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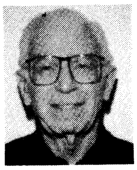
Compatible Stress and Cracking in Reinforced Concrete Membranes with Multidirectional Reinforcement



by Morris N. Fialkow

A methodology is provided for evaluating the design quantities necessary for designing membrane elements of concrete shearwalls and shells with multidirectional reinforcement against in-plane forces. The mode of failure—ductile, ductile-brittle, or brittle—is determined, as are the critical loads and associated stresses in the reinforcement and concrete and the extent of cracking. Toward this end, a set of equations, each involving one unknown parameter, is applied iteratively to convergence, which is shown by example to be rapid. The basis for omitting shear at the crack surface and for using continuum strain equations in the cracked concrete matrix is demonstrated. An illustrative example is included.

Keywords: compressive strength; **cracking (fracturing)**; crack width and spacing; ductility; failure; **reinforced concrete**; shearwalls; shells (structural forms); **stresses**; structural design.

Planar structural elements which transmit in-plane stresses, herein designated as membranes, are basic components of such structures as shearwalls, shells, and folded plates. The behavior of concrete membranes with two-way orthogonal reinforcement has been investigated extensively.[1-4] This paper develops a methodology for determining the response of membranes with multidirectional reinforcement for any loading up to membrane failure in either of the possible failure modes. The development is largely based on the principle of minimum potential.[5-7] The principle is used to develop the equilibrium equations for cracked membranes and to investigate the propriety of omitting shear force between the sides of the crack.

The methodology developed here differs from traditional methods[8,9] in several respects. First, the analysis is extended to determine the mode of failure and the associated brittle or ductile-failure loads. Second, a specific set of equations, each with one unknown, is used cyclically to obtain solutions. Third, the behavior at each critical stage, such as onset of yield in a reinforcement direction, is determined directly for that stage by using specifically applicable equations; loads are not increased by increments.

ASSUMPTIONS

The following are the general assumptions of this paper together with their underlying rationale.

1. Experiments as in Reference 1 show that shear applied to the sides of a membrane element with orthogonal reinforcement parallel to the sides results in coincident tensile stress in the reinforcement and collinear local compressive stress in the concrete. To obviate this inconsistency, it is assumed, as in Reference 3, that the reinforcement and concrete are perfectly bonded at the element boundaries so that no overall slip occurs, but that slip may occur internal to the element.
2. Relative to crack shear, it is assumed that no strain energy exists due to shear force transmitted across the crack. The validity of this assumption is checked by investigating the slip displacement along the crack, which would occur in the absence of shear-resisting force.
3. The strain energy of the reinforced concrete membrane is calculated as the sum of the strain energy of the concrete and the strain energy of the reinforcement. It is assumed that the crack is the first principal plane of the concrete component and the reinforcement stress is uniaxial.
4. Based on experimental results,[1] it is assumed that the crack direction can change as the loading varies.

STRESSES—NOTATION AND SYSTEMS

A definition of each symbol is given where the symbol first appears. Stress notation is illustrated in Fig. 1 and 2; the x,y-directions are longitudinal and transverse, respectively. For the overall membrane, σ_m and v_{mn} are the normal and shear stresses, respectively, and

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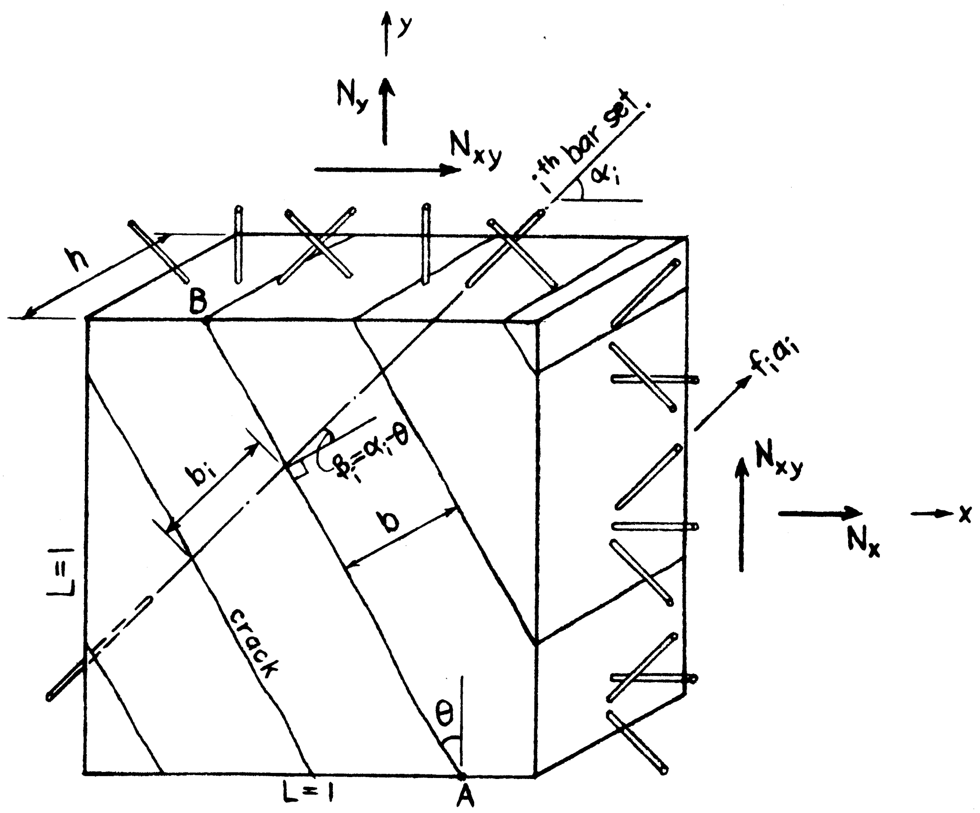
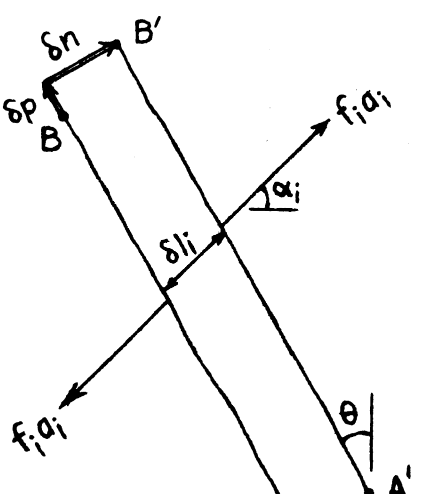
$N_m = \sigma_m h$ and $N_{mn} = v_{mn} h$ are the membrane stress resultants. For the reinforcement, the uniaxial stress and stress resultant in the ith set of reinforcement are designated f_i and F_i, respectively; the direction is defined by its inclination with the x-axis α_i. Concrete compressive and shear stress magnitudes are designated f_c and v_c, respectively. Stresses at the yield or crushing condition are designated by adding the superscripts Y or U, respectively, to the stress symbols.

In line with this, distinction is made between the membrane principal planes and the principal planes in the concrete constituent. Thus, σ_1 and σ_2 are the membrane principal stresses acting on the membrane prin-



a. Membrane forces, cracks, reinforcement



b. Displacements at crack AB

Fig. 1—Membrane element with sides of unit length

cipal planes, and f_{c1} and f_{c2} are the concrete principal stresses acting on the principal planes in the concrete.

Two systems of stresses are identified, namely, the membrane stresses and the component (reinforcement and concrete) stresses. The membrane stress resultants $N_x N_y N_{xy}$ are obtained prior to applying the methodology herein from an equilibrium analysis of the entire structure. The component stresses $f_i f_c v_c$ are obtained from the displacements, which are evaluated to minimize the total potential of the external and internal stresses of the system. Brittle failure herein involves σ_2 directed at the angle ϕ from the y-direction; ϕ is obtained from

$$\tan 2\phi = \frac{2v_{xy}}{\sigma_x - \sigma_y} \qquad (1)$$

Ductile failure is defined in terms of the reinforcement stresses f_i, which depend on the direction of the crack at angle θ from the y-direction. The angles ϕ and θ are independent.

