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Stability Analysis and Design of Concrete Frames

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Introduction

In recent years slender buildings and slender building components have become common making it necessary to consider stability problems and the deflections produced by lateral loads. These deflections can be relatively large when the building has a large height-to-width ratio. As the building deflects laterally, gravity loads acting on the building produce additional moments and forces in the structure. When these effects are taken into account in the structural analysis of the building, the analysis is referred to as a "second-order analysis." Fig. 1 shows a column acted upon by lateral load H and vertical load P. From this figure it is seen that when the structure has deflected laterally vertical load P will also contribute to the lateral sway of the column. When the column or frame has reached its final deflected position, Δ_2, the axial load produces a sway moment commonly referred to as the "$P\Delta$ moment."

This paper examines procedures for carrying out second-order analyses, the use of second-order analyses in the design of concrete structures, and presents a procedure for the design of columns once such an analysis has been carried out.

Second-Order Analysis of Elastic Frames

A rigorous stability analysis of reinforced concrete frames is a rather complicated matter due to the nonlinear load-deformation relationships of concrete members and the variable reinforcement ratios. For this reason, elastic frames will be considered first to develop basic relationships.

In practical frames designed to satisfy practical Δ/h ratios, the effect of axial loads on the member stiffness is very small and can be neglected (10,21). The error due to neglecting this effect seldom, if ever, exceeds 8% in tall

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frames and is less than 1% for more than 90% of all building columns. In the rest of this paper, therefore, only the $P\Delta$ component of the second-order effects will be considered.

Effective Length Factor Method.—Traditionally the effect of frame action has been accounted for in column design by means of effective length factors. These are calculated using elastically derived equations or nomograms based on highly idealized and quite impractical cases (11). The effective length procedure has serious shortcomings in sway frames or partially braced frames where the columns have widely varying effective length factors.

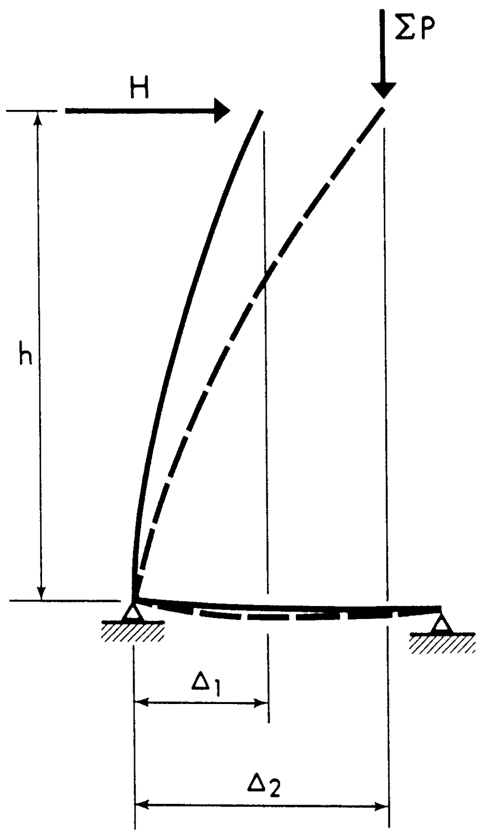

FIG. 1.—Forces on Deflected Column

Some of the problems implicit in the use of this method are illustrated by the frame shown in Fig. 2(*a*). The first-order moments and elastically computed second-order moments in the columns are shown in Fig. 2(*b*). The shaded area shows the design moment envelopes calculated using the American Concrete Institute (ACI) moment magnifier procedure (4,14) using the rotational stiffnesses, *EI*, used in the second-order analysis and the effective length factors for sway frames from the Jackson and Moreland nomographs (11). It is obvious that

there is very little relationship between the actual second-order moments and those estimated using the effective length technique. This is largely due to the effect of the beam moments on the first-order moments and if the magnification were restricted to that portion of the moments resulting from lateral loads, the agreement would be better. Problems also arise in the need to distinguish between "braced" and "sway" frames.

This and other evidence suggests that the traditional effective length factor solution is inadequate for the design of columns in frames.

Approximate Solution for Critical Load of Tall Plane Frames.—Although exact analyses of the critical or bifurication loads of frames exist, they are generally

