Moments in Beam Supported Slabs

By WILLIAM L. GAMBLE

The results of a study of the influence of the stiffness of the supporting beams on the distribution of moments within typical interior panels of reinforced concrete floor slabs are presented. The results are presented in terms of a beam stiffness parameter α and the panel shape. It is shown that once the value of α exceeds 2, as will be the case in a large portion of slabs supported on beams on all four sides of panels, the moment distributions are relatively insensitive to further increases in beam stiffness.

Keywords: beams (supports); bending moments, columns (supports); **concrete slabs;** flat concrete plates; flat concrete slabs; moment distribution; **reinforced concrete; stiffness;** structural analysis; two-way slabs.

■ Reinforced concrete slabs, including those supported directly on columns or capitals and those supported on systems of beams, are common structural elements. While these slabs have been analyzed and successfully designed using many different methods for a long time,[1] the design methods for beam-supported slabs have all been based on the results of analyses of slabs on rigid beams. The possibility of designing, on a reasonable basis, slabs supported on beams of intermediate stiffness, was not even admitted.

ACI 318-71[2] includes provisions relating the distribution of moments within slab panels to the stiffness of the supporting beams, and it is the purpose of this paper to show the theoretical distributions of elastic moments in interior panels as a function of the stiffness of the beams, relative to the slab stiffness. This information should be helpful to the designer trying to use and interpret the Code provisions.

The influence of the size of the supporting columns on the distribution of bending moments is also shown for a few cases.

SCOPE AND DEFINITIONS

Elastic bending moments are presented, in graphical form, for typical interior rectangular

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slab panels such as that shown in Fig 1. Moments are presented for the positive moments at midspan and negative moments at the section shown, in the direction of span l_1, for the beam, slab column, strip, and slab middle strip. All moments are given as percentages of the total static moment M_o acting in the panel in direction l_1.

All moments are for the case of all panels subjected to the same uniformly distributed load. The effects of partial loading patterns have been discussed by Jirsa *et al.*[3]

The slab panel shape is varied from $l_1/l_2 = 0.5$ to 2.0, and the beam stiffness from zero to rigid, that is from the case of no beams to wall-supported slabs.

In the study of the effects of column or capital size, square panels supported on square capitals are considered, with the value of c/l varied from 0 to 0.3. A number of beam stiffnesses are considered.

All moments were evaluated as percentages of the total static moment M_o, which is defined in ACI 318-71 as:

$$M_o = \frac{wl_2l_n^2}{8} = \frac{Wl_1}{8}\left(1 - \frac{c_1}{l_1}\right)^2 \qquad (1)$$

where

w = uniformly distributed load per unit area
l_2 = transverse span
l_n = clear span in direction considered
= $l_1 - c_1$
W = total load on panel = wl_1l_2
l_1 = span considered
c_1 = capital width in direction of span l_1

Since the beam stiffness alone is an almost meaningless quantity, a beam stiffness parameter α, relating the beam stiffness to the stiffness of the width of slab supported by the beam has been adopted. This may be expressed as:

$$\alpha_1 = \frac{E_{cb}I_{b1}}{l_2N} \qquad (2)$$

where $E_{cb}I_{b1} = EI$ of Beam 1.

The flexural stiffness N of unit width of slab is:

$$N = \frac{1}{12}\,\frac{E_{cs}h^3}{(1-\mu^2)} \qquad (3)$$

where

h = slab thickness
E = Young's modulus of elasticity
μ = Poisson's ratio
l_2N = $E_{cs}I_s$ referred to in ACI 318-71

Similarly:

$$\alpha_2 = \frac{E_{cb}I_{b2}}{l_1N} \qquad (4)$$

In most of this study, the beam stiffnesses in the two directions are arbitrarily related to their spans in the following ratio:

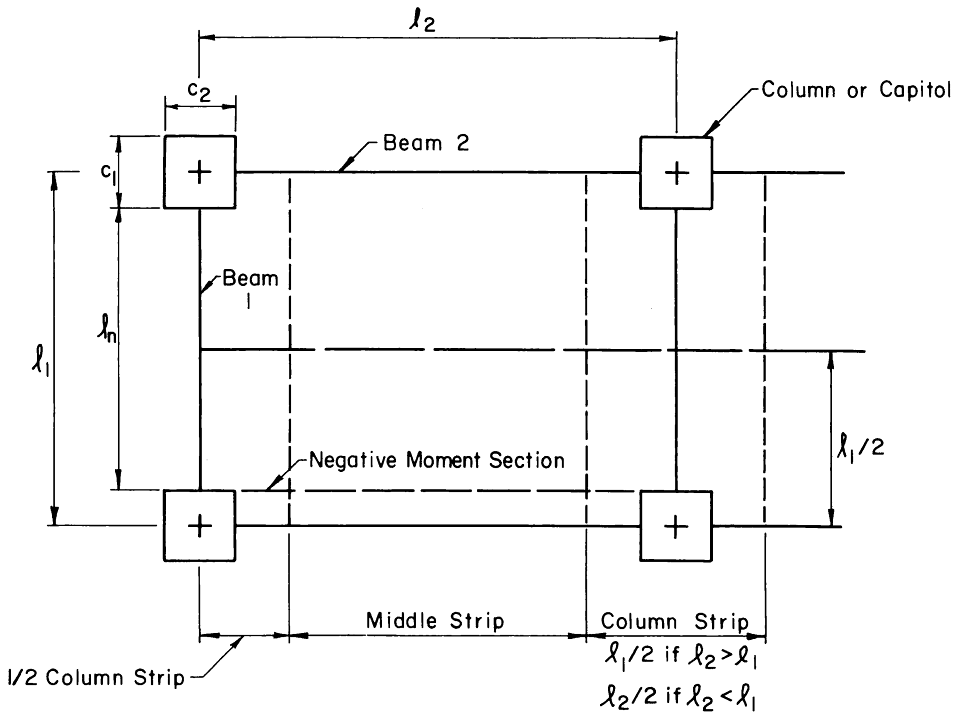

Fig. 1—Layout of typical interior panel

$$\frac{\alpha_1}{\alpha_2} = \left(\frac{l_1}{l_2}\right)^2 \quad (5)$$

This can be reduced to:

