Combined Membrane and Flexural Reinforcement in Plates and Shells

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Abstract: A plate or shell element subjected to membrane forces N_x, N_y, N_{xy} and bending moments M_x, M_y, M_{xy} is considered. Based on equilibrium considerations, equations for capacities of top and bottom reinforcements in two orthogonal directions have been derived. An iterative method is suggested for calculating the design capacities. The proposed equations are more general and rigorous than those derived for membrane reinforcement alone and those for flexure only. For the membrane alone case, the proposed equations degenerate into the previously derived equations. For the latter, it is shown that the present practice of designing flexural reinforcement may underestimate the required capacity.

Introduction

The problem considered here is that of a plate or shell element which is subjected to membrane forces N_x, N_y, N_{xy} and bending moments M_x, M_y, M_{xy} (Fig. 1). The principal directions of the membrane forces and the bending moments in general do not coincide.

The only practical treatment of this problem available in English literature to the knowledge of the writer is a summary report by Brondum-Nielsen (1). The forces and moments are resisted by the net resultants of the tensile forces in the top and bottom reinforcements provided in two directions and by those of the compressive forces developed in compression blocks of concrete. The report is brief, however, and does not establish a general procedure for design. In somewhat vague terms, Baumann (2) suggests resolving the forces and moments into forces in the top and bottom layers, and using an approximate lever arm of $0.8h$, where h is the thickness of the shell.

In the present paper, detailed equations for capacities of the top and bottom reinforcements in the x and y orthogonal directions have been derived based on equilibrium considerations. These equations can be used directly for design purposes. In the particular case when there is no membrane force, it is shown that the present methods for flexural reinforcement design may underestimate the required reinforcement.

Theory

Fig. 2 shows two layers of reinforcement both in x and y directions. The capacities of these reinforcement layers are designated by N^*_{xt}, N^*_{xb}, N^*_{yt}, N^*_{yb}, where subscripts x and y designate the directions, and t

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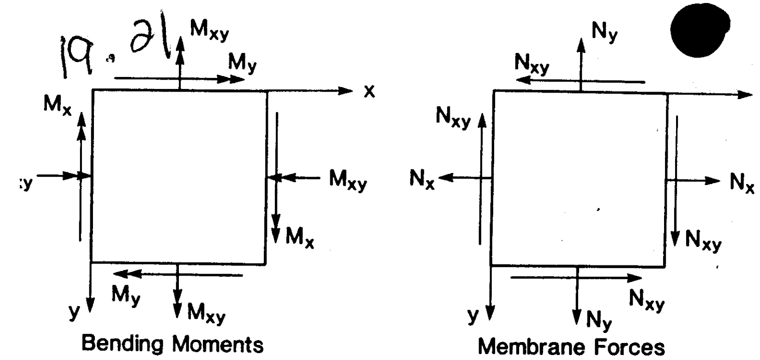

FIG. 1.—Applied Forces and Moments on an Element

and b stand for the top and bottom layers, respectively. A vertical plane of crack, whose normal makes an angle θ_t with the x-axis in the xy plane, penetrates through the top surface. The concrete is under compression parallel to this crack; it is assumed that the depth of Whitney's stress block is a_t. The corresponding crack direction for the bottom surface is

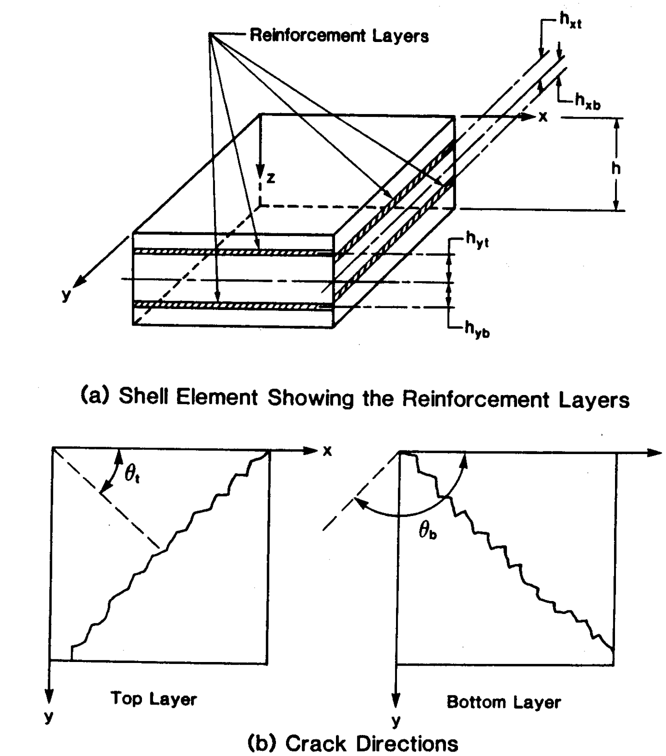

FIG. 2.—Reinforcement Layers and Crack Directions

designated by θ_b and the thickness of the stress block by a_b.

