P_s is the unfactored axial load at midheight including the effects of self-weight, and l_c represents the vertical distance between simple supports. I_e is calculated using Branson's Eq. (1) taking the moment M from Eq. (6), and iteration is required because I_e and M depend on one another.

Slender walls with a central layer of reinforcement typically have a gross reinforcing ratio (relative to the gross concrete area) less than approximately 0.4%, and this results in a very high I_g/I_{cr} ratio between 15 to 30 because the effective depth-to-height (d/h) ratio has dropped down to 0.5 (refer to Fig. 4). Recall that d/h for beams and slabs typically varies from 0.8 to 0.9. When using Branson's equation for I_e, a high I_g/I_{cr} ratio leads to a very stiff response and subsequent underestimation of member deflection as explained earlier. Figure 9 compares computed deflections with the measured response of full size wall tests carried out by a joint Southern California Chapter ACI/Structural Engineers Association of Southern California Task Committee on Slender Walls (SCCACI-SEAOSC 1982). Comparison is made for 7.3 m (24 ft) high tilt-up wall panels that had a thickness of 145 and 185 mm (5.75 and 7.25 in.), and corresponding slenderness (l_c/t) ratio of 50 and 40. The gross reinforcing ratio for the two wall thicknesses was 0.28 and 0.23%, respectively. Other wall thicknesses were also tested, and each wall was subjected to a small eccentric axial load followed by a uniform lateral pressure applied with an air bag.

Results for the 145 mm (I_g/I_{cr} = 15) and 185 mm (I_g/I_{cr} = 22.5) thick walls are compared with the computed response using both Branson's equation for I_e and Bischoff's alternative approach. Calculations are based on the observed cracking moment, and use actual dimensions and measured material properties. The comparison is conclusive in demonstrating the limitations of using Branson's approach when the I_g/I_{cr} ratio of the member cross section exceeds 3, while the approach proposed with Eq. (4) is clearly a suitable alternative. The ACI approach using Branson's expression predicts service load deflections reasonably well for walls with a double layer of reinforcement and subsequent higher reinforcing ratios (SEAOSC 2005), as does the proposed approach.

DESIGN EXAMPLES

Design examples are worked out for 2 and 2.4 m (6.5 and 8 ft) long cantilevered slabs reinforced with either steel or GFRP bars. The concrete is assumed to have a specified compressive strength f_c' of 27.6 MPa (4000 psi). In addition to their own self weight, each slab is subjected to an additional dead load of 0.48 kPa (10 psf), live load of 3.4 kPa (70 psf), and permanent line load of 4.4 kN/m (300 plf) at the end of the slab. This gives a ratio of unfactored dead-to-live load moment at the base of the cantilever of approximately 3. Immediate (short-term) deflection is calculated under full (dead plus live) service load. Long-term deflection is not considered. Results of each design are shown in Fig. 10 and deflection values are summarized in Table 1. Specific details of the slab calculations are provided in the Appendix.

Both of the steel reinforced cantilever slabs have a 200 mm (8 in.) thickness based on the minimum thickness requirement for the shorter 2 m (6.5 ft) slab. This slab just satisfies the minimum thickness requirement of $h_{min} = L/10$ = 200 mm (7.9 in.), whereas the longer 2.4 m (8 ft) slab would need a thickness of 245 mm (9.6 in.) to satisfy the deflection control requirement. The steel reinforced slabs are designed for strength and are lightly reinforced with reinforcing ratios of 0.29 and 0.42% for the shorter and longer spans, respectively. Not surprisingly, the shorter slab exhibits a much larger span-to-deflection (L/Δ) ratio using either the Branson expression or proposed alternative approach. Note, however, that Branson's equation under-predicts deflection in this case by approximately 40% compared with the Bischoff equation. For the longer 2.4 m (8 ft) slab, Branson's equation under-predicts deflection by approximately 30% and this extra stiffness is sufficient to give an L/Δ ratio of approximately 275. Deflection values computed with the proposed alternative equation give a lower L/Δ ratio of 200 and a slab that is less likely to satisfy deflection limits. This demonstrates that potential problems with deflection can arise when using Branson's value of I_e for lightly reinforced members. Other factors such as the assumed value of the cracking moment M_{cr} can also affect deflection calculations.