$$\frac{E_{cb}I_{b1}}{E_{cb}I_{b2}} = \frac{l_1}{l_2} \quad (5a)$$

making the moments of inertia proportional to span. Such a relationship is probably reasonable enough when l_1 is much different than l_2, but in cases where $l_1 \approx l_2$, the most common situation is probably $E_{cb}I_{b1} = E_{cb}I_{b2}$, and this case is considered briefly for $l_1/l_2 = 0.8$ and 1.25.

The beam stiffness parameter naturally has possible values of α ranging from 0 to infinity, but the range of practical values, except for the case of wall-supported slabs, is somewhat limited. For the test structure T-1, of the University of Illinois series,[4] $\alpha_1 = 4$ for the interior beams of the two-way slab structure. This is probably nearly an upper limit on practical beam stiffness as long as the beam is designed to support only the slab loading. The modified two-way slab, T-2,[5] had shallower beams for which $\alpha = 1$. The use of rolled-steel sections would in general result in lower values of α than would be obtained with concrete beams.

Traditionally, when slabs have been divided into various strips for design purposes, the strips have been half the panel width. This is not completely reasonable, however, in the case of panels where the panel width is twice the span considered. In the direction of the short span, the moments are nearly uniformly distributed over a distance considerably greater than half the panel width. Perhaps more importantly, the high moments and gradients existing near the column lines extend substantially less than one-quarter the panel width each side of the column line.

Because of these reasons, the column strip width, half the shorter span of the panel, as shown in Fig. 1, has been adopted.

The moments presented in this paper were obtained from solutions of the governing plate equations previously reported by Sutherland *et al.*,[6] and from unpublished results obtained by Vanderbilt *et al.*, in connection with a study of slab deflections.[7] The former analyses used the Ritz energy method of analysis in conjunction with an infinite series of polynomial functions derived by Duncan.[8] Vanderbilt used the finite difference method of analysis, working with a 20 by 20 grid for each panel. The use of both methods are described by Timoshenko and Woinowsky-Krieger.[9]

In all cases the reported moments are for Poisson's ratio = 0, as is reasonable for the case of cracked reinforced concrete slabs.

RESULTS OF ANALYSES

The results of the basic series of analyses are shown in Fig. 2 through 6. The moments are given for five shapes of panels supported on columns for which $c/l = 0$, and on line-width beams. The beam stiffnesses in the two directions are related by:

$$\frac{E_{cb}I_{b1}}{E_{cb}I_{b2}} = \frac{l_1}{l_2} \quad (6)$$

Fig. 2, for the case of $l_1/l_2 = 2.0$, is typical of the group of five curves, and is the easiest to understand because of the separation of the various curves. In this figure, bending moments as percentages of M_o are plotted versus a beam stiffness factor, $\alpha_1\ (l_2/l_1)$. A linear scale for $\alpha_1\ (l_2/l_1)$ was used as it properly shows the dependence of the various moments on the beam stiffness, especially in the range of $\alpha_1\ (l_2/l_1) < 2$, and the relative insensitivity at higher beam stiffnesses. Other kinds of horizontal scales have been used to include the entire beam stiffness range from zero to infinity but these tend to mask the importance of the values between 0 and 2.0.

The total positive moment in slab plus beam is always $M_o/3$ for $c/l = 0$, regardless of the beam stiffness or panel shape, and this basic moment distribution is shown by the straight, horizontal lines for total positive and negative moments.

Each section moment is divided into three components: Beam, slab column strip, and slab middle strip. In the graphs, the moments are shown by bands, with the beam moment being represented by the lowest curve, the slab column strip moment by the width of the area between the lowest and second curve, and the middle strip moment by the width of the area above the second curve and below the total section moment line. It may be helpful to view the slab column strip plus beam as a unit "beam strip," with the division of moment between the two components depending on the beam stiffness.

The nature of the moment-beam stiffness relationships is largely evident from the figures, and consequently needs only minor comment. As must be expected, there is a strong relationship between beam moment and stiffness, though as pointed out earlier, when $\alpha_1(l_2/l_1) = 4$, moments are nearly those in rigid beams, and slopes of curves are greatly reduced at $\alpha_1(l_2/l_1) = 2$.

In all cases the slab-column strip moments, positive and negative, are significantly reduced as the beam stiffness increases.