The total forces and moments resisted by the reinforcement in the x and y directions are given by

$$N_x^* = N_{xt}^* + N_{xb}^*; \quad N_y^* = N_{yt}^* + N_{yb}^* \qquad (1)$$

$$M_x^* = -N_{xt}^* h_{xt} + N_{xb}^* h_{xb}; \quad M_y^* = -N_{yt}^* h_{yt} + N_{yb}^* h_{yb} \qquad (2)$$

If the average compressive stress in concrete is f^c, the force and moment resultants of the top concrete block are

$$N_t^c = -a_t f_t^c; \quad M_t^c = -\frac{1}{2}(h - a_t) N_t^c \qquad (3)$$

and for the bottom concrete block

$$N_b^c = -a_b f_b^c; \quad M_b^c = \frac{1}{2}(h - a_b) N_b^c \qquad (4)$$

Eqs. 1–4 give the resisting forces and moments. These forces and moments should be under equilibrium with the applied forces and moments. Therefore

$$N_x = N_x^* + N_t^c \sin^2\theta_t + N_b^c \sin^2\theta_b; \quad N_y = N_y^* + N_t^c \cos^2\theta_t + N_b^c \cos^2\theta_b$$

$$N_{xy} = -N_t^c \sin\theta_t \cos\theta_t - N_b^c \sin\theta_b \cos\theta_b$$

$$M_x = M_x^* + M_t^c \sin^2\theta_t + M_b^c \sin^2\theta_b; \quad M_y = M_y^* + M_t^c \cos^2\theta_t + M_b^c \cos^2\theta_b$$

$$M_{xy} = -M_t^c \sin\theta_t \cos\theta_t - M_b^c \sin\theta_b \cos\theta_b \qquad (5)$$

Eqs. 3, 4 and 5 yield

$$-N_t^c = \frac{(h - a_b)N_{xy} - 2M_{xy}}{h_c \sin 2\theta_t}; \quad -N_b^c = \frac{(h - a_t)N_{xy} + 2M_{xy}}{h_c \sin 2\theta_b} \qquad (6)$$

where $h_c = h - (a_t + a_b)/2$. Eqs. 1–6 give

$$N_{xt}^* = N_{xt} + N_{xyt} C_{xtt} \tan\theta_t + N_{xyb} C_{xtb} \tan\theta_b$$

$$N_{yt}^* = N_{yt} + N_{xyt} C_{ytt} \cot\theta_t + N_{xyb} C_{ytb} \cot\theta_b$$

$$N_{xb}^* = N_{xb} + N_{xyt} C_{xbt} \tan\theta_t + N_{xyb} C_{xbb} \tan\theta_b$$

$$N_{yb}^* = N_{yb} + N_{xyt} C_{ybt} \cot\theta_t + N_{xyb} C_{ybb} \cot\theta_b \qquad (7)$$

in which $N_{xt} = \frac{h_{xb}}{h_x} N_x - \frac{M_x}{h_x}; \quad N_{xb} = \frac{h_{xt}}{h_x} N_x + \frac{M_x}{h_x}$

$$N_{yt} = \frac{h_{yb}}{h_y} N_y - \frac{M_y}{h_y}; \quad N_{yb} = \frac{h_{yt}}{h_y} N_y + \frac{M_y}{h_y}$$

$$N_{xyt} = \frac{(h - a_b) N_{xy} - 2M_{xy}}{2h_c}; \quad N_{xyb} = \frac{(h - a_t) N_{xy} + 2M_{xy}}{2h_c} \qquad (8)$$

and $C_{xtt} = \dfrac{h_{xb} + \frac{1}{2}(h - a_t)}{h_x}; \quad C_{xtb} = \dfrac{h_{xb} - \frac{1}{2}(h - a_b)}{h_x}$

$$C_{ytt} = \frac{h_{yb} + \frac{1}{2}(h - a_t)}{h_y}; \quad C_{ytb} = \frac{h_{yb} - \frac{1}{2}(h - a_b)}{h_y}$$