DISPLACEMENTS AND STRAINS

Fig. 1 shows the adopted cracking pattern and crack displacements in the unit element. It is divided into subelements by equidistant cracks. The normal distance between cracks is b. The number of cracks per unit distance normal to the crack is $k = 1/b$; the number per unit distance in the x and y directions are $k_x = k\cos\theta$ and $k_y = k\sin\theta$. The number of cracks per unit distance along Bar i is $k_i = 1/b_i = k\cos\beta_i$, where $\beta_i = \alpha_i - \theta$.

The displacements across one crack in the coordinate directions are δx and δy. These displacements result in an increase in length of Bar i equal to δl_i and displacement components δp and δn tangential and normal to the crack, respectively. In determining δl_i, only the compo-

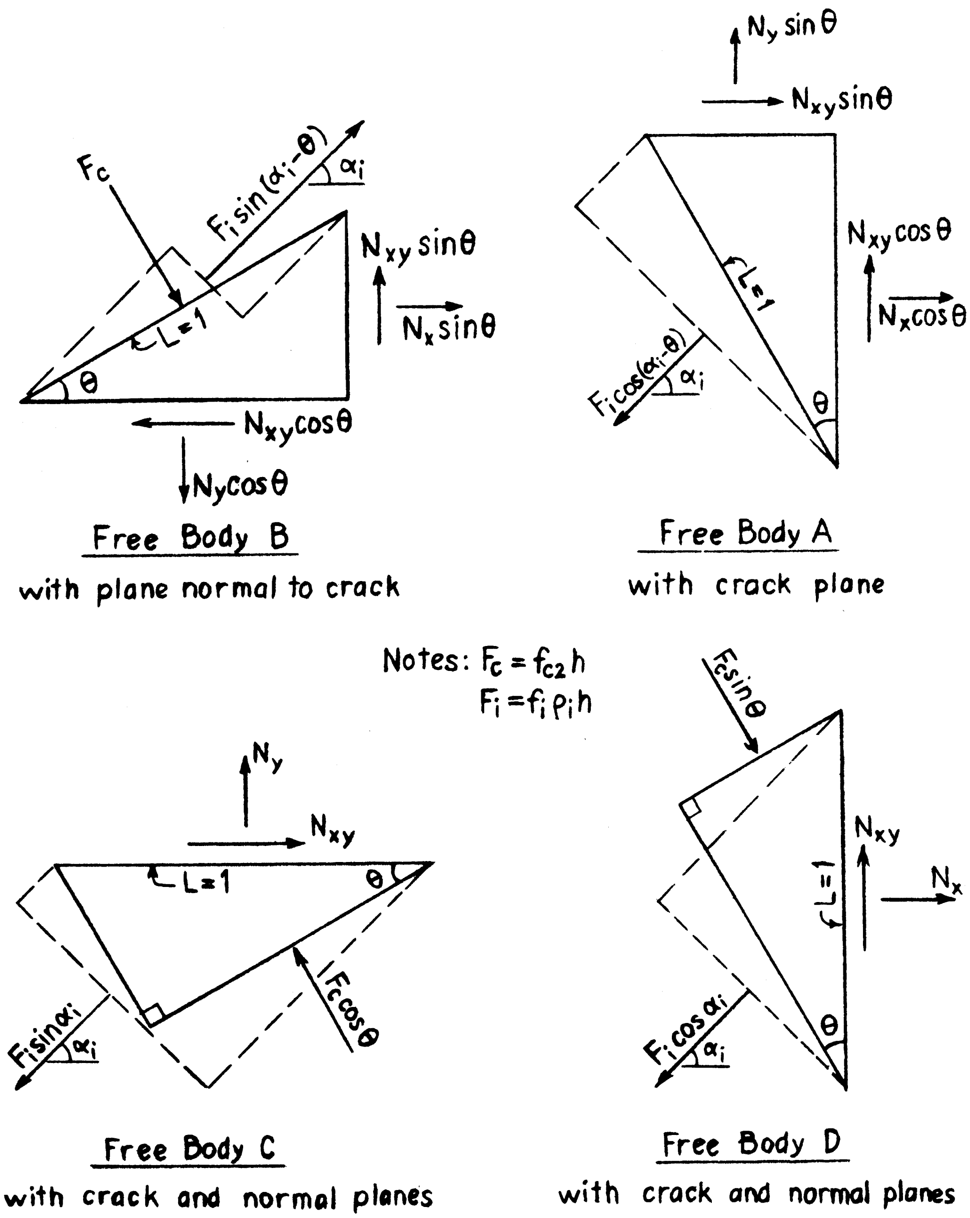


Fig. 2—Membrane-free bodies with hypotenuse of unit length

nents of displacement parallel to the bar are included, as it is assumed that the effect of the crack on the bar direction extends for a distance along the bar that is many times the small dimensions of the crack displacements

$$\delta l_i = \delta x \cos\alpha_i + \delta y \sin\alpha_i \tag{2a}$$

$$\delta p = -\delta x \sin\theta + \delta y \cos\theta \tag{2b}$$

$$\delta n = \delta x \cos\theta + \delta y \sin\theta \tag{2c}$$

From Eq. (2b), the tangential displacement vanishes if $\tan\theta = \delta y/\delta x$. Whether this is really so is investigated in the following.

The unit strain of Bar i due to crack displacements ϵ_i' is

$$\epsilon_i' = \frac{\delta l_i}{b_i} = k(\delta x \cos(\alpha_i + \delta y \sin\alpha_i) \cos(\alpha_i - \theta) \tag{3a}$$

The strain is written in terms of the components of the total crack displacement in unit distance normal to the crack

$$\epsilon_i' = (e_x \cos\alpha_i + e_y \sin\alpha_i) \cos(\alpha_i - \theta) \tag{3b}$$

where $e_x = k\delta x$; $e_y = k\delta y$; and $e_n = k\delta n$.
If δp is zero, $e_x = e_n \cos\theta$; $e_y = e_n \sin\theta$; and ϵ_i' transforms as in a continuum

$$\epsilon_i' = e_n \cos^2(\alpha_i - \theta) = e_n \cos^2\beta_i \tag{3c}$$

The displacements and strains in each concrete subelement are those of a continuum. The crack is the first principal plane for stress and strain with zero tension normal to the crack. The stress-strain relation up to crushing is assumed to be linear. The principal concrete strain which is parallel to the crack is defined as $-e_2$; the principal concrete strain normal to the crack is μe_2, where μ is Poisson's ratio. The strain transformation equations for a continuum apply to give the concrete strains ϵ_{cx}, ϵ_{cy}, γ_{cxy}, and ϵ_{c2} in the coordinate and reinforcement directions

$$\epsilon_{cx} = \mu e_2 \cos^2\theta - e_2 \sin^2\theta \tag{4a}$$

$$\epsilon_{cy} = \mu e_2 \sin^2\theta - e_2 \cos^2\theta \tag{4b}$$

$$\gamma_{cxy} = (\mu e_2 + e_2) \sin 2\theta \tag{4c}$$

$$\epsilon_{c_i} = \mu e_2 \cos^2(\alpha_i - \theta) - e_2 \sin^2(\alpha_i - \theta) \tag{4d}$$

Since it has been assumed that no overall slip occurs, the total strain ϵ_i in Bar i is the sum of the strains due to the crack displacements and concrete strains. The parameters e_x, e_y, e_2, and θ are taken to be the independent displacements in the membrane

$$\epsilon_i = (e_x \cos\alpha_i + e_y \sin\alpha_i) \cos(\alpha_i - \theta) + \mu e_2 \cos^2(\alpha_i - \theta) - e_2 \sin^2(\alpha_i - \theta) \tag{5a}$$