In the direction of the short span $(l_1 < l_2)$, increasing the beam stiffness decreases the slab middle strip moments, while in a square panel increasing α_1 causes small increases in negative

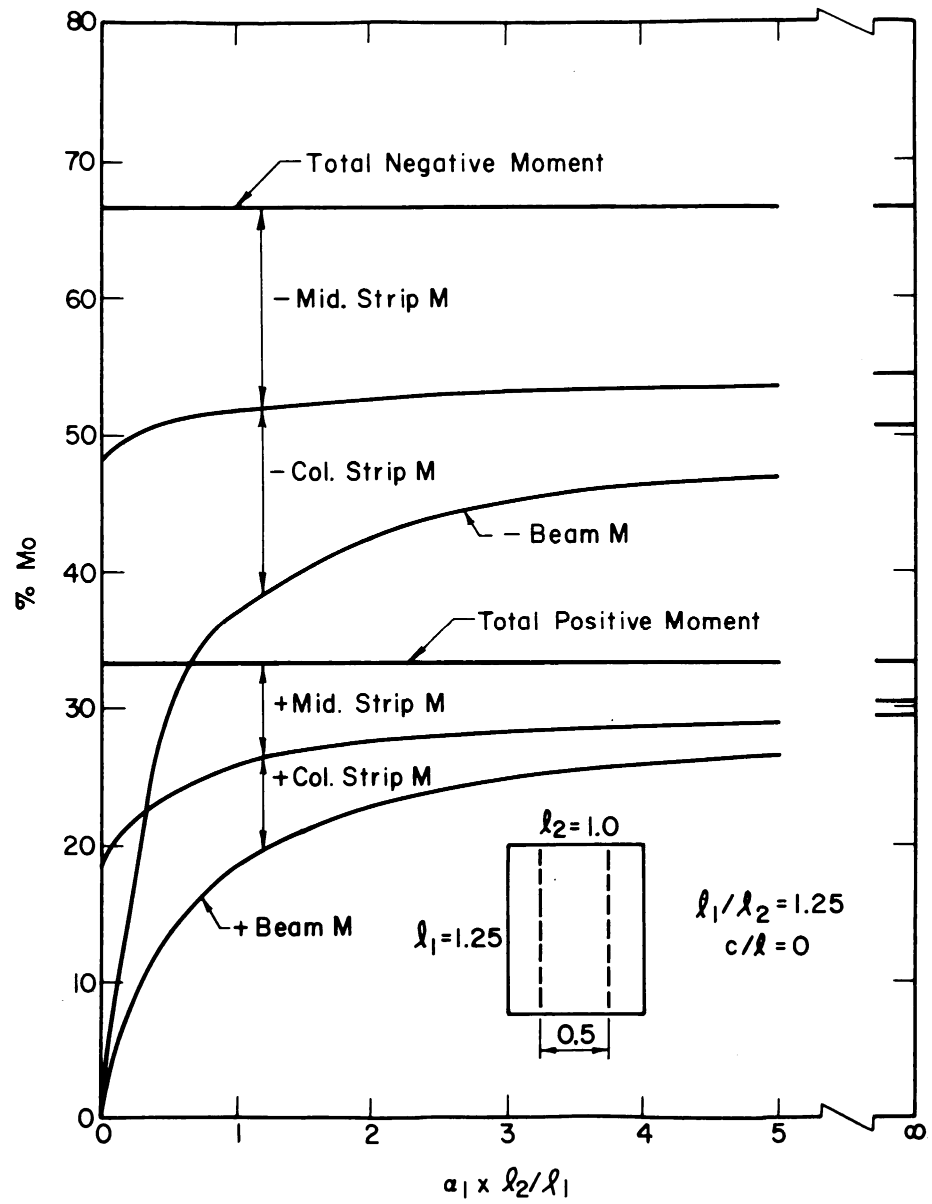

Fig. 3—Moments versus beam stiffness for $l_1/l_2 = 1.25$

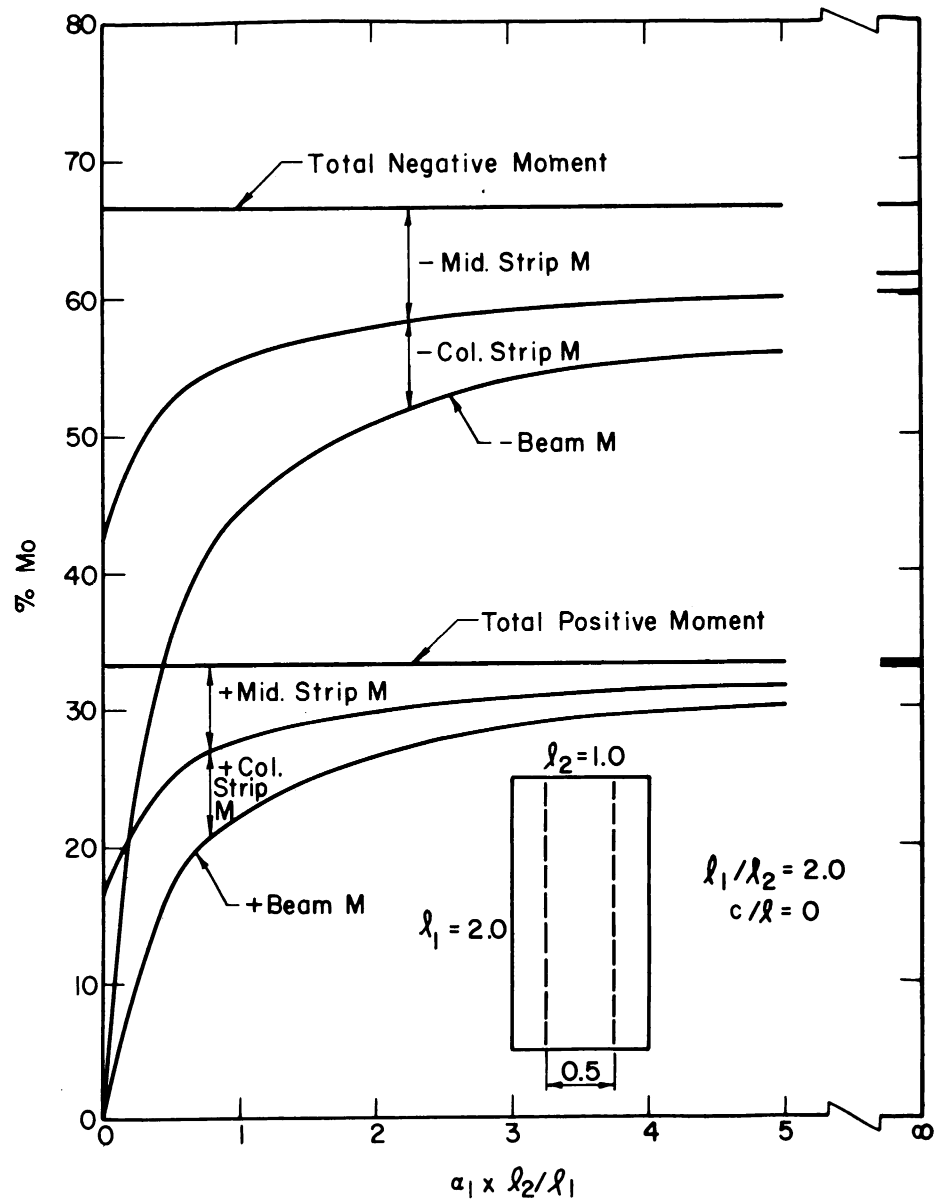

Fig. 2—Moments versus beam stiffness for $l_1/l_2 = 2.0$

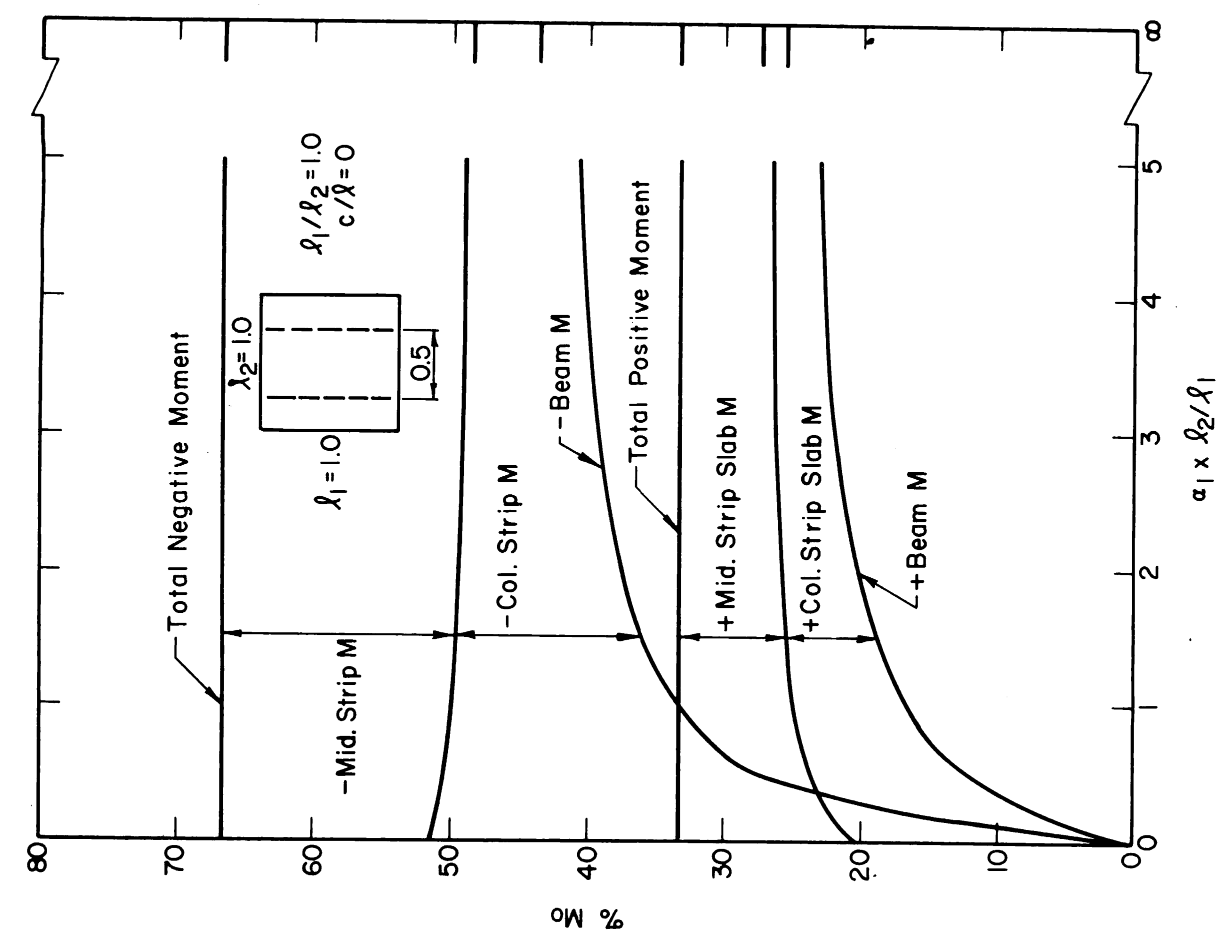

Fig. 4—Moments versus beam stiffness for $l_1/l_2 = 1.0$

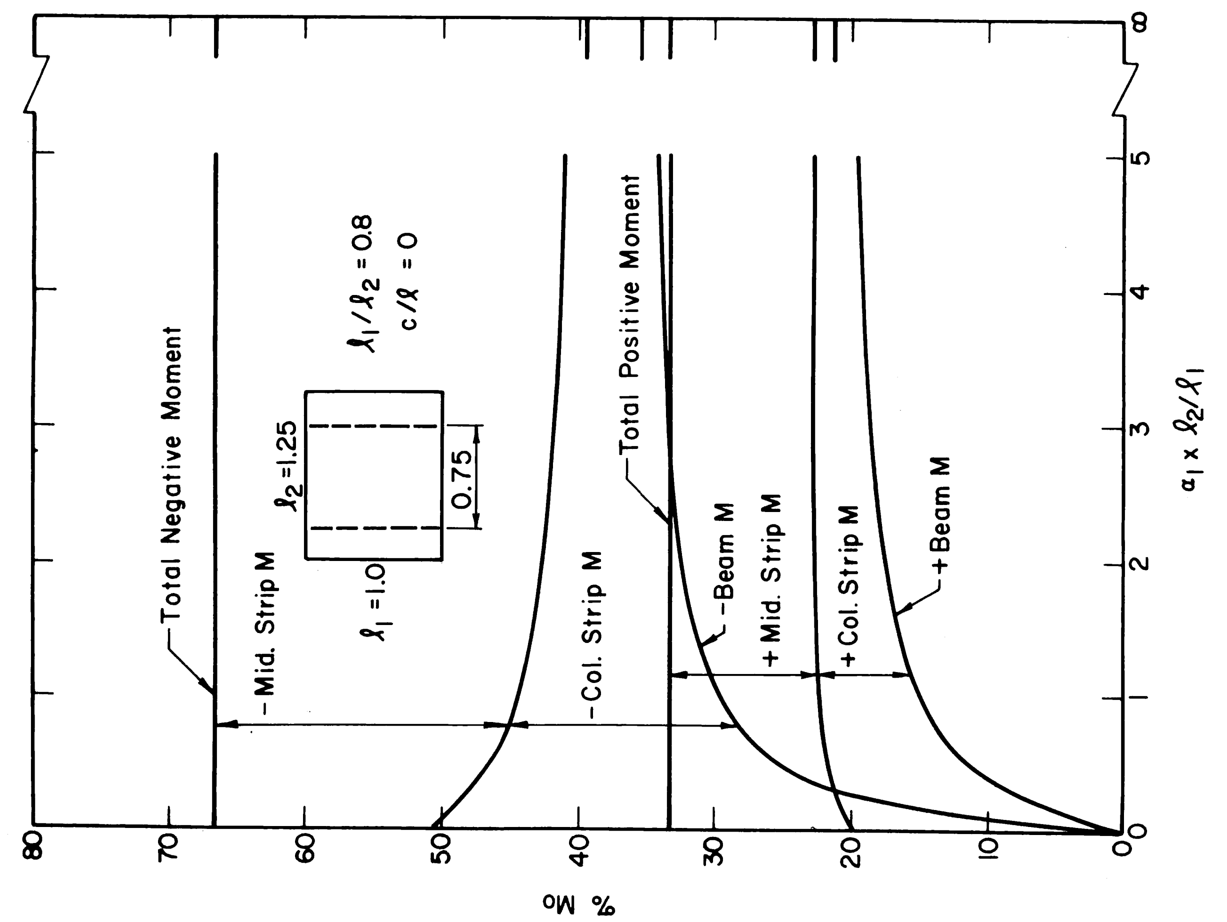

Fig. 5—Moments versus beam stiffness for $l_1/l_2 = 0.8$