$$C_{xbt} = \frac{h_{xt} - \frac{1}{2}(h - a_t)}{h_x}; \quad C_{xbb} = \frac{h_{xt} + \frac{1}{2}(h - a_b)}{h_x}$$

$$C_{ybt} = \frac{h_{yt} - \frac{1}{2}(h - a_t)}{h_y}; \quad C_{ybb} = \frac{h_{yt} + \frac{1}{2}(h - a_b)}{h_y}$$

$$h_x = h_{xt} + h_{xb}; \quad h_y = h_{yt} + h_{yb} \qquad (9)$$

Eqs. 7 constitute the desired design equations for calculating the reinforcement capacities. If the cross-coefficients, C_{xtb}, C_{xbt}, C_{ytb}, C_{ybt}, were zero, we could visualize the plate-shell element as consisting of two membrane layers. The first two equations give the design reinforcement for the top membrane, and the remaining two for the bottom membrane. The cross terms are introduced because reinforcements in the x and y directions are not concentric ($h_{xt} \neq h_{yt}$, $h_{xb} \neq h_{bt}$), nor are the centeroids of concrete compression blocks concentric with either reinforcement. The compressive forces in concrete can be obtained from Eqs. 6 and 8, and are given by

$$-N_t^c = \frac{2N_{xyt}}{\sin 2\theta_t}; \quad -N_b^c = \frac{2N_{xyb}}{\sin 2\theta_b} \qquad (10)$$

The compressive stresses can be calculated from Eqs. 3 and 4.

Design Method

The quantities of interest are the reinforcement capacities N_{xt}^*, N_{yt}^*, N_{xb}^*, N_{yb}^*. The other unknowns are a_t, a_b and θ_t, θ_b. Ideally, these quantities should be selected so that the total capacity is the minimum possible.

Temporarily, to simplify the design equations, we assume

$$h_{xt} = h_{yt} = h_{xb} = h_{yb} = 0.5h_x = 0.5h_y = 0.4h \qquad (11)$$

Eqs. 7 now become

$$N_{xt}^* = N_{xt} + N_{xyt} C_{tt} \tan\theta_t + N_{xyb} C_{tb} \tan\theta_b$$

$$N_{yt}^* = N_{yt} + N_{xyt} C_{tt} \cot\theta_t + N_{xyb} C_{tb} \cot\theta_b$$

$$N_{xb}^* = N_{xb} + N_{xyt} C_{bt} \tan\theta_t + N_{xyb} C_{bb} \tan\theta_b$$

$$N_{yb}^* = N_{yb} + N_{xyt} C_{bt} \cot\theta_t + N_{xyb} C_{bb} \cot\theta_b \qquad (12)$$

From Eqs. 8

$$N_{xt} = 0.5N_x - \frac{M_x}{0.8h}; \quad N_{xb} = 0.5N_x + \frac{M_x}{0.8h}$$

$$N_{yt} = 0.5N_y - \frac{M_y}{0.8h}; \quad N_{yb} = 0.5N_y + \frac{M_y}{0.8h} \quad \ldots\ldots (13)$$

Equations for N_{xyt} and N_{xyb} do not change. Also, from Eqs. 9

$$C_{tt} = C_{xtt} = C_{ytt} = 1.125 - \frac{0.625\, a_t}{h}$$

$$C_{tb} = C_{xtb} = C_{ytb} = -0.125 + \frac{0.625\, a_b}{h}$$

$$C_{bt} = C_{xbt} = C_{ybt} = -0.125 + \frac{0.625\, a_t}{h}$$

$$C_{bb} = C_{xbb} = C_{ybb} = 1.125 - \frac{0.625\, a_b}{h} \quad \ldots\ldots (14)$$

We still need to iterate to evaluate a_t and a_b. In the first iteration we may set $a_t = a_b = 0.2h$, which would eliminate C_{tb} and C_{bt}, and $C_{tt} = C_{bb} = 1$. Eqs. 12 become

$$N^*_{xt} = N_{xt} + N_{xyt} \tan \theta_t; \quad N^*_{yt} = N_{yt} + N_{xyt} \cot \theta_t$$

$$N^*_{xb} = N_{xb} + N_{xyb} \tan \theta_b; \quad N^*_{yb} = N_{yb} + N_{xyb} \cot \theta_b \quad \ldots\ldots (15)$$

As we had mentioned earlier, Eqs. 15 are the perfect two membrane plate equations. These equations can be solved by standard techniques (3,4). Typically, for minimum capacity, we have $\theta_t = \pm\pi/4$ and $\theta_b = \pm\pi/4$. The signs of θ_t and θ_b would depend upon the signs of N_{xyt} and N_{xyb}, respectively. When the $\pi/4$ angle leads to a negative capacity, the particular capacity is set to zero, and the corresponding angle is calculated accordingly. It is possible that one or more of the preceding capacities are zero.