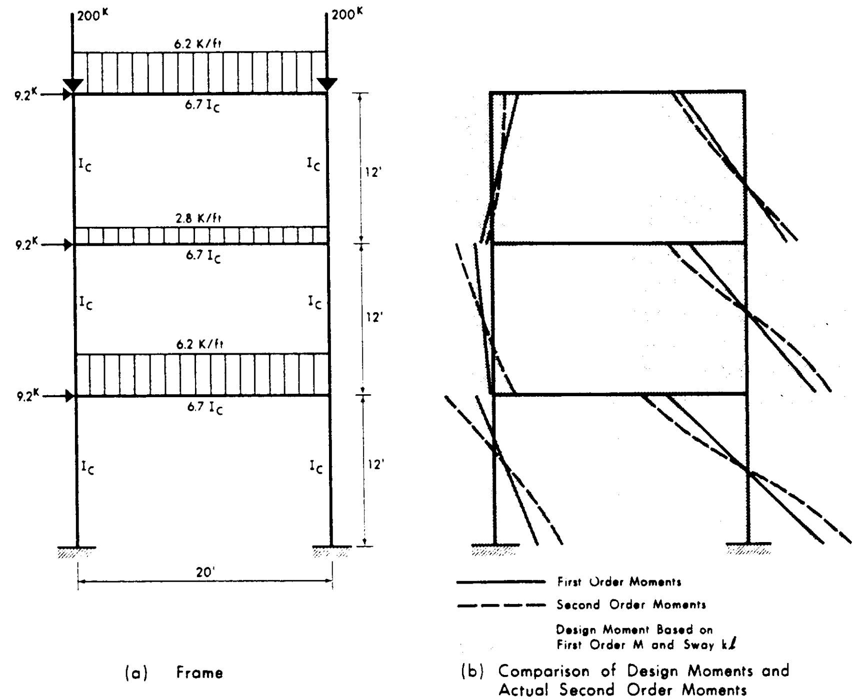

FIG. 2.—Comparison of Moments in Frame by Two Methods of Analysis

not suited for design office use. For this reason a number of authors have proposed simplified approximate calculations of the critical loads of tall frames. Based on simple second-order analyses Rosenblueth (17), Goldberg (9), and Stevens (20) have shown that the critical load of the *i*th story in a sway frame is approximately equal to

$$\Sigma P'_{ci} = \frac{K_{li} h_i}{\gamma} = \frac{H_i}{\gamma} \frac{h_i}{\Delta_{1i}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

in which γ is a factor accounting for the deflected shape of the columns and varies from 1.0 for stiff columns and flexible beams to 1.22 for flexible columns and stiff beams. In general, γ approaches 1.0 in the lower story of a tall building and will be taken equal to 1.0 in this paper.

Eq. 1 shows the strong relationship between critical load ΣP_c and lateral stiffness K_l, or the first-order deflection index, Δ_1/h, for a given lateral load, H.

If the first-order deflections of a building are known from a structural analysis, Eq. 1 can be used to estimate the total load in all the columns in a story at instability. This can be done story by story until the lowest critical load is known.

A similar equation can be derived to estimate the torsional critical load of a story:

$$\Sigma(P_{ct} r^2) = \frac{T_1 h}{\theta_1} \quad \text{(2)}$$

Note that both the magnitude of vertical load ΣP and its spacial distribution, r^2, must be known before the torsional critical load, $\Sigma(P_{ct} r^2)$, can be estimated.

Review of Procedures for Second-Order Analyses.—In a second-order analysis the equilibrium equations are formulated for the deformed frame. Five separate procedures for computing second-order moments and deflections are presented in the following sections.

Moment Magnifier Solution for Second-Order Effects.—The method of analysis described in this section is approximate and is primarily of use in preliminary design. The basic assumption in this method of analysis is that the shapes of the first and second-order deflections are similar. Bending moment M at the base of the column shown by the dashed lines in Fig. 1 is

$$M = Hh + \Sigma P \Delta_2 \quad \text{(3)}$$

If the critical load for this column is P_{cr} and $Q' = \Sigma P/P_{cr}$ then:

$$\Delta_2 = \left(\frac{1}{1 - Q'}\right)\Delta_1 \quad \text{(4)}$$

and $$M_2 = Hh + \Sigma P \Delta_1 \left(\frac{1}{1 - Q'}\right) \quad \text{(5)}$$

As shown earlier the critical load of a story or a column can be approximated using Eq. 1. Thus, second-order moment M_2 can be approximated by Eq. 6, in which Q' has been replaced by $Q = \Sigma P \Delta_1/Hh$:

$$M_2 = Hh + Hh\left(\frac{Q}{1 - Q}\right) = Hh\left(\frac{1}{1 - Q}\right) \quad \text{(6)}$$

Thus, for a given lateral loading pattern leading to first-order frame moments, total second-order moment M_2 at any point in the frame is approximately

$$M_2 = M_1 \delta \quad \text{(7)}$$

in which $$\delta = \frac{1}{1 - Q} = \frac{1}{1 - \dfrac{\Sigma P \Delta_1}{Hh}} \quad \text{(8)}$$

A number of authors have shown that the same moment magnifier can be applied to all sections of the story including beams, columns, and walls provided the moments result only from lateral loads (9,13,14,18). Since the gravity load moments must satisfy statics in any given span, they are not magnified.

Note that the lateral stiffness must correspond to the failure state being considered including any significant inelastic action or foundation deformation that develops before failure.

P-Δ Method

Iterative P-Δ Analyses.—For tall buildings designed for normal deflection limitations, an acceptable estimate of the second-order shears, moments, and forces in an elastic structure can be obtained by an iterative calculation including the "sway forces" induced by the *P*-Δ moments (19,21,22). The computation of sway forces for the combined loading case is relatively simple. The lateral and vertical loads are applied to the structure and the relative first-order lateral displacements, Δ_{1j}, in each story are computed. The additional story shears due to the vertical loads are computed as shown in Fig. 3. At a given floor level, the sway force will be the algebraic sum of the story shears from the columns above and below the floor as shown in Fig. 3. The sway forces are

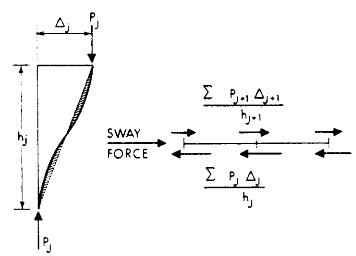

FIG. 3.—Calculation of Sway Forces

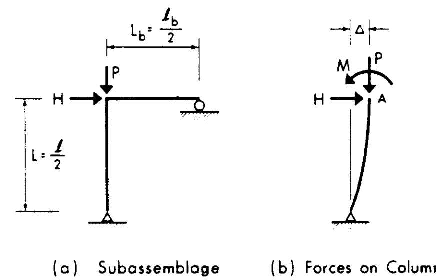

FIG. 4.—Sway Subassemblage

added to the applied lateral loads and the total forces and moments in the structure can be computed. Generally, one or two cycles of iteration are adequate for elastic structures of reasonable stiffness.