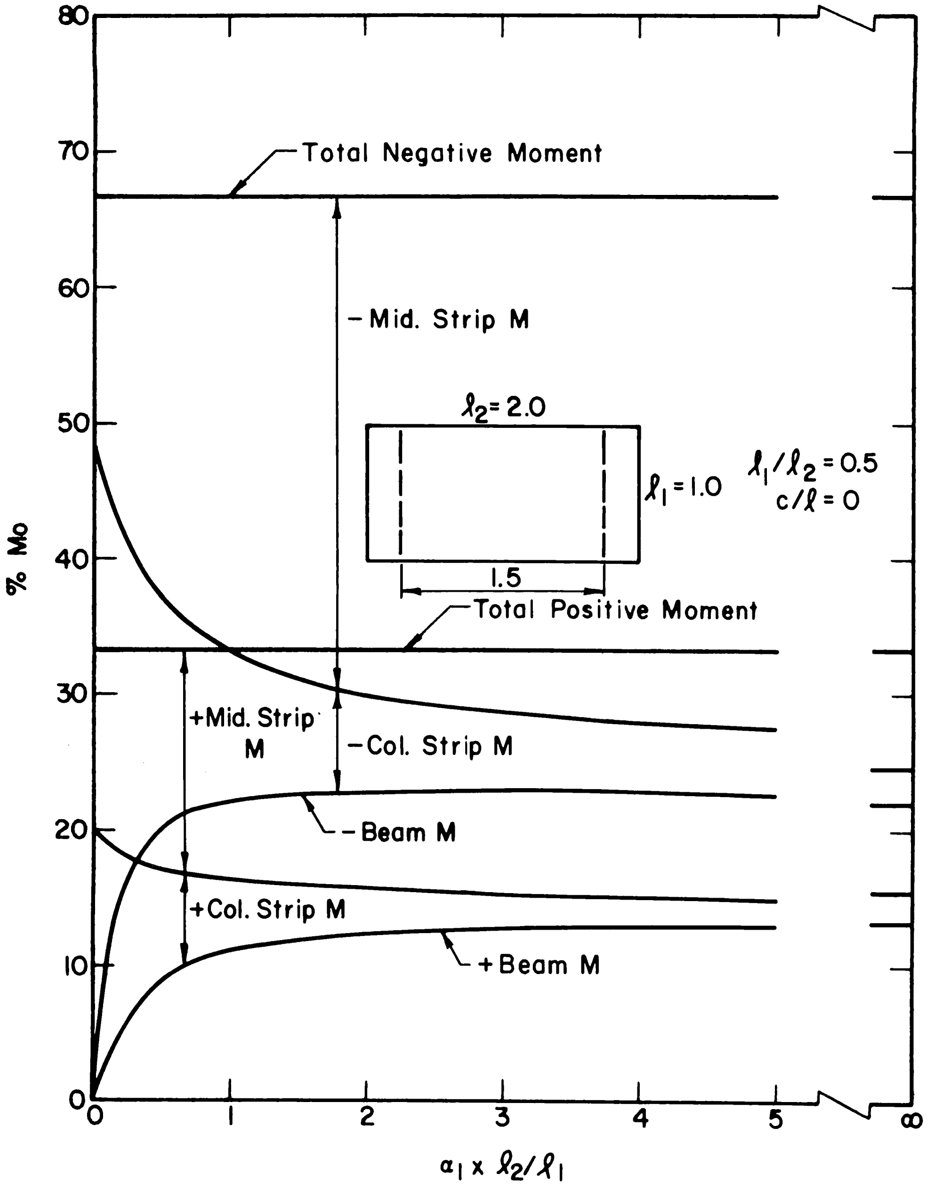

Fig. 6—Moments versus beam stiffness for $l_1/l_2 = 0.5$

moment and decreases in positive moments in the middle strip.

In the direction of the long span ($l_1 > l_2$), increasing beam stiffness leads to significant increases in middle strip negative moments but only small changes in the positive slab moments. The increase in the negative moment is due to the bracing or propping effect on the slab by the beam which spans in the l_2 direction.

In the case of $l_1/l_2 = 0.5$ (Fig. 6), the increase in middle strip negative moment with increasing beam stiffness is so great that the beam in the l_1 direction is in effect "unloaded," as its negative moment decreases very slightly as α_1 (l_2/l_1) exceeds 3.

Effect of support size

The preceding information was all based on the assumption that the slab was supported on point columns and zero-width beams, while this is of course never the actual case. In order to study the effect of column size, Fig. 7 was prepared. These figures show the various section moments, as percentages of the static moment M_o, as functions of the column size ratio c/l, for a number of values of beam stiffness. All information is for square panels supported on rigid square columns or capitals, and only line beams were considered.

One common factor appears in the curves: No matter what the beam stiffness, increasing c/l from 0 to 0.3 reduces the total negative moment from 2/3 M_o to about 0.6 M_o, with a corresponding increase in positive moment, though one must remember that M_o for $c/l = 0.3$ is just 49 percent of M_o for $c/l = 0$, for the same load and span.

For the beamless slab case, the positive to negative moment ratio changes appear to be spread more or less uniformly through the panel.