Once θ_t and θ_b are evaluated, the compressive forces in concrete can be calculated from Eqs. 10. We substitute the values of N^c_t and N^c_b into Eqs. 3 and 4 along with the allowable compressive stress in concrete $f^c_{\text{allowable}}$, which in turn yields new values of a_t and a_b.

Next, we return to Eqs. 12–14 with new a_t and a_b values, calculate the reinforcement capacities, compute concrete compressive forces from Eqs. 10, and compute new values of a_t and a_b from Eqs. 3 and 4, if necessary. If new values of a_t and a_b are calculated, we would go back to Eqs. 12–14, and so on. Finally, we have reinforcement capacities in accordance with the values of h_{xt}, h_{yt}, etc. assumed in Eqs. 11. The actual values of h_{xt}, h_{yt}, etc. are different, and therefore, the reinforcement capacities need be adjusted accordingly.

Say, for h'_{xt}, h'_{yt}, h'_{xb}, h'_{yb}, h'_x, h'_y the calculated capacities are $N^{*\prime}_{xt}$, $N^{*\prime}_{yt}$, $N^{*\prime}_{xb}$, $N^{*\prime}_{yb}$. We need to calculate N^*_{xt}, N^*_{yt}, N^*_{xb}, N^*_{yb} for h_{xt}, h_{yt}, h_{xb}, h_{yb}, h_x, h_y. The necessary transformation can be achieved in accordance with Eqs. 1 and 2. We note that for equilibrium, N^*_x, N^*_y and M^*_x, M^*_y remain same for both sets. We have

$$\begin{Bmatrix} N^*_{xt} \\ N^*_{xb} \end{Bmatrix} = \frac{1}{h_x} \begin{bmatrix} h_{xb} - h'_{xt} & h_{xb} - h'_{xb} \\ h_{xt} - h'_{xt} & h_{xt} + h'_{xb} \end{bmatrix} \begin{Bmatrix} N^{*\prime}_{xt} \\ N^{*\prime}_{xb} \end{Bmatrix}$$

and $$\begin{Bmatrix} N^*_{yt} \\ N^*_{yb} \end{Bmatrix} = \frac{1}{h_y} \begin{bmatrix} h_{yb} - h'_{yt} & h_{yb} - h'_{yb} \\ h_{yt} - h'_{yt} & h_{yt} + h'_{yb} \end{bmatrix} \begin{Bmatrix} N^{*\prime}_{yt} \\ N^{*\prime}_{yb} \end{Bmatrix} \quad \ldots\ldots (16)$$

If any of the new capacities is negative, we may have to repeat the calculations to avoid the negative quantity.

EXAMPLE

Given, $N_x = -2{,}000$ lb/in., $N_y = 1{,}700$ lb/in., $N_{xy} = 1{,}000$ lb/in., $M_x = -13{,}500$ lb-in./in., $M_y = 2{,}700$ lb-in./in., $M_{xy} = 200$ lb-in./in., $f^c_{\text{allowable}} = 1{,}000$ psi, $h = 10$ in.

Since h_{xt}, etc. are not specified, we shall assume Eq. 11 holds. For the first iteration we assume $a_t = a_b = 0.2h = 2$ in.

Eqs. 8: $N_{xt} = 688$ lbs/in., $N_{yt} = 512$ lbs/in., $N_{xyt} = 475$ lbs/in.
$N_{xb} = -2{,}688$ lbs/in., $N_{yb} = 1{,}188$ lbs/in., $N_{xyb} = 525$ lbs/in.

Eqs. 15: $N^*_{xt} = 688 + 475 \tan \theta_t$, $N^*_{yt} = 512 + 475 \cot \theta_t$
$N^*_{xb} = -2{,}688 + 525 \tan \theta_b$, $N^*_{yb} = 1{,}188 + 525 \cot \theta_b$

Take $\theta_t = \pi/4$, $N^*_{xt} = 1{,}163$ lbs/in., $N^*_{yt} = 987$ lbs/in.