Note that all analyses which use sway forces to approximate the *P*Δ effect lead to slightly incorrect column shears.

Direct P-Δ Analyses.—Although the iterative *P*-Δ analysis just described has the advantage of being easily understood, several cycles of iteration may be required for convergence in very slender structures. It is possible, however, to derive an equation for the final second-order deflections. Let H and Δ_1 represent the applied lateral load and the corresponding first-order deflection, respectively. Also, let the axial load be ΣP, the deflection caused by a unit lateral load be F_l, and let $H_i (i = 1, 2, 3, \ldots, \infty)$ be the sum of the applied and additional sway force in the ith cycle. Then, for first-order analysis:

$$\Delta_1 = F_l H \quad \text{(9)}$$

for the first cycle of iteration:

$$\Delta_{(i=1)} = F_l H_{(i=1)} = F_l H \left(1 + \frac{\Sigma P F_l}{h}\right) \quad (10)$$

and for the ith cycle of iteration:

$$\Delta_i = F_l H \left[1 + \left(\frac{\Sigma P F_l}{h}\right) + \left(\frac{\Sigma P F_l}{h}\right)^2 + \ldots \left(\frac{\Sigma P F_l}{h}\right)^i\right] \quad (11)$$

This is a geometric series that converges if $\Sigma P F_l / h < 1.0$, in which case the sum of the infinite series is

$$\Delta_2 = \frac{F_l H}{1 - \dfrac{\Sigma P F_l}{h}} \quad (12)$$

or since $F_l H = \Delta_1$, the final second-order deflection is

$$\Delta_2 = \frac{\Delta_1}{1 - \dfrac{\Sigma P \Delta_1}{H h}} \quad (13)$$

This equation combines Eqs. 1 and 4 or 8 and is identical to equations previously derived by Fey (6), Parme (16), and Goldberg (9).

Limitations on the Use of P-Δ Analysis.—The accuracy of the iterative P-Δ procedure and Eq. 13 were studied by Hage (10) for 690 mathematically generated reinforced concrete column-beam subassemblages similar to Fig. 4. This subassemblage represents a portion of a sway frame. The columns studied included geometric slenderness ratios L/t from 5 to 40, end restraints ranging from fully fixed to values representative of the lower floors of an unbraced flat plate building, and tied and spiral cross sections. The subassemblage response was obtained by numerical integration of moment-curvature diagrams for the cross sections considered (5).

The response of two such subassemblages are compared in Fig. 5(a) and 5(b). In each case the axial load was held constant at 0.4 times the pure axial load capacity, P_o, of the column cross section while the end rotation of the top of the column was increased. From the geometry and the moment curvature relationships it was possible to calculate the corresponding lateral load, H. Both columns were restrained by a beam having a stiffness representative of a T-beam in a sway frame. The sloping line in each figure represents ultimate moment capacity M_u of the column cross section at $P_u = 0.4\ P_o$ minus the $P\Delta$ moment for any given lateral deflection. The curved line in each figure is the lateral load-deflection curve for the column. The column with $L/t = 15$ [Fig. 5(a)] failed when the moments at point A in the subassemblage in Fig. 4 reached the capacity of the cross section. This is called a "material failure." The longer column [$L/t = 30$, Fig. 6(b)] developed a "stability failure" prior to failure of the cross section. The sway deflection index, Δ/L, at which the column became unstable was 0.012 or 1/80 of the story height.

If the *EI* used to calculate the deflections corresponded to the failure of the column considered [i.e., corresponded to point A in Fig. 5(a)], the iterative

$P\Delta$ procedure and Eq. 13 were found to give accurate results provided the columns developed material failures.

In Hage's study, all the stability failures in frames similar to Fig. 4 developed at sway deflections greater than $\Delta/L = 1/300$ at working loads or $\Delta/L = 1/200$ at ultimate as shown in Figs. 6 and 7. A more severe case would exist if a cast-in-place frame braced a pin-jointed precast or steel frame as shown in the inset in Fig. 8. Even for this severe case, there were no stability failures