If δp vanishes, ϵ_i transforms as in a continuum as in

$$\epsilon_i = (e_n + \mu e_2) \cos^2(\alpha_i - \theta) - e_2 \sin^2(\alpha_i - \theta) \tag{5b}$$

The displacements of each side of the unit element relative to the opposite sides are now determined. For the side x = constant

$$\Delta x_x = e_x \cos\theta + \mu e_2 \cos^2\theta - e_2 \sin^2\theta \tag{6a}$$

$$\Delta y_x = e_y \cos\theta + \frac{1}{2}(\mu e_2 + e_2) \sin 2\theta \tag{6b}$$

and for the side y = constant

$$\Delta x_y = e_x \sin\theta + \frac{1}{2}(\mu e_2 + e_2) \sin 2\theta \tag{7a}$$

$$\Delta y_y = e_y \sin\theta + \mu e_2 \sin^2\theta - e_2 \cos^2\theta \tag{7b}$$

PRINCIPLE OF MINIMUM POTENTIAL

The total potential of the force system U_T consists of the potential energy of the internal forces (the strain energy V) and the potential energy of the applied forces U. The principle states that, of all possible displacements that satisfy stress-strain compatibility and the physical constraints, the actual displacements minimize U_T.

The components of U_T for the unit reinforced concrete membrane element are evaluated in the following. The terms ρ, h, and E are in accord with conventional notation. The strain energy of those reinforcing bars in which the stress is less than yield, with area per unit length of membrane $A_i = \rho_i h$, is termed V_i

$$V_i = \sum_i \frac{1}{2} \rho_i h E_s \epsilon_i^2, \qquad \epsilon_i < f_i^Y/E_s \tag{8a}$$

For the bars in a state of yield with area $A_r = \rho_r h$, the strain energy V_r is

$$V_r = \sum_r \rho_r h f_r^Y \epsilon_r - \sum_r \frac{1}{2} \rho_r h [f_r^Y]^2/E_s, \qquad \epsilon_r \geqslant f_r^Y/E_s \tag{8b}$$

The strain energy of the concrete component V_c is given by

$$V_c = \frac{1}{2} h E_c e_2^2 \tag{8c}$$

and the potential of the applied forces U is

$$U = -N_x \Delta x_x - N_y \Delta y_y - N_{xy} (\Delta x_y + \Delta y_x) \tag{8d}$$

The total potential for the unit element U_T is given by

$$U_T = V_i + V_r + V_c + U \tag{8e}$$

The total potential U_T is minimized by Eq. (9)

$$\frac{\partial U_T}{\partial e_j} = 0, \qquad e_j = e_x, e_y, e_2, \theta \tag{9}$$

Eq. (9), in turn, represents the four equations of equilibrium corresponding to the four independent displacements.

EQUILIBRIUM EQUATIONS IN TERMS OF STRESS

The equilibrium equations are derived and expressed in terms of the internal forces exerted by the reinforcement and concrete, as shown in Appendix B. Three of these equations from Appendix B, which can be expressed in terms of stresses and crack angle, are stated as Eq. (11), (12), and (13). They enforce equilibrium for different free bodies shown in Fig. 2. The following notation is used

$$F_i = E_s \epsilon_i \rho_i h = f_i A_i \tag{10a}$$

$$F_r^Y = f_r^Y \rho_r h = f_r^Y A_r \tag{10b}$$

$$F_c = E_c e_2 h = f_c h \tag{10c}$$

$$\beta_i = \alpha_i - \theta, \qquad \beta_r = \alpha_r - \theta \tag{10d}$$

Eq. (11) and (12) enforce the equilibrium of Free Body A in Fig. 2 relative to forces in the x and y directions, respectively

$$N_x \cos\theta + N_{xy} \sin\theta = \sum_i F_i \cos\beta_i \cos\alpha_i + \sum_r F_r^Y \cos\beta_{r} \cos\alpha_r \tag{11}$$

$$N_y \sin\theta + N_{xy} \cos\theta = \sum_i F_i \cos\beta_i \sin\alpha_i + \sum_r F_r^Y \cos\beta_r \sin\alpha_r \tag{12}$$

Eq. (13a) enforces the equilibrium of forces on Free Body B in the direction of F_c

$$N_x \sin^2\theta + N_y \cos^2\theta - N_{xy} \sin 2\theta + F_c = \sum_i F_i \sin^2\beta_i + \sum_r F_r^Y \sin^2\beta_r \tag{13a}$$

Eq. (13b) enforces the equilibrium of forces on Free Body C in the y-direction together with the equilibrium of forces on Free Body D in the x-direction

$$N_x + N_y + F_c = \sum_i F_i + \sum_r F_r^Y \tag{13b}$$

Eq. (13c) can be obtained either from equilibrium of forces in the x-direction on Free Body C or from equilibrium of forces in the y-direction on Free Body D

$$F_c \sin 2\theta = 2N_{xy} - \sum_i F_i \sin 2\alpha_i - \sum_r F_r^Y \sin 2\alpha_r \tag{13c}$$

The equilibrium equation corresponding to $\partial U_T/\partial\theta$ is given in Appendix B. It is not repeated here because it includes the displacements e_x, e_y, e_2, and θ in addition to the membrane and component material forces.

Note that the three stress-equilibrium equations are statically indeterminate and do not permit direct determination of the unknown forces F_i, F_c and crack angle θ. For this purpose, substitution is required for the forces in terms of the independent displacements.

CRACK SHEAR AND STRAIN TRANSFORMATION

It has been shown that no crack-shear displacement occurs and that continuum strain-transformation equations apply to the reinforcement strains in the membrane with crack openings, if the ratio of the crack displacements e_y/e_x equals $\tan\theta$. In this section, e_y/e_x is evaluated. Toward this end, the four equilibrium equations are expressed in terms of e_x, e_y, e_2, and θ. This is done in Appendix B.

Because of the complexity of the equations, the following evaluation of e_y/e_x is accomplished for membranes with orthogonal two-way reinforcement. For the elastic case of two-way orthogonal reinforcement, the equations as stated here are obtained. Using Eq. (11), (12), and (13c), in turn, we obtain

$$e_x = \frac{N_x \cos\theta + N_{xy} \sin\theta - hE_s \rho_x e_2 (\mu\cos^2\theta - \sin^2\theta)\cos\theta}{h\, E_s\, \rho_x \cos^2\theta} \tag{14}$$

$$e_y = \frac{N_y \sin\theta + N_{xy} \cos\theta - h\, E_s \rho_y e_2 (\mu\sin^2\theta - \cos^2\theta) \sin\theta}{h\, E_s\, \rho_y \sin^2\theta} \tag{15}$$

$$e_2 = \frac{N_{xy}}{hE_c \sin\theta \cos\theta} = \frac{nN_{xy}}{h\, E_s \sin\theta \cos\theta}, \qquad n = \frac{E_s}{E_c} \tag{16}$$

The equation obtained directly from $\partial U_T/\partial\theta = 0$ [Eq. (B9)] involves the four displacements and is not stated here because of its complexity. By substituting the values of e_x, e_y, and e_2 from Eq. (14), (15), and (16) into Eq. (B9), the following is obtained for the membrane with two-way reinforcement

$$\frac{1}{\rho_y} N_y \tan\theta + N_{xy}\left(\frac{1}{\rho_y} + n\right) = N_{xy}\left(\frac{1}{\rho_x} + n\right)\tan^4\theta + \frac{1}{\rho_x} N_x \tan^3\theta \tag{17}$$