Eqs. 10: $-N^c_t = 950$ lbs/in., Eqs. 3: $a_t = 0.95$, say 1 in.
$N^*_{xb} = 0$, $\tan \theta_b = 5.12$, $N^*_{yb} = 1{,}291$ lbs/in.

Eqs. 10: $-N^c_b = 2{,}791$ lbs/in., Eqs. 4: $a_b = 2.8$, say 3 in.

<u>Next Iteration</u>

Eqs. 14: $C_{tt} = 1.0625$, $C_{bt} = -0.0625$, $C_{bb} = 0.9375$, $C_{tb} = 0.0625$

Eqs. 8: $2h_c = 16$ in., $N_{xyt} = 413$ lbs/in., $N_{xyb} = 588$ lbs/in.

Eqs. 12: $N^*_{xt} = 688 + 439 \tan \theta_t + 37 \tan \theta_b$
$N^*_{yt} = 512 + 439 \cot \theta_t + 37 \cot \theta_b$
$N^*_{xb} = -2{,}688 - 26 \tan \theta_t + 551 \tan \theta_b$
$N^*_{yb} = 1{,}188 - 26 \cot \theta_t + 551 \cot \theta_b$

Take $\theta_t = \pi/4$, $N^*_{xb} = 0$, $\tan \theta_b = 4.926$, $N^*_{yb} = 1{,}274$ lbs/in.

Eqs. 10: $-N^c_b = 3{,}016$ lbs/in., Eqs. 4: $f^c_b = 1{,}005$ psi
$N^*_{xt} = 1{,}309$ lbs/in., $N^*_{yt} = 959$ lbs/in.

Eqs. 10: $-N^c_t = 826$ lbs/in., Eqs. 3: $f^c_t = 826$ psi.

The value of f^c_b is a bit too high (>1,000 psi), and f^c_t is less than allowable. We may increase a_b slightly and reduce a_t. The calculations of the second iteration will be repeated.

APPLICATION TO MEMBRANE REINFORCEMENT

When the shell is subjected to membrane forces only ($M_x = M_y = M_{xy} = 0$), there will be only one vertical crack passing through the shell element. Only one layer of reinforcement is required in any direction, although if thickness permits, it is desirable to place membrane reinforcement in any direction in two layers in order to provide bending resistance against uncalculated accidental moments. In Eqs. 6, we can substitute $N_t^c = N^c$, $a_t = h$, $\theta_t = 0$, and $N_b^c = 0$, $a_b = 0$; hence

$$-N^c = \frac{2N_{xy}}{\sin 2\theta} \quad (17)$$

Eqs. 5 with Eq. 17 give

$$N_x^* = N_x + N_{xy} \tan\theta; \quad N_y^* = N_y + N_{xy} \cot\theta \quad (18)$$

Eqs. 17 and 18 are identical to equations for membrane capacity (3,4).

Application to Flexural Reinforcement

In this case $N_x = N_y = N_{xy} = 0$. Eqs. 8 yield

$$-N_{xt} = N_{xb} = \frac{M_x}{h_x}; \quad -N_{yt} = N_{yb} = \frac{M_y}{h_y}; \quad -N_{xyt} = N_{xyb} = \frac{M_{xy}}{h_c} \quad (19)$$

Eqs. 7 become

$$N_{xt}^* = -\frac{M_x}{h_x} - \frac{M_{xy}}{h_c} C_{xtt} \tan\theta_t + \frac{M_{xy}}{h_c} C_{xtb} \tan\theta_b$$

$$N_{yt}^* = -\frac{M_y}{h_y} - \frac{M_{xy}}{h_c} C_{ytt} \cot\theta_t + \frac{M_{xy}}{h_c} C_{ytb} \cot\theta_b$$

$$N_{xb}^* = \frac{M_x}{h_x} - \frac{M_{xy}}{h_c} C_{xbt} \tan\theta_t + \frac{M_{xy}}{h_c} C_{xbb} \tan\theta_b$$

$$N_{yb}^* = \frac{M_y}{h_y} - \frac{M_{xy}}{h_c} C_{ybt} \cot\theta_t + \frac{M_{xy}}{h_c} C_{ybb} \cot\theta_b \quad (20)$$