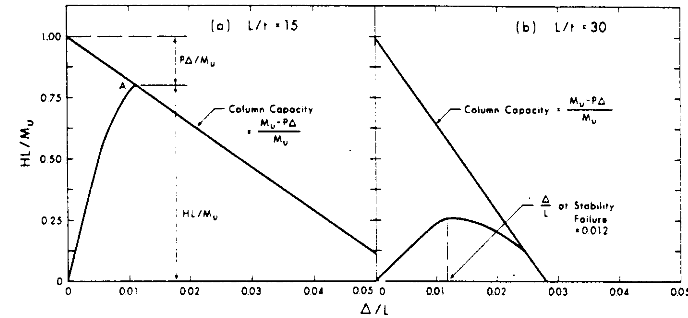

FIG. 5.—Moments at Point A in Subassemblage of Fig. 8

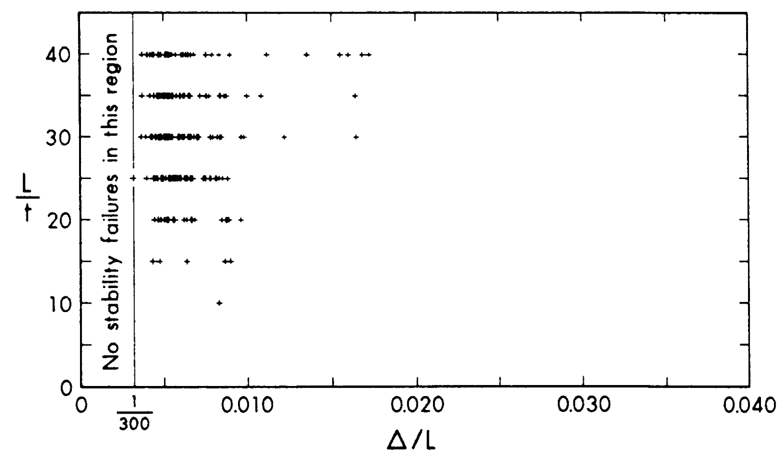

FIG. 6.—Deflection Indices at Minimum Service Load Versus L/t for Stability Failures

detected at Δ/L values less than 0.00225 or 1/450 at ultimate as shown in Fig. 8.

Thus, it may be concluded that the P-Δ method and Eq. 13 can be used to analyze stability effects in continuous frames designed for sway deflections of 1 in 300 or less at working loads, or for reinforced concrete frames bracing pin-jointed structures with sway deformations limited to 1 in 500 at ultimate. As shown by Fintel (9) most concrete buildings are considerably stiffer than this. Other limits on sway deflections may be required for serviceability reasons.

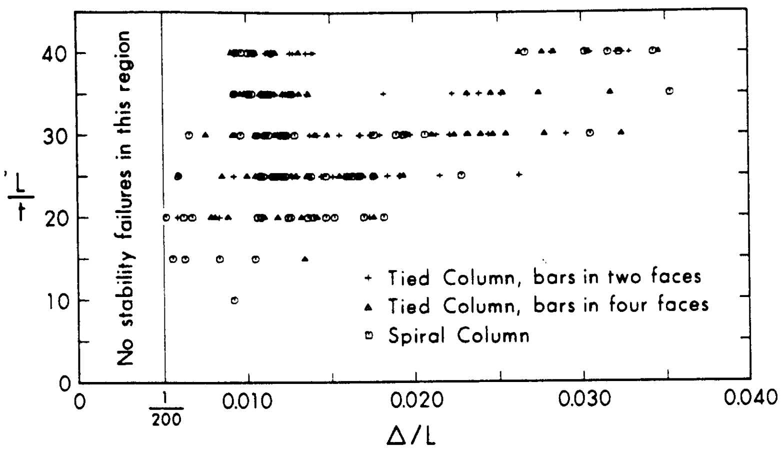

FIG. 7.—Deflection Indices at Ultimate Versus L/t for Stability Loads

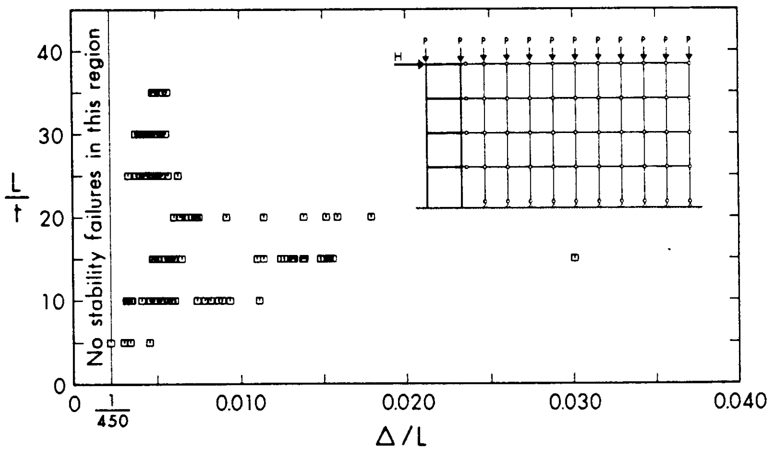

FIG. 8.—Deflection Indices at Ultimate for Leaned Frame

If the iteration process is considered to have converged when the deflection in the ith cycle is within 5% of the final deflection (i defined in conjunction with Eqs. 10 and 11), Eq. 13 can be rearranged (10) to give the number of iterations, i, required:

$$i + 1 = \frac{-1.30}{\log\left(\dfrac{\Sigma P\Delta}{H\,h}\right)} \quad \text{(14)}$$

Alternatively, the check of the iteration process may also be based on the convergence of the moments. For convergence of moments within 5%, Eqs. 3, 11, and 13 can be used to show that convergence will occur in i cycles if

$$\frac{\Sigma P\Delta^{i-1}}{H\,h} \geq 0.0475 \quad \text{(15)}$$

in which $i = 0$ corresponds to a first-order analysis. Using Eqs. 11, 13, and 14 it can be shown that for a convergence limit of 0.05, the deflection will be within 5% of the second-order deflection if convergence is obtained in less than six iterations.