The ratio e_y/e_x is now formed from Eq. (14) and (15) after inserting the value of e_2 from Eq. (16)

$$\frac{e_y}{e_x} = \frac{(1/\rho_y)\,[N_y \tan\theta] + N_{xy}(1/\rho_y + n) - n\mu\, N_{xy} \tan^2\theta}{N_{xy}\,(1/\rho_x + n)\tan^3\theta + 1/\rho_x\,(N_x \tan^2\theta) - n\mu\, N_{xy} \tan\theta)}$$

When the right-hand side of Eq. (17) is substituted for the terms in the numerator corresponding to the left-hand side of Eq. (17), the preceding equation reduces to

$$e_y/e_x = \tan\theta \tag{18}$$

The ratio e_y/e_x is now evaluated for the elastic-plastic phase, that is, with $f_x = f_x^Y$ and $f_y < f_y^Y$. The stress-equilibrium Eq. (11), (12), and (13c) are adapted for this case to give

$$\rho_x h f_x^Y - N_x = N_{xy} \tan\theta \quad (19)$$

$$\rho_y h f_y - N_y = N_{xy} \cot\theta \quad (20)$$

$$h E_c e_2 \sin\theta \cos\theta = N_{xy} \quad (21)$$

In the equation obtained from $\partial U_T/\partial\theta = 0$, terms corresponding to the left-hand side of Eq. (19) and (20) are replaced by their equivalents on the right-hand side, giving

$$e_x N_{xy} (\sin\theta\tan\theta + \cos\theta) - e_y N_{xy} (\cos\theta\cot\theta + \sin\theta) + e_2 N_{xy}(1 + \mu) \sin2\theta(\tan\theta - \cot\theta + 2\cot2\theta) = 0 \quad (22)$$

The ratio e_y/e_y is formed from this equation to give $e_y/e_x = \tan\theta$.

The preceding development shows that, for a membrane in which the reinforcement carries uniaxial stress and the crack is the first principal concrete stress plane, the potential energy of the membrane is a minimum when $e_y/e_x = \tan\theta$. From Eq. (2b), this is equivalent to no tangential displacement along the crack, even though no resistance to such displacement is postulated. This leads to the conclusion that, with the adopted stress system in the reinforcement and concrete, no crack-shear force is developed even if crack-shear resistance exists.

The previous strain transformation Eq. (5a) is rewritten on the basis of $e_y/e_x = \tan\theta$ developed for two-way orthogonal reinforcement but assumed here to be applicable to multidirectional reinforcement. The resulting reinforcement strain transformation equation is the same as that for a continuum

$$\epsilon_i = e_1 \cos^2 \beta_i - e_2 \sin^2 \beta_i \quad (23)$$

where $e_1 = e_n + \mu e_2$ and $\beta_i = \alpha_i - \theta$.

MODES OF MEMBRANE BEHAVIOR

The modes of membrane behavior considered are those in which cracking is feasible; this implies that one or both of the membrane principal stress resultants, N_1 or N_2, are tensile.

As membrane loading is increased after the concrete has cracked, experience has shown that failure may occur in one of three modes. The initial behavior of the reinforced concrete membrane after cracking is elastic, with the reinforcement together with the concrete carrying the load. As the load is increased, the behavior becomes either ductile due to yielding of the reinforcement or brittle due to concrete crushing.

The brittle failure may occur before any reinforcement yield (Mode B), or after reinforcement yield in some, but not all, reinforcement directions (Mode DB). Ductile failure is that associated with yielding of the reinforcement in all directions (Mode DD). At critical stages in the loading, the methodology developed in subsequent sections determines the behavior mode, loading, crack angle, and equilibrium stress-strain systems. The response is determined for the following behavioral phases:

Elastic action—Characteristics prevailing before yield or crushing are determined.

Ductile phases—Response is calculated, in turn, as the different directions of reinforcement successively begin to yield. In this paper, yield n refers to the stage at which $n - 1$ sets of reinforcement are already at yield stress and the nth set is at incipient yield.

Crushing phases—The membrane crushing strengths and associated failure loads are determined for the stress-strain patterns associated with the critical elastic and ductile phases.

EQUATIONS FOR ELASTIC AND DUCTILE BEHAVIOR

The working equations are developed by using Eq. (18) and (23) in the equilibrium equations in Appendix B. The equations are applied in a cyclical process described later. In the equations for the calculation of e_1 and θ, the term involving e_2 with its coefficient, a function of θ, is treated as known. This follows from the cyclical process in which the values from the preceding cycle are used to evaluate this term. This procedure was adopted because it was found that variation of e_2 has relatively minor effect on the values of e_1 and θ.

Constant terms

In this section, constants that define the membrane reinforcement in the successive phases are introduced. The following apply to bars not yet in the yield state, including incipient yield

$$\begin{aligned} A &= \sum_i \rho_i \cos^4\alpha_i \\ B &= \sum_i \rho_i \cos^3\alpha_i \sin\alpha_i \\ C &= \sum_i \rho_i \cos^2\alpha_i \sin^2\alpha_i \\ D &= \sum_i \rho_i \cos\alpha_i \sin^3\alpha_i \\ E &= \sum_i \rho_i \sin^4\alpha_i \end{aligned} \quad (24a)$$

The following apply to bars that are in the yield state, including all bars at final yield

$$\begin{aligned} F &= \sum_r \rho_r f_r^Y \cos^2\alpha_r \\ G &= \sum_r \rho_r f_r^Y \cos\alpha_r \sin\alpha_r \\ H &= \sum_r \rho_r f_r^Y \sin^2\alpha_r \end{aligned} \quad (24b)$$

The following apply to the bar set that is at incipient yield, the kth set, for use in Eq. (28)

$$J = \cos^2\alpha_k$$
$$K = \cos\alpha_k \sin\alpha_k \quad (24c)$$
$$L = \sin^2\alpha_k$$

The applied membrane loading represents a constant loading pattern defined by the ratios c_x, c_y, and c_{xy} in Eq. (25). The first principal stress resultant N_1 is taken as the index of the load level

$$c_x = N_x/N_1$$
$$c_y = N_y/N_1 \quad (25)$$
$$c_{xy} = N_{xy}/N_1$$

Equations for e_1

The equations in Appendix B corresponding to $\partial U_T/\partial e_x = 0$ and to $\partial U_T/\partial e_y = 0$ each are solved for e_1 after insertion of Eq. (18)

$$hE_s e_1 = \frac{(N_x - hF)\cos\theta + (N_{xy} - hG)\sin\theta + hE_s X e_2}{A\cos^3\theta + 3B\cos^2\theta\sin\theta + 3C\cos\theta\sin^2\theta + D\sin^3\theta} \quad (26a)$$

where

$$X = C\cos^3\theta + (D - 2B)\cos^2\theta\sin\theta + (A - 2C)\cos\theta\sin^2\theta + B\sin^3\theta \quad (26b)$$

$$hE_s e_1 = \frac{(N_y - hH)\sin\theta + (N_{xy} - hG)\cos\theta + hE_s Y e_2}{B\cos^3\theta + 3C\cos^2\theta\sin\theta + 3D\cos\theta\sin^2\theta + E\sin^3\theta} \quad (27a)$$

where

$$Y = D\cos^3\theta + (E - 2C)\cos^2\theta\sin\theta + (B - 2D)\cos\theta\sin^2\theta + C\sin^3\theta \quad (27b)$$