The thickness of the GFRP reinforced slabs was initially based on the ACI Committee 440 (2006) recommendation

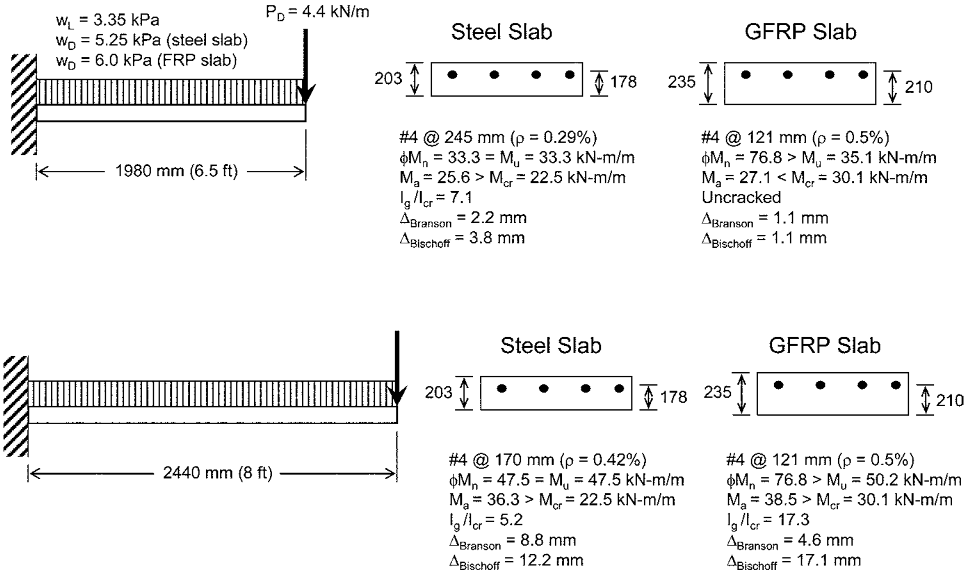

Fig. 10—Cantilever slab design examples. (Note: 1 mm = 0.04 in.; 1 m = 3.28 ft, 1 kN-m/m = 224.8 in.-lb/in., 1 kPa = 20.89 psf, and 1 kN/m = 68.5 plf).

for minimum thickness giving h_{min} = L/5.5 = 360 mm (14.2 in.) for the shorter slab. This recommendation was much too conservative and gave a slab with a cracking moment M_{cr} that exceeded even the ultimate factored moment M_u. A more reasonable thickness of 235 mm (9.25 in.) is used for this example, giving a span-to-depth (L/h) ratio of 8.4 for the shorter slab. Even at this thickness, the shorter slab does not crack under full service load with M_{cr} = 30.1 kN-m/m (6.8 in.-kip/in.) and M_a = 27.1 kN-m/m (6.1 in.-kip/in.). Deflections based on the gross uncracked section easily satisfy deflection criteria with L/Δ equal to 1800.

Serviceability often governs design with the lower stiffness FRP bars (Bischoff 2005a), and an initial estimate of ρ = 0.5% is used for the longer 2.4 m (8 ft) slab with the same thickness of 235 mm (9.25 in.). This gives an over-reinforced beam with ρ/ρ_b = 1.17 and I_g/I_{cr} = 17.3. Branson's equation only gives approximately 1/4 of the expected deflection compared with the proposed approach because of the high I_g/I_{cr} ratio. This results in a high L/Δ ratio greater than 500 because of the unrealistically stiff response, whereas the L/Δ ratio is less than 150 using the deflection value obtained with Bischoff's approach. Note that the calculated design strength of 76.8 kN-m/m (17.3 in.-kip/in.) is more than adequate to resist the factored ultimate moment of 50.2 kN-m/m (11.3 in.-kip/in.). Creep rupture stress limits are also satisfied. Increasing the reinforcing ratio to approximately 1% (ρ/ρ_b = 2.3) has little effect on deflection values calculated with Branson's equation, but decreases deflection significantly using Bischoff's approach to give an L/Δ ratio of 240 for this particular example. It should be noted that in all examples, the slab is assumed fixed at the support. In most design situations it would be necessary to add the contribution of support rotation to obtain the total deflection at the end of the cantilever span.