Negative Bracing Member Method.—Nixon, et al. (15) have shown that a direct solution of the second-order deflections and moments can be obtained using a standard first-order structural analysis program by inserting a fictitious diagonal brace of negative area in each story as shown by the dashed lines

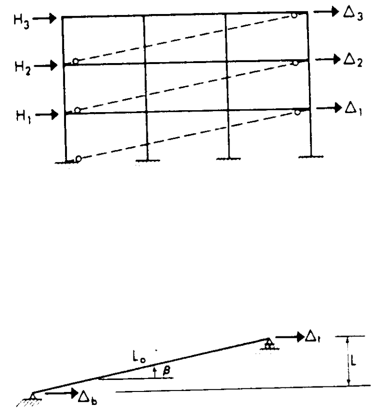

FIG. 9.—Frame with Negative Bracing Members

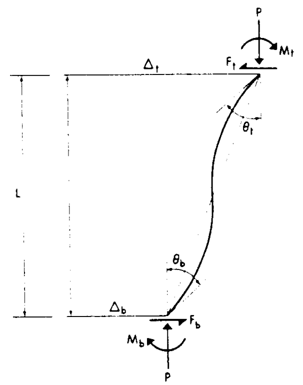

FIG. 10.—Deflected Column

in Fig. 9. The area of this brace can be obtained from an examination of the stiffness matrix for the column shown in Fig. 10. From statics

$$F_t = -\frac{M_t + M_b}{L} - \frac{P(\Delta_t - \Delta_b)}{L} = -F_b \quad \text{(16)}$$

Substituting the slope deflection equations for M_t and M_b into Eq. 16 and writing the equations in matrix form gives:

$$\begin{bmatrix} M_t \\ M_b \\ F_t \\ F_b \end{bmatrix} = \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{6EI}{L^2} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} - \dfrac{P}{L} & -\dfrac{12EI}{L^3} + \dfrac{P}{L} \\ \dfrac{6EI}{L^2} & \dfrac{6EI}{L^2} & -\dfrac{12EI}{L^3} + \dfrac{P}{L} & \dfrac{12EI}{L^3} - \dfrac{P}{L} \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta_b \\ \Delta_t \\ \Delta_b \end{bmatrix} \quad \text{(17)}$$

This is a second-order stiffness matrix because equilibrium is based on the deflected shape.

If a first-order program is used to analyze the frame shown in solid lines in Fig. 9, the stiffness matrix for a column would contain all the terms in Eq. 17 except the four involving P/L. If the structure contained bracing members of length L_o, as shown by the dashed lines in Fig. 9, the program would generate a stiffness matrix corresponding to the degrees-of-freedom shown in Fig. 10, i.e.

$$\begin{bmatrix} F_t \\ F_b \end{bmatrix} = \frac{AE}{L_o} \begin{bmatrix} \cos^2\beta & -\cos^2\beta \\ -\cos^2\beta & \cos^2\beta \end{bmatrix} \begin{bmatrix} \Delta_t \\ \Delta_b \end{bmatrix} \quad (18)$$

and add these terms to the overall stiffness matrix in the same position that the P/L terms would occupy in the second-order matrix. The required area of fictitious bracing member for a given story is obtained by equating $(AE/L_o \cos^2\beta)$ and $(-P/L)$:

$$A = -\frac{P}{L}\frac{L_o}{E\cos^2\beta} \quad (19)$$

in which P = the sum of the axial loads in the columns in the story.

The area found by Eq. 19 is generally very small and is negative. Although bracing members normally stiffen the structure, the artificial bracing members make the structure more flexible. The increased flexibility is due to the $P\Delta$ effect.

This analysis gives a direct calculation of the deflections and moments. Axial loads and shears in the columns are slightly in error, however, because of the horizontal and vertical components of the force in the bracing members, but can be easily corrected using statics. The effect of the vertical component can be reduced using long bracing members, as shown by the dashed lines in Fig. 9, since the horizontal component, i.e., the $\Sigma P\Delta/L$ term, is constant for a given story.

Second-Order Finite Element Analysis.—Aas-Jakobsen (1) has proposed a finite element approach to solve for second-order effects under linear-elastic conditions. The stiffness matrix $[K]$ is assumed to be the sum of two stiffness matrices $[K_1]$ and $[K_2]$, in which $[K_1]$ is the first-order stiffness matrix and $[K_2]$ is obtained through an iteration procedure. When unit displacements are applied to the member the axial load required to maintain equilibrium is unknown and can only be obtained by trial and error. Aas-Jakobsen suggests that the axial load be set equal to zero in the first cycle.

The first-order axial forces obtained in the first cycle are used in the second cycle. The process is repeated until the axial load found in one cycle is close to the value computed in the previous cycle. Since second-order effects will not change the axial loads in the columns significantly the process will usually converge rapidly so that two cycles are generally sufficient.

Procedures for Second-Order Analysis—Summary.—Five alternative procedures for carrying out second-order analysis have been presented in the preceding sections. The first is proposed primarily for preliminary design. The iterative and direct $P\Delta$ methods and the Negative Bracing Method seem well suited for office use in conjunction with standard computer analyses. The fifth method is promising but is not included in most computer programs available for design office use. Its inclusion in such programs requires a relatively simple program change.

Design of Reinforced Concrete Buildings Considering Stability

Stability Index.—The term $Q = \Sigma P\Delta_1/Hh$ will be referred to as the "stability index." This term, which appears frequently in the first part of this paper, is a relatively good estimate of the ratio $\Sigma P/\Sigma P_{cr}$ for either a single story or in many cases for the entire building depending on the definition of the terms. This assumes that Eq. 1 can be used to estimate P_{cr}. In general, this equation is a good estimate of the critical load of a story and, if the vertical load per story, the lateral load per story, and sway angle Δ/h are relatively constant over the height of the building, Q is a good measure of $\Sigma P/\Sigma P_{cr}$ for the entire building. If these things are not true, Q can be used story by story to estimate the critical loads of each story. A "torsional stability index," $Q_t = \Sigma(Pr^2)\theta_1/T_1 h$ can also be expressed, based on Eq. 2.