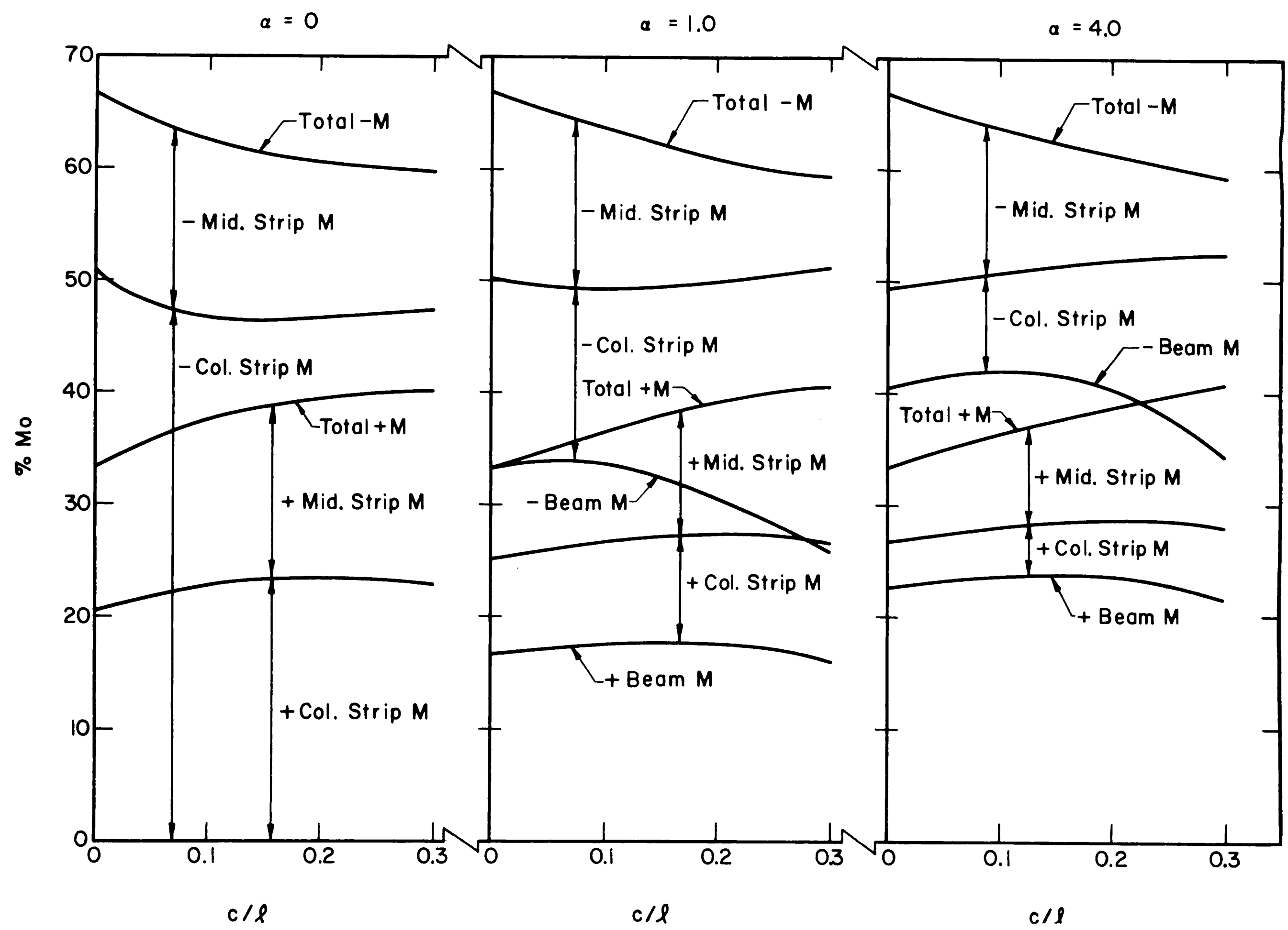

Fig. 7—Effects of column or capital size on slab and beam moments in square panels

When the slab panels were supported on beams, the beam positive moments and the positive and negative beam strip (beam plus slab column strip) moments, in terms of M_o, were nearly insensitive to changes in c/l.

This means that most of the increase in the positive moment occurred in the middle strip. In the negative moment regions, considerable redistribution of moment with increasing c/l is indicated, with the most marked change, for all values of α, being the drastic reduction in the negative beam moments for $c/l > 0.1$. While this is an interesting redistribution, it is probably of no great practical consequence since slabs with large values of c/l usually do not have beams. It is a reflection of a "bracing" effect by the very large capital on the slab near the capital, resulting in a change in the effective load distribution on the beam.

There has been a trend in recent construction to use beamless slabs supported on rectangular columns which are very elongated in cross section, such as 1 x 8 ft, even though the panel may be approximately square. The above data do not apply to such slabs since it is based on square capitals or columns. However, information on this useful case is contained in a recent paper by Simmonds.[10]

Effect of transverse beam stiffness

In the preceding material concerning rectangular panels, it was assumed that beam stiffnesses in the two orthogonal directions were always related in a set manner, while in reality any combination of stiffnesses is possible. For example, the use of the same beam cross section in the two directions is probably the most common case, especially when the panel is not far from square.

A review of available solutions for rectangular slab panels supported on beams with different combinations of moments of inertia in the two directions showed that the moment distributions were not very sensitive to changes in the stiffness of the transverse beam. The stiffness of the beam in the direction of the span being considered is the most important stiffness.

Of the various slab and beam sections, the slab middle strip negative moment is the most sensitive to the stiffness of the transverse beam, and the two sets of curves of Fig. 8 show that, at least for $l_1/l_2 = 0.8$ and 1.25, there are no important differ-