Eqs. 20 with Eqs. 9 give

$$M_{xb}^* = \Delta M_{xb}^* + M_x + M_{xy} \tan\theta_b; \quad M_{yb}^* = \Delta M_{yb}^* + M_y + M_{xy} \cot\theta_b \quad (21)$$

for bottom reinforcement, and

$$M_{xt}^* = \Delta M_{xt}^* + M_x + M_{xy} \tan\theta_t; \quad M_{yt}^* = \Delta M_{yt}^* + M_y + M_{xy} \cot\theta_t \quad (22)$$

for top reinforcement; where

$$M_{xb}^* = C_{xtt} h_x N_{xb}^*; \quad M_{yb}^* = C_{ytt} h_y N_{yb}^*$$

$$M_{xt}^* = -C_{xbb} h_x N_{xt}^*; \quad M_{yt}^* = -C_{ybb} h_y N_{yt}^*$$

$$\Delta M_{xb}^* = C_{xbt} h_x N_{xt}^*; \quad \Delta M_{yb}^* = C_{ybt} h_y N_{yt}^*$$

$$\Delta M_{xt}^* = -C_{xtb} h_x N_{xb}^*; \quad \Delta M_{yt}^* = -C_{ytb} h_y N_{yb}^* \quad (23)$$

When the flexural capacity is calculated, it is common to ignore ΔM^* terms (5). As is obvious from Eqs. 21 and 22, doing so is unconservative. The ΔM^* terms introduce the effect of interaction between the top and bottom reinforcements in the same direction, the effect which is commonly ignored. Of course, when only top or bottom reinforcement is needed in any direction, the interaction does not exist, and the present practice is correct in that case.

Consider an example when, $M_x = M_y = 100$, $M_{xy} = 150$. Taking $\theta_b = -\theta_t = \pi/4$, Eqs. 21 and 22 give

$$M_{xb}^* - \Delta M_{xb}^* = M_{yb}^* - \Delta M_{yb}^* = 250; \quad M_{xt}^* - \Delta M_{xt}^* = M_{yt}^* - \Delta M_{yt}^* = -50$$

Eqs. 23 give

$$\Delta M_{xb}^* = -\frac{C_{xbt}}{C_{cbb}} M_{xt}^*; \quad \Delta M_{xt}^* = -\frac{C_{xtb}}{C_{xtt}} M_{xb}^*$$

Assuming, $h_{xt} = h_{yt} = h_{xb} = h_{yb} = 0.5h_x = 0.5h_y = 0.4h$ as in Eq. 11, and $a_t = a_b = 0.4h$, the preceding equations become

$$\Delta M_{xb}^* = \frac{-1}{7} M_{xt}^*; \quad \Delta M_{xt}^* = \frac{-1}{7} M_{xb}^*$$

Similarly $\Delta M_{yb}^* = \frac{-1}{7} M_{yt}^*; \quad \Delta M_{yt}^* = \frac{-1}{7} M_{yb}^*$

Hence $M_{xb}^* = M_{yb}^* = 262.5; \quad M_{xt}^* = M_{yt}^* = 87.5$

The conventional method (5) would yield $M_{xb}^* = M_{yb}^* = 250$, $M_{xt}^* = M_{yt}^* = 50$. For the present example, the conventional method yielded the capacity for the bottom reinforcement which is 5% too low, and the capacity for the top reinforcement which is 43% too low. These estimates will vary for individual cases.

Summary and Conclusions

Equations have been derived for calculating reinforcement capacities for plates and shells subjected to combined membrane and bending. The actual design procedure would be that of trial and error. A reasonable design can be achieved in a few iterations. The combined membrane and bending equations degenerate into the familiar equations for the membrane only case. The same is not true for the pure flexture case. The equations in the literature (5) for the pure flexture case do not account for the interaction between the top and bottom reinforcements in the same direction. This may lead to underestimation of required reinforcement capacities as shown in this paper.

Appendix.—References

1. Brondum-Nielsen, T., "Optimum Design of Reinforced Concrete Shells and Slabs," *Report No. R44*, Structural Research Laboratory, Univ. of Denmark, Copenhagen, 1974.
2. Bauman, T., "Zur Frage der Netzbewehrung von Flächentragwerken," *Der Bauingenieur*, Vol. 47, No. 10, 1972, pp. 367–377.
3. Gupta, A. K., "Membrane Reinforcement in Shells," *Journal of the Structural Division*, ASCE, Vol. 107, No. ST1, Jan., 1981, pp. 41–56.
4. Gupta, A. K., "Membrane Reinforcement in Concrete Shells: A Review," *Nuclear Engineering and Design*, Vol. 82, Oct., 1984, pp. 63–75.
5. Gupta, A. K., and Sen, S., "Design of Flexural Reinforcement in Concrete Slabs," *Journal of the Structural Division*, ASCE, Apr., 1977.