Whereas it is recognized that time-dependent deflection caused by creep and shrinkage comprise a significant part of the total deflection experienced by a reinforced concrete flexure member, the intent herein is to highlight the differences between the two approaches and demonstrate the relative ease with which deflection can be calculated using the proposed approach. Hence, only short-term deflections are considered in the examples provided. Effects of long-term behavior can be easily evaluated using the long-term multiplier from the ACI 318 Code, and for a dead-to-live load ratio of 3:1 would give an additional long term deflection that is one-and-a-half times the short-term value (assuming no sustained live load and a worst case scenario using a deflection multiplier of 2.0 for sustained loads of 5 years or more). The total deflection occurring after attachment of the non-structural elements would then equal 1.75 times the computed short-term values for the examples considered in this paper. For these calculations, it is assumed that both the dead and live load deflection values are obtained with the same effective moment of inertia under full dead plus live load. In other words, the member has been previously loaded up to this load level during construction.

CONCLUSIONS AND RECOMMENDATIONS

The adoption of Branson's effective moment of inertia expression in the 1971 and subsequent editions of the ACI Code (ACI Committee 318 1971, 2005) was a significant advance in recognizing the gradual transition from an uncracked section to cracked transformed section response with increasing load beyond the cracking load. This response replaced the abrupt transition at $\rho = 500/f_y$ in psi ($3.45/f_y$ in MPa) as previously assumed. Branson's expression was verified for steel reinforcement ratios greater than 1%, but does not work well for lower steel reinforcement ratios nor for beams reinforced with FRP bars. Service load deflections are also underestimated for slender walls with a central layer of reinforcement. In this paper, it has been demonstrated that an alternative formulation of the effective moment of inertia as given by Eq. 4(a) or (b) is applicable to steel reinforced flexure members at all ranges of reinforcement ratio as well as FRP beams without the need to apply correction factors. It is recommended that the effective moment of inertia expression given in ACI 318-05 (ACI Committee 318 2005) be replaced with an equation of the form $I_e = I_{cr}/[1 - \eta(M_{cr}/M_a)^2]$ where $\eta = 1 - I_{cr}/I_g$. This equation is simple and as easy to use as Branson's original expression for control of deflection.

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NOTATION

A_b = reinforcing bar area
b = beam or slab width
c_{cr} = neutral axis depth for cracked section
d = effective depth of reinforcement
E_b = elastic modulus of reinforcing bar
E_c = elastic modulus of concrete
f_b = bar stress
f_c' = specified compressive strength of concrete
f_r = concrete rupture modulus
f_u = ultimate (design) strength of FRP bar
f_y = yield stress of reinforcing steel
h = beam height or slab thickness
h_{min} = minimum beam height or slab thickness for deflection control
I = moment of inertia
I_{cr} = cracked transformed moment of inertia
I_e = effective moment of inertia
I_g = gross moment of inertia
k = spring stiffness (k_1 or k_2) used in spring models
k_{cr} = normalized neutral axis depth (c_{cr}/d) of cracked section
k_e = equivalent spring stiffness for spring models
L = beam or slab span length
l_c = vertical span for walls
M = moment (includes $P\Delta$ effects for slender walls)
M_a = applied service load moment
M_{cr} = cracking moment
M_D = dead load moment
M_L = live load moment
M_n = nominal moment capacity
M_{sa} = maximum (unfactored) wall moment
M_u = factored moment
m = power coefficient in Eq. (4) set equal to 2
n = modular ratio (E_b/E_c)
P = axial load or moment used in spring model
P = applied beam load or axial wall load
P_D = dead line load
P_s = axial load at mid-height of wall
R_n = nominal flexural resistance factor (M_n/bd^2)
t = wall thickness
w = uniformly distributed load (w_D and w_L for dead and live loads, respectively)
α_1 = rectangular stress block factor for stress
β_1 = rectangular stress block factor for depth of compression zone
β_d = correction coefficient used in modified Branson expression (Eq. (5))
Δ = deflection
Δ = spring displacement or rotation used in spring model
Δ_s = wall deflection
ε_b = bar strain
ε_{cu} = ultimate compressive strain in concrete (3000 $\mu\varepsilon$)
η = stiffness reduction coefficient ($1 - I_{cr}/I_g$)
ϕ = curvature
ϕ = strength reduction factor
ϕ_{cr} = uncracked curvature at M_{cr}
ρ = reinforcing ratio (A_b/bd)
ρ_b = balanced reinforcing ratio
ρ_g = gross reinforcing ratio (A_b/bh)