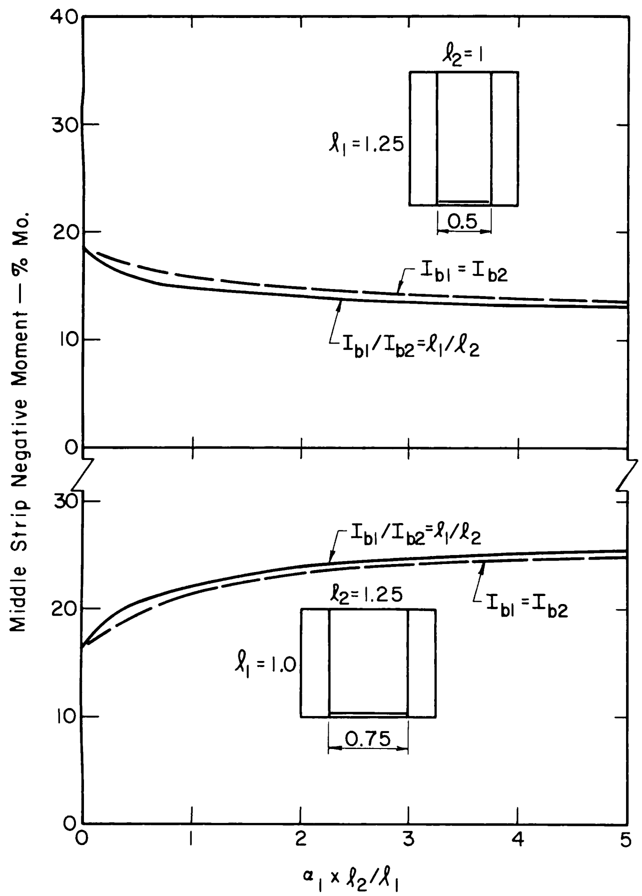

Fig 8—Effects of beam stiffness on middle strip negative moment

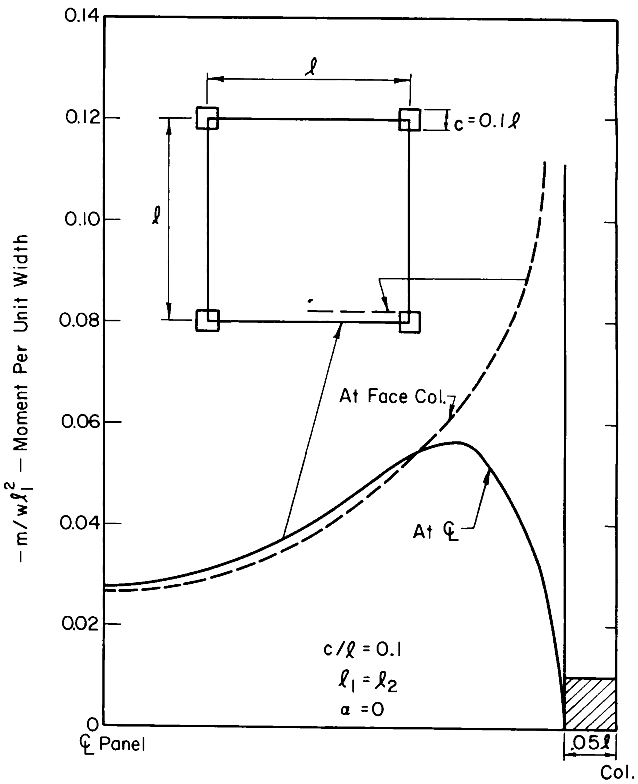

Fig. 9—Distributions of negative moments across panel borders ($c/l = 0.1$)

ences between the cases where $I_{b1}/I_{b2} = l_1/l_2$ and where $I_{b1} = I_{b2}$.

Effect of location of critical negative moment section

ACI 318-71 locates the critical negative moment section at the face of the supporting columns, while the 1963[11] and earlier versions of the Code had considered, at least implicitly, a section which followed either the face of the beams or the line connecting centers of supporting columns between columns, and then was offset out to the face of the columns.

The clear span static moment used by ACI 318-71, as shown in Eq. 1, is slightly smaller, for finite values of c_1/l_1, than that originally derived by Nicholas[12] for flat slabs:

$$M_o = \frac{Wl_1}{8}\left[1 - 2/3\,\frac{c_1}{l_1}\right]^2 \qquad (7)$$

However, the clear span static moment is somewhat larger than the value of M_o given by the 1963 and earlier Codes for flat slabs. The clear span static moment is considerably smaller than the total design moment for beam-supported slabs designed according to the 1963 Code.

Because of this change in location of critical negative moment section, it may be useful to look at the implications. The strongest reason for the change was possibly simplicity. Justification for the change can be obtained from elastic moment distributions and also from the fact that, at collapse, the negative moment yield lines will form at or near the faces of the supports, and theoretically will not contain the offsets.[10,13-15]

Distributions of elastic moments across the line connecting the column centers and across the line connecting column faces are shown for a square panel without beams and with $c/l = 0.1$ in Fig. 9. The moment midway between columns is slightly higher on the line connecting column centers, but the total area under the moment curve is appreciably higher for the section at the face of column. Similar distributions are obtained for larger values of c/l.

The addition of beams to the system would complicate the picture, but large values of c/l in combination with narrow beams are not usually found in beam-supported slabs. No solutions of cases with finite-width beams are known to the author, and they would be necessary for any definitive study of this problem.

The moment distributions shown in the figure supports the concept of using the clear span in computing the static moment to be used in design of slab structures.

DISCUSSION AND CONCLUSIONS

The effects on the elastic moment distributions of the stiffnesses of the beams supporting slab panels have been presented, and these data should be of use to design engineers. However, it must be remembered that the moments are for the case of all panels loaded, and the effects of partial loading patterns may require some modifications to these distributions.

The moments in panels at the edges of structures have not been considered, and the distributions of moments in such panels obviously depend on factors such as column flexural stiffness and edge member torsional stiffness, in addition to beam flexural stiffnesses and panel shape.

It also must be observed that it is never necessary to arrange the reinforcement to exactly match the computed elastic moments to obtain a satisfactory structure. A reinforced concrete slab is not ideally elastic, and the actual moment distributions will vary from the theoretical distributions because of creep and shrinkage strains, cracking at some sections but not others, support movements, and many other factors. This fact has of course long been recognized, and is given explicit recognition in the ACI Code provisions, past and present, allowing the designer to arbitrarily increase or decrease the moment assigned to any section by up to 10 percent as long as the total moment capacity of the panel is not decreased.

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APPENDIX—NOTATION

E = Young's modulus of elasticity of material
α = relative flexural stiffness of beam
I = moment of inertia of supporting beam
M_o = total static moment in direction of span l_1
N = flexural stiffness of unit width of slab
W = total uniformly distributed load in a panel
c_1 = capital width in direction of span l_1
l_1 = span, center-to-center of supports, in the direction considered
l_2 = transverse span, center-to-center of supports
l_n = clear span in direction of span l_1
$= l_1 - c_1$
h = slab thickness
w = uniformly distributed load per unit area
μ = Poisson's ratio

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