Either of the preceding equations can be used to calculate e_1 when the loading level is known. However, when the loading associated with incipient reinforcement yield is to be determined, the following method, developed for incipient yield of Bar k, is used. The yield value of $\epsilon_k^Y = f_k^Y/E_s$ is substituted in Eq. (23) with the following evaluation for e_1

$$e_1 = \frac{f_k^Y/E_s + e_2 g_2}{g_1} \quad (28a)$$

where

$$g_1 = J\cos^2\theta + 2K\cos\theta\sin\theta + L\sin^2\theta \quad (28b)$$

$$g_2 = L\cos^2\theta - 2K\cos\theta\sin\theta + J\sin^2\theta \quad (28c)$$

Equation for θ

The values of $hE_s e_1$ from Eq. (26) and (27) are set equal to give

$$K_4\tan^4\theta + K_3\tan^3\theta + K_2\tan^2\theta + K_1\tan\theta + K_0 = hE_s Z e_2 \quad (29a)$$

where

$$K_4 = N_1(Ec_{xy} - Dc_y) + h(DH - EG)$$

$$K_3 = N_1(Ec_x - 3Cc_y + 2Dc_{xy}) + h(3CH - 2DG - EF)$$

$$K_2 = N_1(3Dc_x - 3Bc_y) + 3h(BH - DF) \quad (29b)$$

$$K_1 = N_1(3Cc_x - Ac_y - 2Bc_{xy}) + h(2BG - 3CF + AH)$$

$$K_0 = N_1(Bc_x - Ac_{xy}) + h(AG - BF)$$

$$Z = \cos^2\theta\,(M_6\tan^6\theta + M_5\tan^5\theta + M_4\tan^4\theta + M_3\tan^3\theta + M_2\tan^2\theta + M_1\tan\theta + M_0)$$

where

$$M_6 = CD - BE$$

$$M_5 = 3C^2 - 2BD - 2D^2 + 2CE - EA$$

$$M_4 = 3BC - 2CD + 2BE - 3AD$$

$$M_3 = 2B^2 - 2D^2 + 2CE - 2AC$$

$$M_2 = 2BC - 3CD + 3BE - 2AD$$

$$M_1 = 2B^2 + 2BD - 3C^2 - 2AC + AE$$

$$M_0 = AD - BC \quad (29c)$$

In the cyclical solution process, Eq. (29) is first used in the elastic phase with the given loading. However, at incipient yield of a reinforcing bar, the loading index N_1 must be established prior to using Eq. (29).

Equation for N_1

In the elastic phase, the loading is given and the loading level N_1 and the principal membrane compression N_2 are calculated from the given stress resultants

$$N_1, N_2 = \frac{N_x + N_y}{2} \pm \left[\left(\frac{N_x - N_y}{2}\right)^2 + N_{xy}^2\right]^{1/2} \quad (30)$$

When the loading level associated with the yield of a reinforcing bar is sought, the value of N_1 is determined in the solution process from previously calculated values of the material stresses. Derivation of Eq. (31) follows from the equilibrium of membrane and material forces acting on the free body bounded by the crack, the normal to the crack, and the first principal membrane plane[4]

$$N_1 = \sum_j F_j\cos^2(\alpha_j - \phi) - F_c\sin^2(\theta - \phi) \quad (31)$$

In this equation, F_j represents all the reinforcement forces F_i and F_r^Y.

Equation for e_2

The stress-equilibrium equations Eq. (13b) and (13c) each are used in calculating the value of F_c, resulting in F_{cb} and F_{cc}

$$F_{cb} = \sum_i F_i + \sum_r F_r^Y - N_x - N_y \quad (32a)$$

$$F_{cc} = \frac{1}{\sin 2\theta} \left(2N_{xy} - \sum_i F_i \sin 2\alpha_i - \sum_r F_r \sin 2\alpha_r\right) \quad (32b)$$

$$F_c = 0.5\,(F_{cb} + F_{cc}) \quad (32c)$$

The displacement e_2 is then evaluated from the average value

$$e_2 = \frac{F_c}{E_c\,h} \quad (33)$$

Equations for θ and N_1 at yield of last set of bars

Solution for loading when all the reinforcement has reached yield follows the method presented by Nielsen[10] and is based on Eq. (11) and (12). For determining θ

$$(Gc_y - Hc_{xy})\tan^2\theta + (Fc_y - Hc_x)\tan\theta + Fc_{xy} - Gc_x = 0 \quad (34)$$

and for determining N_1

$$N_1^2\,(c_{xy}^2 - c_x c_y) + N_1 h(Fc_y + Hc_x - 2Gc_{xy}) + h^2(G^2 - FH) = 0 \quad (35)$$

CALCULATION PROCEDURE FOR ELASTIC AND DUCTILE BEHAVIOR

The procedure given here provides for the calculation of the crack angle, material strains and stresses, and load level (if not specified) at the critical stages of nonbrittle behavior. The procedure for checking these results against possible brittle failure by concrete crushing is discussed in the next section.

The calculation procedure is iterative, and the procedure is defined for a typical cycle. Examples indicate rapid convergence; the calculated results converge to within one percent of the values in the preceding cycle by the third cycle. The steps in one cycle of each phase are given. At each step of the cycle, the previously calculated value of each parameter is to be used.

Yield 0—Elastic action with initial loading

Equation numbers for each calculation step are listed.

Preliminary calculation

Reinforcement characteristics	(24a)
Load level N_1 (principal membrane force) and direction ϕ	(30), (1)
Stress ratios	(25)

Cycle 1—Start by assuming $e_2 = 0$ and calculate in order

Crack angle θ	(29a)
Strain e_1	(26) or (27)
Reinforcement strains ϵ_i and forces F_i	(23), (10a)
Principal concrete compression F_c	(32a,b,c)
Strain e_2	(33)
Parameter Z	(29b)

Cycle 2 and succeeding cycles—Start with values of e_2 and Z from preceding cycle. The calculation steps are the same as for Cycle 1.

Analysis of these equations applicable to elastic action up to first yield shows that θ remains constant and that the stresses and strains vary linearly with the load.

Yield 1—Incipient yield of first set of bars

Elastic action continues to incipient yield of the set of bars that yields first. This set of reinforcement A_m is ascertained as the mth set with the largest ratio f_m/f_m^Y of elastic-action stress to yield stress. The stresses and strains at incipient yield of A_m are obtained by multiplying the corresponding results from Yield 0 by the inverse ratio f_m^Y/f_m. Angle θ at Yield 1 remains the same as for Yield 0.

Yield 2 and succeeding yields k up to final yield phase

Yield 2 (Yield k) represents the conditions when the second (kth) set of reinforcing bars is at incipient yield. The next set to yield is established by the results of preceding phase. Calculation steps and the associated equations are as follows.

Preliminary calculations

Reinforcement characteristics based on yielding in bars of the preceding phase and elastic action in all other bars	(24b,a)

Cycle 1—Start by assuming the values of e_2, θ, and F_c to be those determined in the preceding phase. Then, calculate the following in order

Strain e_1	(28a,b,c)
Reinforcement strains ϵ_i and forces F_i, F_r^Y	(23), (10a,b)

Load level N_1	(31)
Principal concrete compression F_c	(32a,b,c)
Strain e_2	(33)
Parameter Z	(29b)
Crack angle θ	(29a)

Cycle 2 and succeeding cycles—Start by assuming the values of e_2, θ, and F_c to be those determined in the preceding cycle. Repeat steps in Cycle 1.

Final yield phase—Incipient yield of last set of bars

This phase represents conditions when the final set of bars is at incipient yield so that the condition is that of ductile failure; it is designated Type DD failure. Calculation results are obtained directly, without iteration. The calculation steps with associated equations are

Reinforcement characteristics, yielding in all bars	(24b)
Crack angle θ	(34)
Load level N_1	(35)
Membrane stress resultants N_x N_y N_{xy}	(25)
Principal concrete compression F_c	(32a,b,c)
Strain e_2	(33)
Strain e_1, with last set of bars at incipient yield	(28a,b,c)
Reinforcement strains	(23)

CRUSHING STRENGTH AND FAILURE LOAD

In the overall calculation procedure, the preceding calculations, which define the ductile behavior of the membrane, have been made on the assumption that earlier brittle failure by crushing does not intervene. To ascertain the actual failure mode, the membrane crushing strength must be determined. This is done subsequent to the preceding calculations because information about reinforcement stresses and crack size is needed to determine crushing strength.