Tests of Need for Second-Order Analyses.—Since a second-order analysis requires additional analysis time and expenditures, it is desirable to have some means of determining in advance whether such an analysis is required. Alternatively, it may be desirable to control the stiffness of a building such that the second-order effects are not too significant in the overall response of the building.

Eq. 15 can be used to derive tests of whether second-order effects can be ignored. If it is assumed that first-order moments will be sufficiently accurate for design if they are within 5% of the second-order moments, Eq. 15 shows that a second-order analysis can be ignored if $Q \leq 0.0475$. If Q is between 0.0475 and 0.22 sufficiently accurate moments will be obtained if the second-order moments are calculated directly from the first-order deflections.

Based on analyses of hinged frames braced by a shear wall acting as a vertical cantilever, Beck and König (3) have defined a similar stability parameter α:

$$\alpha = h\sqrt{\frac{P}{EI_g}} \quad (20)$$

in which P and h refer to the total load and the total building height.

If $\alpha \leq 0.6$, Beck and König suggest that a first-order analysis is sufficient and if $\alpha > 0.9$ a second-order analysis is necessary. These limits can be shown to correspond to Q equal to about 0.0476 and 0.1, respectively.

Dicke (private communication) has shown that if foundation rotations occur, the critical load tends to decrease making the stability test based on Eq. 20 unconservative. If foundation deformations have been considered in the computation of the first-order deflections, Δ_1, however, the stability checks based on Q (e.g., Eq. 15) do not need any further correction.

It is also important to set an upper limit on Q to ensure adequate safety and stiffness. Dicke has shown that for large values of $Q' = \Sigma P/P_{cr}$, the factor $1/(1 - Q')$ is strongly affected by errors in calculating P_{cr} and Q'. Such errors could result from using the wrong flexural rigidities, EI, or from a mistake in determining the rotational rigidity of the foundation. Based on assumed statistical distributions of loads and EI values, Dicke has shown that for $Q' < 0.1$ the probability of failure of a structure designed using a second-order analysis is relatively constant while for $Q' > 0.2$ the probability of failure increases rapidly. For this reason he recommends an absolute upper limit of 0.2 for Q' which is equivalent to limiting $\Sigma P/P_{cr}$ for a story or frame to 0.2 or less. A similar limit should be imposed on Q.

To summarize, limitations based on stability index Q have been proposed for determining whether second-order analyses are required. Second-order effects can be ignored in design if $Q < 0.0475$ for instability about all three axes of the building. For Q between 0.0475 and 0.22 the error in second-order moments will be less than 5% if moments of $P\Delta$ moments are based on first-order deflections Δ_1. The value of Q should not be taken greater than about 0.2. For concrete building frames Q will generally be less than 0.0475 if the sway index is less than 1/500 at ultimate.

This concept can be taken a step further. If a braced frame is defined as a frame in which the $P\Delta$ moments are so small they can be neglected, the factor Q can be used to define when a frame is braced or free to sway. Based on the assumption that a magnification of less than 5% is negligible (14), a braced frame can be defined as one for which $Q < 0.0475$.

Design Oriented Second-Order Analyses.—In each of the following analyses the limitations on Δ/h and Q presented in preceding sections must be observed. Thus, Δ/h must not exceed 1/200 at ultimate and 1/300 at service in fully continuous frames, and 1/500 at ultimate in frames bracing pin-jointed structures. Generally, this is not a serious problem since it is often necessary to limit Δ/h to 1/500 or so for serviceability reasons. In addition, Q must not exceed 0.2 as mentioned earlier.

P-Δ Analyses.—The basic iterative *P-Δ* procedure described in the first part of this paper will always give a good estimate of the second-order moments if the columns develop material failures rather than stability failures.

Alternatively, Eq. 13 can be used to compute the second-order deflections from the first-order deflections. A second-order analysis suitable for design would include: (1) A first-order analysis to determine Δ_1 in each story; (2) computation of the second-order deflection in each story using Eq. 13; (3) evaluation of the sway forces as outlined previously but using story deflection Δ_2—note that the sway forces may be positive or negative; and (4) another first-order frame analysis for the frame subjected to the applied vertical and lateral loads plus the sway forces from step 3, gives the second-order moments and forces.

Moment Magnifier Method.—For columns in sway frames the ACI Code (4) requires the use of the moment magnifier given by Eq. 21 to magnify the moments in a given story:

$$\delta = \frac{1}{1 - \dfrac{\Sigma P}{\Sigma P_c}} \qquad (21)$$

in which $P_c = \pi^2 EI/(KL)^2$. In Eq. 21 the critical load of the story is approximated by ΣP_c based on the free to sway effective lengths of each column in the story (4,14). A much simpler and equally accurate procedure results from taking

$\Sigma P/\Sigma P_c = Q$ as given in Eq. 8 (10). A second-order analysis suitable for design would include:

1. A first-order analysis for lateral loads to determine Δ_1 in each story. This can be carried out at service load levels as part of the serviceability check. If the analysis is carried out at the service load level the first-order ultimate wind moments will be $\lambda_w M_{1s}$, in which λ_w is the wind load factor and M_{1s} is the first-order service load wind moment.
2. Compute $Q = \Sigma P_u \Delta_1 / H_1 h$ for each story, in which Δ_1 and H_1 are both for the same load level, either service or ultimate; and ΣP_u is the sum of the ultimate loads in the story.
3. The second-order moments at ultimate due to lateral loads can be computed as $\delta \lambda_w M_{1s}$, in which

$$\delta = 1.0 \quad \text{when} \quad Q \le 0.0475 \qquad (22a)$$

$$\delta = \frac{1}{1 - Q} \quad \text{when} \quad 0.0475 < Q \le 0.2 \qquad (22b)$$