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APPENDIX

Design details of steel and FRP reinforced concrete slabs

This appendix provides a detailed explanation of the designs carried out for the cantilever slab examples. Design requirements for steel reinforced concrete are based on ACI 318-05 (ACI Committee 318 2005), and requirements for the FRP reinforced concrete follow the recommendations of ACI 440.1R-06 (ACI Committee 440 2006). Whereas the steel reinforced concrete slabs are under-reinforced to ensure yielding of the steel before the concrete crushes, design of the FRP slabs is based on an over-reinforced section using an equation for bar stress f_b based on flexure strength analysis (see below). In this case, the concrete crushes before the bar ruptures. The flexure capacity ϕM_n for design is then calculated using the flexure resistance equation for nominal strength $M_n = R_n bd^2$. Creep rupture of the glass FRP bars under sustained loading is also considered by limiting the bar stress under sustained service loads to $0.2f_u$. Other requirements such as shear and bond strength are outside the scope of this study.

Normalweight concrete with a specified compressive strength of f_c' = 27.6 MPa (4000 psi) is used with either Grade 60 steel reinforcement having f_y = 415 MPa (60 ksi) and E_b = 200 GPa (29,000 ksi), or GFRP bars with a design tensile strength f_u = 690 MPa (100 ksi) and elastic bar modulus E_b of 40 GPa (5800 ksi). Table 2 provides a detailed summary of calculated values for each design example.

Concrete properties

$$E_c = 4730\sqrt{f_c'} \text{ and } f_r = 0.62\sqrt{f_c'} \text{ in MPa}$$
$$E_c = 57{,}000\sqrt{f_c'} \text{ and } f_r = 7.5\sqrt{f_c'} \text{ in psi}$$

Stress block

$$\alpha_1 = 0.85 \text{ and } \beta_1 = 0.85 - 0.05(f_c' - 27.6)/6.9 \geq 0.65$$
$$\text{(in MPa) at } \varepsilon_{cu} = 3000\mu\varepsilon$$

$$\beta_1 = 0.85 - 0.05(f_c' - 4000)/1000 \geq 0.65 \text{ (in psi)}$$

Strength reduction factor for steel (ACI 318-05)

$$\phi = 0.65 \leq [3.95 - 2(\rho/\rho_b)]/3 \leq 0.90$$

Strength reduction factor for GFRP (ACI 440.1R-06)

$$\phi = 0.55 \leq 0.3 + 0.25(\rho/\rho_b)] \leq 0.65$$

Flexural strength analysis

$$\phi M_n \geq M_u = 1.2M_D + 1.6M_L$$

- Reinforcing bar ratio

$$\rho = A_b/bd$$

- Balanced reinforcement ratio

$$\rho_b = \alpha_1\beta_1\frac{f_c'}{f_u}\frac{\varepsilon_{cu}E_b}{\varepsilon_{cu}E_b + f_u}$$

(note that f_u is replaced by f_y for steel).

- Bar stress

$$f_b = 0.5E_b\varepsilon_{cu}[\sqrt{1 + 4\alpha_1\beta_1 f_c'/(\rho E_b\varepsilon_{cu})} - 1] \leq f_u \text{ or } f_y$$

- Nominal flexural resistance factor

$$R_n = \frac{M_n}{bd^2} = \rho f_b[1 - \rho f_b/(2\alpha_1 f_c')]$$

Serviceability analysis

$$f_b = n\frac{M_a}{bd^2}\frac{(1 - k_{cr})}{(k_{cr}^3/3 + n\rho(1 - k_{cr})^2)} = \frac{M_a}{\rho bd^2(1 - k_{cr}/3)}$$

- Cracked section properties

$$I_{cr}/bd^3 = k_{cr}^3/3 + n\rho(1 - k_{cr})^2$$

with

$$c_{cr} = k_{cr}d, k_{cr} = \sqrt{(n\rho)^2 + 2n\rho} - n\rho \text{ and } n = E_b/E_c$$

Deflection calculations under full (dead + live) service load:

Table 2—Cantilever slab design details

	Steel reinforced slab		GFRP reinforced slab	
	2 m cantilever	2.4 m cantilever	2 m cantilever	2.4 m cantilever
L: m (ft)	1981 (6.5)	2438 (8)	1981 (6.5)	2438 (8)
h_{min} = L/10[*] or L/5.5[†]	198 (7.8)	244 (9.6)	360 (14.2)	443 (17.5)
h: mm (in.)	203 (8)	203 (8)	235 (9.25)	235 (9.25)
d: mm (in.)	178 (7)	178 (7)	210 (8.25)	210 (8.25)
M_{cr}: kN-m/m (k-ft/ft)	22.5 (5.1)	22.5 (5.1)	30.1 (6.8)	30.1 (6.8)
M_a: kN-m/m (k-ft/ft)	25.6 (5.8)	36.3 (8.2)	27.1 (6.1)	38.5 (8.7)
M_u: kN-m/m (k-ft/ft)	33.3 (7.5)	47.5 (10.7)	35.1 (7.9)	50.2 (11.3)
M_a/M_{cr}	1.14	1.61	0.9	1.28
ρ	0.29%	0.42%	0.5%	0.5%[‡]
A_b	No. 4 at 245 mm (9.6 in.)	No. 4 at 170 mm (6.7 in.)	No. 4 at 121 mm (4.75 in.)	No. 4 at 121 mm (4.75 in.)
ρ/ρ_b	0.10	0.15	1.17	1.17
ϕ	0.9	0.9	0.59	0.59
M_n: kN-m/m (k-ft/ft)	37.0 (8.3)	52.8 (11.9)	129.8 (29.2)	129.8 (29.2)
ϕM_n: kN-m/m (k-ft/ft)	33.3 (7.5)	47.5 (10.7)	76.8 (17.3)	76.8 (17.3)[§]
M_a/M_n	0.69	0.69	0.21	0.30
$f_{b@M_a}$: MPa (ksi)	298 (43.2)	296 (43.0)	—	183 (26.5)
$\varepsilon_{b@M_a}$: $\mu\varepsilon$	1489	1482	—	4568
I_g: mm^4/m (in.^{4}/ft)	6.99×10^8 (512)	6.99×10^8 (512)	10.81×10^8 (791.5)	10.81×10^8 (791.5)
I_{cr}: mm^4/m (in.^{4}/ft)	0.99×10^8 (72.55)	1.35×10^8 (99.0)	0.63×10^8 (45.9)	0.63×10^8 (45.9)
I_g/I_{cr}	7.1	5.2	17.3	17.3
Service load behavior	Cracked	Cracked	Uncracked	Cracked
$I_{e,Branson}$: mm^4/m (in.^{4}/ft)	5.08×10^8 (371.6)	2.70×10^8 (197.5)	10.81×10^8 (791.5)	5.49×10^8 (401.2)
$I_{e,Bischoff}$: mm^4/m (in.^{4}/ft)	2.95×10^8 (215.9)	1.96×10^8 (143.5)	10.81×10^8 (791.5)	1.47×10^8 (107.8)
$\Delta_{Branson}$: mm (in.)	2.2 (.087)	8.8 (.348)	1.1 (.043)	4.6 (.181)
$(L/\Delta)_{Branson}$	895	276	1811	532
$\Delta_{Bischoff}$: mm (in.)	3.8 (.150)	12.2 (.479)	1.1 (.043)	17.1 (.673)
$(L/\Delta)_{Bischoff}$	521	201	1811	143

[*]Minimum thickness requirement for steel reinforced concrete cantilever slab (ACI 318-05).

[†]Minimum thickness requirement for FRP reinforced concrete cantilever slab (ACI 440.1R-06).

[‡]For ρ = 0.95% or No. 4 at 63.5 mm (2.5 in.): ϕM_n = 110 kN-m/m (24.7 k-ft/ft); M_a/M_n = 0.23; $f_{b@M_a}$ = 98 MPa (14.2 ksi) with bar strain of 2440 $\mu\varepsilon$; $f_{b,sus}$ = 72.3 MPa (10.5 ksi); I_g/I_{cr} = 9.7; $\Delta_{Branson}$ = 4.4 mm (0.173 in.); $\Delta_{Bischoff}$ = 10.2 mm (0.401 in.); and $L/\Delta_{Bischoff}$ = 240.

[§]Bar stress for sustained loading, $f_{b,sus}$ = 135 MPa (19.6 ksi) $\leq 0.2f_u$ = 138 MPa (20 ksi).

- Distributed loads

$$\Delta = wL^4/8E_cI_e = ML^2/4E_cI_e$$

- Concentrated end load

$$\Delta = PL^3/3E_cI_e = ML^2/3E_cI_e$$