Experimental results such as those in Reference 1 have shown that compression failure in reinforced concrete membranes occurs at stresses substantially smaller than the uniaxial cylinder strength f_c' when membrane tension exists perpendicular to the principal membrane compression. A method to evaluate this reduced strength is presented here for membranes with multidirectional reinforcement. This method is an extension of that developed for membranes with two-way orthogonal reinforcement[4] on the basis of experimental results and theoretical considerations.

The membrane crushing strength is defined here as the principal membrane compressive stress at membrane crushing $\sigma_2^U = N_2^U/h$. Crushing strength is evaluated for two types of failure involving crushing. In the first, Type B, sudden membrane failure by crushing occurs prior to any reinforcement yield. With the second, Type DB, increasing loading results first in ductile yielding of the reinforcement in one or more directions. Subsequently, as loading is increased, failure occurs by crushing prior to ductile yielding in all directions; this ductile brittle-type failure is designated Type DB. Ductile failure by yielding of all the reinforcement is designated Type DD.

Type B failure

Prior to reinforcement yield, in the presence of perpendicular tension, the membrane crushing strength σ_2^U depends on the concrete cylinder strength f_c' and on the membrane stresses as characterized by two parameters. The first parameter s is equal to the negative ratio of the principal membrane tensile stress to the principal membrane compressive stress. The second parameter s' depends on the tensile forces in the reinforcement; it is equal to the ratio between the normal component of the reinforcement forces acting on the tension face of the principal membrane element and the normal component of these forces acting on the compression face.

In the evaluation procedure that follows, parameters s and s' are calculated by Eq. (36a) and (36b). In the calculation for s', the values of F_i obtained in the phases Yield 0 and Yield 1 are applicable

$$s = -\sigma_1/\sigma_2 = -N_1/N_2 \qquad (36a)$$

$$s' = \frac{\sum_i F_i \cos^2(\alpha_i - \phi)}{\sum_i F_i \sin^2(\alpha_i - \phi)} \qquad (36b)$$

A material strength reduction factor R' depends on s

$$R' = 0.14 + 1/6\,(2.0 - s)^{2.3} \qquad (37a)$$
$$0 \leqslant s \leqslant 1.0$$

$$(37b)$$
$$R' = 0.20 + 1/9\,(2.0 - s)^2$$
$$1.0 \leqslant s \leqslant 2.0 \qquad (37c)$$

$$R' = 0.20, \qquad 2.0 \leqslant s$$

The membrane crushing strength for Type B failure is then evaluated by Eq. (38)

$$\sigma_{2B}^U = -R f_c', \qquad R = \frac{R'(1 + s)}{1 + s/s'} \qquad (38)$$

The load level N_{1B} associated with this crushing strength is evaluated by Eq. (39)

$$N_{1B} = -s\sigma_{2B}^U h = -sN_{2B}^U \qquad (39)$$

The load level N_1, calculated with elastic action at Yields 0 and 1, is designated generically as N_{1E}. This load level is attained and may possibly be exceeded if

$$N_{1E} < N_{1B} \qquad (40)$$

If the inequality is reversed, failure is by brittle crushing, Type B, and the failure load level is N_{1B}. If the inequality holds, behavior at yield of the next set of bars is investigated.

Type DB failure

When crushing occurs after yield in one or more directions has begun, the membrane crushing strength, designated σ^U_{2DB}, depends on the preceding factors s, s', and R', and also on Δe_n, the increase in the normal crack opening over that at first incipient yield. The following evaluation of σ^U_{2DB} is applicable at loads beyond Yield 1, up to the final yield phase.

Parameter s is still defined by Eq. (36a), but evaluation of s' is now by Eq. (36c), since some of the bars are at yield stress

$$s' = \frac{\sum_i F_i \cos^2(\alpha_i - \phi) + \sum_r F^Y_r \cos^2(\alpha_r - \phi)}{\sum_i F_i \sin^2(\alpha_i - \phi) + \sum_r F^Y_r \sin^2(\alpha_r - \phi)} \quad (36c)$$

Evaluation of the factor R' is by Eq. (37a), (37b), and (37c), as it was previously.

The normal crack opening e_n at any loading where the strains are e_1 and e_2 is given by

$$e_n = e_1 - \mu e_2 \quad (41a)$$

At first incipient yield (Yield 1), the crack opening is designated $e_n^{(1)}$. The increase in normal crack opening at any loading over that at Yield 1 is

$$\Delta e_n = e_n - e_n^{(1)}, \qquad e_n^{(1)} = e_1^{(1)} - \mu e_2^{(1)} \quad (41b)$$

Due to Δe_n, further reduction in crushing strength $r\sigma^U_{2B}$ occurs where r is defined by

$$r = 1.0 - 40.0\,\Delta e_n \quad (42a)$$
$$0 \leqslant \Delta e_n \leqslant 0.0125$$

$$r = 0.50, \qquad 0.0125 \leqslant \Delta e_n \quad (42b)$$

The concrete crushing strength σ^U_{2DB} is then evaluated

$$\sigma^U_{2DB} = r\,\sigma^U_{2B} = -\frac{rR'(1+s)}{1+s/s'} f'_c \quad (43)$$

The load level N_{1DB} associated with this crushing strength is evaluated by Eq. (44)

$$N_{1DB} = -s\,\sigma^U_{2DB} h = -s N^U_{2DB} \quad (44)$$

The load level N_1, calculated with incipient yield of reinforcement subsequent to yield of the first set of bars, is designated generically as N_{1D}. This load level is attained and may possibly be exceeded if

$$N_{1D} < N_{1DB} \quad (45)$$

If the inequality is true, behavior at yield of the next set of bars is checked for crushing. Eventually, if the inequality applies at incipient yield of the last set of bars, the membrane failure is ductile in Type DD and the failure load is N_{1D}, determined for the last set of bars. However, if inequality [Eq. (45)] is first found to be reversed when checking the nth set of bars, failure occurs by ductile brittle crushing (Type DB) and the failure load level is determined as described in the following section.

Crushing at intermediate phase after Yield 1

By the preceding calculations, crushing strength load levels corresponding to each of the critical yield stages are available. However, if inequality [Eq. (45)] is found to be reversed at Yield n, membrane crushing occurs before Yield n. In this case, the magnitude of load level N_{1DB} associated with crack increment Δe_n is not valid. The proper value of load level N_{1DB} must be established so that, as the applied load, it causes a crack increment that, in turn, defines a crushing strength corresponding to the same applied load level.

Toward this end, it is assumed that the rates of change of the load level and the crushing strength between Yield $n-1$ and Yield n are the same; thus, when the load level has changed by the fraction p of the increment between the yields, the crushing strength has also changed by p. The basis for this assumption is that both the load and the crushing strength are functions of the displacement parameters; the load level has been shown by the calculation procedure to depend on these parameters, and the crushing strength does so through dependence on Δe_n and on the stress ratios, which, in turn, are functions of the displacement parameters. The associated equations developed below were checked against test Specimen PV20 of Reference 1, which is an illustrative example in References 3 and 4. The ratio of test failure load to predicted failure load calculated by Eq. (46) is 0.97.