4. The moments from step 3 must be added to those from a vertical load analysis at the appropriate load factors.

Amplified Lateral Load Analysis.—An even simpler method of analysis involves using Eq. 8 to amplify the lateral loads prior to carrying out the first-order frame analysis. If this is done, the resulting moments will approximate the second-order moments. The lateral loads to be used in analysis are taken as $1/(1 - Q_u)$ times the actual lateral loads, in which $Q_u = (\Sigma P_u / H_u)(\Delta_u / h)$. At this stage, Q_u is based on assumed drift Δ_u/h. If the resulting value of Δ_u/h exceeds the assumed value the structural framing should be adjusted to reduce Δ_u/h to the desired value. The accuracy of this method of analysis decreases if there are large differences in the lateral stiffness of various stories.

In this procedure the moments and deflections are being calculated at the ultimate limit state. For serviceability reasons it is necessary to know the second-order deflections at service loads. Studies of the effective flexural stiffness values for partially cracked beams and columns and studies of a number of typical frames suggest that the *EI* at service loads is roughly 1.7 times that at ultimate loads (10), assuming the ACI value of E and the gross moment of inertia are applicable at service loads. For the wind loading case, the ACI load factors are 1.05 dead + 1.28 live + 1.28 wind. Thus, the first-order deflection at ultimate is $1.28 \times 1.7 = 2.2$ times the first-order service load deflection. However, the deflection causing uncomfort or damage is the total or second-order deflection at service loads. From Eq. 13 the relationship between service load and ultimate load deflections can be estimated:

$$\frac{\Delta_{2s}}{h} = \frac{\dfrac{\Delta_{1s}}{h}}{1 - \dfrac{P_s \Delta_{1s}}{H_s h}} \qquad (23)$$

and $$\frac{\Delta_{2u}}{h} = \frac{\dfrac{\Delta_{1u}}{h}}{1 - \dfrac{P_u \Delta_{1u}}{H_u h}} \qquad (24)$$

in which subscripts s and u refer to service and ultimate loads, respectively. If we let $(\Delta_{2s}/h) = C(\Delta_{2u}/h)$, $H_u = 1.28\ H_s$, and $P_u = (1.05\text{-}1.28)P_s$ then from Eqs. 23 and 24 we get:

$$C = \frac{1}{2.2}\left[\frac{1 - Q_u}{1 - \dfrac{Q_u}{(1.05 \text{ to } 1.28) \times 1.7}}\right] \qquad (25)$$

If Q_u is not allowed to exceed 0.2, $C \leq 0.40$ to 0.41 and thus the deflections at service loads should be at least 40% of those at ultimate.

A design procedure based on the amplified lateral load analysis procedure would entail:

1. Establish the maximum allowable Δ_s/h in a story at service loads for serviceability and to limit nonstructural damage. In a multistory building it may be assumed that the Δ_s/h for the entire building will be 85% or less of the maximum story sway.
2. Compute the overall Δ_u/h at ultimate as $0.85/0.40 = 2.125$ times the maximum story sway at service loads. This should not exceed the limits given earlier.
3. Compute $Q_u = (\Sigma P_u/H_u)(\Delta_u/h)$ and compute the amplified lateral load, $H_{2u} = H_u/(1 - Q_u)$; Q_u should not exceed 0.2.
4. Analyze the frame using amplified lateral load H_{2u} and the factored vertical loads.
5. If the resulting overall Δ_u/h exceeds that computed in step 2 or if the Δ_u/h in any story exceeds $1/0.85$ times the computed value the structure should be stiffened to reduce Δ/h.

Flexural Stiffnesses for Use in Second-Order Analyses.—The major problem in any stability or second-order analysis of concrete structures is the choice of a suitable mathematical model of flexural stiffness EI under various loading conditions. Ideally the EI value should reflect the amount of reinforcement, the extent of cracking, axial loads, creep, and the inelastic behavior of the steel and concrete. Furthermore, the EI values should reflect the variation of stiffness along the entire length of each member taking into account cracked and uncracked regions and should not merely represent the most highly loaded section. Clearly, however, when dealing with a 20-story building with more than 1,500 members and more than 4,000 critical sections it is not economically feasible for designers to go into this detail and simplified methods must be used to compute EI.

If elastic second-order analyses are to be used in the design of columns in a building it is important that the deflections be representative of those at the factored design loads. Thus the EI values required for such an analysis should be those at the stage immediately prior to the onset of yielding at the critical sections in the members.

Kordina (12) and Hage (10) have studied the variation of stiffness for various types of frame members subjected to gravity load moments, wind load moments, and combinations of the two. Based on these studies, a reasonable estimate of EI for second-order analyses would be based on the ACI value of E and $I = 0.4\ I_g$ for beams and $0.8\ I_g$ for columns. (These correspond to the assumptions made in deriving Eq. 25.)