In line with the preceding, the proper value of the intermediate load level N_{1p} is determined by equilibrating the load level and the crushing strength

$$N_{1p} = N^{(n-1)}_{1DB} + p[N^{(n)}_{1DB} - N^{(n-1)}_{1DB}] \quad (46a)$$
$$= N^{(n-1)}_{1D} + p[N^{(n)}_{1D} - N^{(n-1)}_{1D}]$$

where

$$p = \frac{N^{(n-1)}_{1D} - N^{(n-1)}_{1DB}}{N^{(n)}_{1DB} - N^{(n)}_{1D} + N^{(n-1)}_{1D} - N^{(n-1)}_{1DB}} \quad (46b)$$

SUMMARY AND CONCLUSIONS

This paper provides a methodology for calculating the design quantities necessary for designing membrane elements in concrete shearwalls and shells with multidirectional reinforcement against in-plane forces. Since the material stress systems developed are in equilibrium with the applied forces and are in compliance with ductile and brittle strength requirements, the basis for safe design is provided. An illustrative example is included.

The membrane behavior is determined in terms of failure mode (ductile or brittle); load level; and stresses, strains, and cracking characteristics at critical stages of ductile and brittle behavior. Toward this end, a set of equations, each involving one unknown parameter, is applied iteratively; examples show the convergence to be rapid. As the basis for the equations, the paper demonstrates the propriety of omission of shear force between the sides of cracks in the concrete.

CONVERSION FACTORS

1 in. = 25.4 mm
1 kip/in. = 0.175 kN/mm
1 ksi = 6.9 MPa

REFERENCES

1. Vecchio, F. J., and Collins, M. P., "Response of Reinforced Concrete to In-Plane Shear and Normal Stresses," *Publication* No. 82-03, Department of Civil Engineering, University of Toronto, Mar. 1982, 332 pp.
2. Gupta, A. K., and Akbar, H., "Cracking in Reinforced Concrete Analysis," *Journal of Structural Engineering,* ASCE, V. 110, No. 8, Aug. 1984, pp. 1735-1746.
3. Vecchio, Frank J., and Collins, Michael P., "Modified Compression—Field Theory for Reinforced Concrete Elements Subjected to Shear," ACI JOURNAL, *Proceedings* V. 83, No. 2, Mar.-Apr. 1986, pp. 219-231.
4. Fialkow, M. N., "Behavior of RC Membranes with Compatible Stress and Cracking," ACI JOURNAL, *Proceedings* V. 87, No. 5, Sept.-Oct. 1990, pp. 571-582.
5. Zienkiewicz, O. C., *Finite Element Method in Structural and Continuum Mechanics,* McGraw-Hill Publishing Co. Ltd., London, 1967, pp. 18-19.
6. Hodge, P. G., Jr., *Plastic Analysis of Structures,* McGraw-Hill Book Co., Inc., New York, 1959, p. 28.
7. Desai, Chandrakant S., and Abel, John F., *Introduction to the Finite Element Method,* Van Nostrand Reinhold Co., New York, 1972, pp. 58-59.
8. Duchon, Nicholas B., "Analysis of Reinforced Concrete Membrane Subject to Tension and Shear," ACI JOURNAL, *Proceedings* V. 69, No. 9, Sept. 1972, pp. 578-583.
9. Medwadowski, Stefan J., "Multidirectional Membrane Reinforcement," *ACI Structural Journal,* V. 86, No. 5, Sept.-Oct. 1989, pp. 563-569.
10. Nielsen, M. P., "Yield Conditions for Reinforced Concrete Shells in the Membrane State," *Proceedings,* IASS Symposium on Non-Classical Shell Problems, Warsaw, 1963, pp. 1036-1038.

APPENDIX A—EXAMPLE

The behavior of a reinforced concrete membrane with three-way reinforcement is investigated as the loading is increased to membrane failure; the membrane is the same as that in Example 2 in Reference 9. Physical characteristics of membrane initial loading are

$$N_x = 0.5 \text{ kip/in.}; N_y = -0.5 \text{ kip/in.}; N_{xy} = 1.0 \text{ kip/in.}$$

$$E_s = 30{,}000 \text{ ksi}; E_c = 3500 \text{ ksi}; f^y = 40 \text{ ksi}$$

$$\alpha_1 = 10 \text{ deg}; \alpha_2 = 70 \text{ deg}; \alpha_3 = 130 \text{ deg}; h = 3 \text{ in.}; \mu = 0.17$$

$$\rho_1 = \rho_2 = \rho_3 = 0.01; \epsilon^Y = 40/30{,}000 = 0.00133$$

Elastic and ductile behavior

The section "Calculation Procedure for Elastic and Ductile Behavior" in the main body of this paper lists the calculation steps with associated equations to be used. These steps are followed in order in the solution that follows. Equation numbers are listed to the right of the calculation results.

Yield 0—Behavior under initial loading, elastic action

Preliminary calculations—i = 1, 2, 3; *r* = 0

Table A1 — Yield 0: Initial loading, elastic action

Variable	Cycle 1	Cycle 2	Cycle 3
θ, deg	31.72	31.72	31.72
$e_1 \times 10^3$	1.104	1.181	1.175
$\epsilon_1 \times 10^3$; F_1, kip/in.	0.953; 0.857	1.00; 0.900	0.996; 0.896
$\epsilon_2 \times 10^3$; F_2, kip/in.	0.680; 0.612	0.673; 0.606	0.674; 0.606
$\epsilon_3 \times 10^3$; F_3, kip/in.	0.023; 0.021	−0.115; −0.103	−0.104; −0.094
F_c, kip/in.	1.49	1.37	1.38
$e_2 \times 10^3$	0.142	0.131	0.132
Z	0	0	0

1 kip/in. = 0.175 kN/mm.

$$A = 0.01125; B = 0; C = 0.00375; D = 0; E = 0.01125 \quad (24a)$$

$$N_{1,2} = \pm 1.118 \text{ kip/in.}; \phi = 31.72 \text{ deg} \quad (30)(1)$$

$$c_x = 0.4472; c_y = -0.4472; c_{xy} = 0.8944 \quad (25)$$

Cyclical calculations—Three iterative cycles of calculations suffice to attain substantive convergence within one percent. The cyclical results are listed in Table A1. The final results are those listed in Cycle 3. In beginning Cycle 1, e_2 is taken equal to zero.

Yield 1—Incipient yield of first set of bars—With the given loading, bar No. 1 (α_1 = 10 deg) has the largest strain ($\epsilon_1^{(0)}$ = 0.000996) and therefore is the first bar to yield. The results at Yield 1 are those at Yield 0 multiplied by the ratio $\epsilon_1^{(Y)}/\epsilon_1^{(0)}$ = 1.339. The following results apply at Yield 1

$$\theta = 31.72 \text{ deg}; e_1 = 0.00157; e_2 = 0.000177$$

$$\epsilon_1 = 0.00133; \epsilon_2 = 0.000902; \epsilon_3 = -0.000140$$

$$F_1 = 1.20 \text{ kips/in.}; F_2 = 0.812 \text{ kips/in.}; F_3 = -0.127 \text{ kips/in.}; F_c = 1.847 \text{ kips/in.}$$

$$N_1 = 1.496 \text{ kips/in.}; N_x = -N_y = 0.669 \text{ kip/in.}; N_{xy} = 1.338 \text{ kips/in.}$$

Yield 2—Incipient yield of second set of bars—Based on the results of Yield 1, bar No. 2 (α_2 = 70 deg) yields next, as its strain is the second largest. Hence, the reinforcement characteristics are calculated on the basis that the stress in bar No. 1 is the yield stress (40 ksi), the strain in bar No. 2 is the yield strain (0.00133), and bar No. 3 is at less than yield stress.

Preliminary calculations—i = 2, 3; *r* = 1

$$A = 0.001844; B = -0.001658; C = 0.003458; D = 0.000052; E = 0.01124 \quad (24a)$$

$$F = 0.3879 \text{ ksi}; G = 0.0684 \text{ ksi}; H = 0.0121 \text{ ksi} \quad (24b)$$

Cyclical calculations—Three cycles of calculations suffice, as previously, with final results in Cycle 3. Cycle 1 starts by using the values of e_2 (0.000177) and θ (31.72 deg) from Yield 1. The cyclical results are listed in Table A2.