Tests and analyses of multistory steel buildings reported by Baker (2) showed that sway tended to reduce the dead-load column moments by an average of 2% and up to 6%. Because of this and because lateral deflections due to dead loads are generally small compared to the instantaneous deflections due to wind or earthquake loads, E can be taken as the short time modulus of elasticity of concrete. In the rare case where sustained lateral loads exist (buildings providing reactions for arch roofs, etc.), or if the creep deflections of one side of a building exceed those on the other side due to poor design (gravity drift), it will be necessary to include the creep deflections in the second-order analysis.

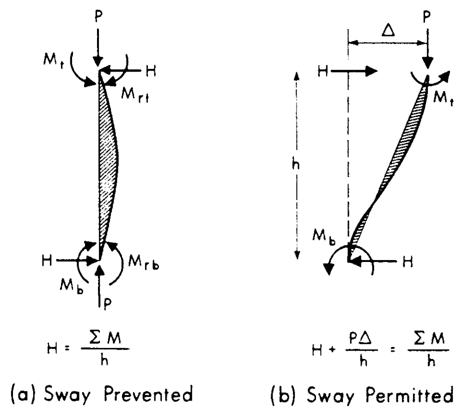

FIG. 11.—Forces in Deflected Columns

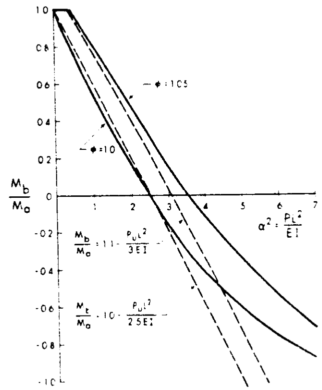

FIG. 12.—Limits on Columns with Maximum Moment at End of Column

Column Design Following Second-Order Analysis.—Fig. 11 shows columns with and without lateral displacements of the ends. If translation is prevented, the deflected shape is as shown in Fig. 11(a). Moments M_t and M_b are the applied end moments while M_{rt} and M_{rb} are restraining moments caused by the rotations of the end restraints as the column deflects. Horizontal forces H are present if the end moments are unequal. At midheight there are secondary moments equal to the axial load times the deflections shown shaded. To account for restraining moments M_{rt} and M_{rb} in the design of this "braced" column an effective length less than the real length is used to compute the lateral deflections.

If, however, the column is free to sway laterally as shown in Fig. 11(b), moments M_t and M_b must equilibriate not only any horizontal load, H, but also a moment $P\Delta$. The secondary moments in this column can be divided into two components, one due to the additional horizontal reaction or sway force, $P\Delta/h$, necessary to resist the axial force in the deformed position and the second equal to the axial load times the deflections from the chord line, shown shaded. Traditionally these have been accounted for in design by using the elastic effective length factors for the unbraced case in designing the column.

On the other hand, if a second-order structural analysis is carried out including the effects of both the applied loads and the sway forces, the latter have been accounted for in the analysis and need not be considered a second time in evaluating the effective length. Since the maximum moment theoretically may occur away from the ends of the column it may be necessary to use a moment magnifier calculation to estimate this moment.

It is desirable to have a means of determining whether a given column can be designed for the maximum end moment or whether the moments between the ends of the column will exceed those at the ends. Galambos (8) has shown that the maximum moment in an elastic beam column loaded with an axial load and end moments M_a and M_b is

$$M_{\max} = M_a \delta \qquad (26)$$

in which $$\delta = \sqrt{\frac{1 + \left(\frac{M_b}{M_a}\right)^2 - 2\left(\frac{M_b}{M_a}\right)\cos\alpha}{\sin\alpha}} \qquad (27)$$

If it is assumed that stability effects can be disregarded if $M_{\max}$ is not more than 1.05 M_a, i.e., $\delta = 1.05$, as done in the ACI Code (14), Eq. 27 can be solved to determine the combinations of M_b/M_a and α corresponding to $\delta = 1.05$. These are plotted with the solid line for $\delta = 1.05$ in Fig. 12. Combinations of M_b/M_a and α^2 falling below this line can be designed for the second-order end moments without a further stability calculation. This line and the corresponding line for $\delta = 1.0$ can be approximated by the equations shown in Fig. 12. Thus, if

$$\frac{M_b}{M_a} < \left(1.1 - \frac{P_u L^2}{3EI}\right) \qquad (28)$$

the maximum moment will always be less than 1.05 times that at the end of the column.

If Eq. 28 shows that the maximum column moment occurs away from the end of the column, column design should be based on amplified moments based on the ACI moment magnifier with C_m taken for the braced frame case using the ratio of end moments obtained from the second-order analysis and with $k = 1.0$. This can be demonstrated by setting the moment nagnifier from Eq. 29 equal to the ACI moment magnifier and solving for k.

Conclusions

This paper reviews the current state-of-the-art of second-order analysis of reinforced concrete frames and presents a series of design-oriented procedures for carrying out such analyses.

A stability index, $Q = \Sigma P \Delta_1/Hh$, is presented for determining whether second-order analyses are required. It is concluded that if $Q < 0.0475$, second-order effects can be ignored and the structure can be designed as a braced frame. For Q between 0.0475 and 0.2, second-order analyses are recommended. Frames having $Q > 0.2$ are not recommended. The value of Q should be computed at the ultimate load level using stiffnesses representative of this load level.

If a second-order analysis has been carried out, the column design should be based on the second-order end moments. Eq. 28 can be used to check whether the maximum moment occurs at the end of the column. If it occurs between the ends of the column, the maximum moment can be computed using the ACI moment magnifier procedure with C_m based on the ratio of end moments from the second-order analysis and with the effective length, $k = 1.0$.

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