Yield 3—Incipient yield of last set of bars—i = 0; *r* = 1, 2, 3

At incipient yield of bar No. 3, all the reinforcement stresses are known, and the system is determinate. The solution follows the order of the calculation procedure

Reinforcement characteristics, in ksi: $F = 0.60; G = 0; H = 0.60$ (24b)

Reinforcement forces, in kips/in.: $F_1 = 1.2; F_2 = 1.2; F_3 = 1.2$

Crack angle: $\theta = 31.72$ deg (34)

Load level, in. kips/in.: $N_1 = 1.8$

Table A2 — Yield 2: Incipient yield of second set of bars

Variable	Cycle 1	Cycle 2	Cycle 3
$e_1 \times 10^3$	2.274	2.668	2.667
$\epsilon_1 \times 10^3$; F_1, kip/in.	1.938; 1.2	2.42; 1.2	2.41; 1.2
$\epsilon_2 \times 10^3$; F_2, kip/in.	1.333; 1.2	1.333; 1.2	1.333; 1.2
$\epsilon_3 \times 10^3$; F_3, kip/in.	−0.126; −0.114	−0.065; −0.058	−0.083; −0.074
N_1, kip/in.	1.773	1.759	1.759
F_c, kip/in.	2.193	2.348	2.325
$e_2 \times 10^3$	0.209	0.224	0.222
$Z \times 10^3$	−0.0111	−0.0110	−0.0110
θ, deg	27.07	27.23	27.23

1 kip/in. = 0.175 kN/mm.

Stress resultants, in. kips/in.: $N_x = 0.805$; $N_y = -0.805$; $N_{xy} = 1.61$ (25)

Principal compressive force, in. kips/in.: $F_c = 3.6$ (32)

Strain parameters: $e_2 = 0.000343$; $e_1 = 0.08025$ (33)(28)

Reinforcement strains: $\epsilon_1 = 0.06921$; $\epsilon_2 = 0.04932$; $\epsilon_3 = 0.00133$ (23)

Crushing strength and failure load

The calculation results follow the procedure and the equations as outlined in the section, "Crushing Strength and Failure Load."

Type B failure

$$s = 1.0;\ R' = 0.31;\ s' = 4.36;\ R = .504 \quad (36)(37)(38)$$

$$f_c' = \left[\frac{E_c}{57{,}000}\right]^2 = \left[\frac{3{,}500{,}000}{57{,}000}\right]^2 = 3770 \text{ psi} = 3.77 \text{ ksi}$$

$$\sigma_{2B}^U = -0.504 \times 3.77 = -1.90 \text{ ksi},\ N_{1B} = 5.70 \text{ kips/in.} \quad (38)(39)$$

$$N_{1E} = 1.496 \text{ kips/in., from Yield 1}$$

$$N_{1E} = 1.496 < 5.70 = N_{1B},\ \text{true} \quad (40)$$

Since inequality [Eq. (40)] is true, the load level associated with Yield 1 is attained and behavior at Yield 2 is investigated next.

Type DB failure at Yield 2

$$s = 1.0;\ R' = 0.31;\ s' = 3.21;\ R = 0.473;\ f_c' = 3.77 \text{ ksi} \quad (36a,c)(37)$$

$$e_n^{(1)} = 0.00154 \text{ at Yield 1};\ e_n^{(2)} = 0.00263 \text{ at Yield 2} \quad (41a)$$

$$\Delta e_n^{(2)} = 0.00109;\ r = 0.956 \quad (41b)(42)$$

$$\sigma_{2DB}^U = -0.452 \times 3.77 = 1.70 \text{ ksi};\ N_{1DB} = 5.10 \text{ kips/in.} \quad (43)(44)$$

$$N_{1D} = 1.76 \text{ kips/in., from Yield 2}$$

$$N_{1D} = 1.76 < 5.10 = N_{1DB},\ \text{true} \quad (45)$$

Since inequality [Eq. (45)] is true, the load level associated with Yield 2 is attained and behavior at Yield 3 is investigated next.

Type DB failure at Yield 3

$$s = 1.0;\ R' = 0.31;\ s' = 1.0;\ R = 0.31;\ f_c' = 3.77 \text{ ksi} \quad (36a,c)(37)$$

$$e_n^{(1)} = 0.00154 \text{ at Yield 1};\ e_n^{(3)} = 0.08019 \text{ at Yield 3} \quad (41a)$$

$$\Delta e_n^{(3)} = 0.0787;\ r = 0.50 \quad (41b)(42)$$

$$\sigma_{2DB}^U = -0.155 \times 3.77 = -0.584 \text{ ksi};\ N_{1DB} = 1.75 \text{ kip/in.} \quad (43)(44)$$

$$N_{1D} = 1.80 \text{ kips/in., from Yield 3}$$

$$N_{1D} = 1.80 < 1.75 = N_{1DB},\ \text{not true} \quad (45)$$

Since inequality [Eq. (45)] is not true, the load level associated with Yield 3 is not attained. Failure occurs by ductile brittle crushing between Yield 2 and Yield 3.

Type DB failure between Yields 2 and 3

$$p = \frac{1.76 - 5.10}{1.75 - 1.80 + 1.76 - 5.10} = 0.9825 \quad (46b)$$

$$N_{1p} = 5.10 + 0.98525\,(1.75 - 5.10) = 1.7994 \text{ kips/in.} \quad (46a)$$

The membrane failure load level equals 1.7994 kips/in. and the failure mode is by Type DB crushing. Failure occurs after yielding of bars No. 1 and 2, but before yielding of bar No. 3.

APPENDIX B—EQUILIBRIUM EQUATIONS

Equations in terms of stress

The equilibrium equations corresponding to each independent displacement are obtained via Eq. (9). For $e_j = e_x$, the following equilibrium equation is obtained

$$\frac{\partial U_T}{\partial e_x} = \sum_i \rho_i h E_s \epsilon_i \frac{\partial \epsilon_i}{\partial e_x} + \sum_r \rho_r h f_r^Y \frac{\partial \epsilon_r}{\partial e_x} - N_x \cos\theta - N_{xy}\sin\theta = 0 \quad \text{(B1a)}$$

The partial derivatives of the strains are evaluated; Eq. (10) is then used to express Eq. (B1b) in terms of forces and β or θ

$$N_x\cos\theta + N_{xy}\sin\theta = \sum_i F_i \cos\beta_i \cos\alpha_i + \sum_r F_r^Y \cos\beta_r \cos\alpha_r \quad \text{(B1b)}$$

Similarly, for $e_j = e_y$, the following force equilibrium equation is obtained

$$N_y\sin\theta + N_{xy}\cos\theta = \sum_i F_i \cos\beta_i \sin\alpha_i + \sum_r F_r^Y \cos\beta_r \sin\alpha_r \quad \text{(B2)}$$

For $e_j = e_2$, the following equilibrium equation also involves the forces and θ

$$N_x(\mu\cos^2\theta - \sin^2\theta) + N_y(\mu\sin^2\theta - \cos^2\theta) + N_{xy}\,(1 + \mu)\sin 2\theta - F_c = \sum_i F_i(\mu\cos^2\beta_i - \sin^2\beta_i) + \sum_r F_r^Y(\mu\cos^2\beta_r - \sin^2\beta_r) \quad \text{(B3)}$$

For $e_j = \theta$, the resulting force equilibrium Eq. (B4) includes all four displacement parameters

$$\begin{aligned} &- N_x[e_x\sin\theta + e_2(1 + \mu)\sin 2\theta] + N_y[e_y\cos\theta + e_2(1 + \mu)\sin 2\theta] \\ &+ N_{xy}[e_x\cos\theta - e_y\sin\theta + 2e_2\,(1 + \mu)\cos 2\theta] \\ &= \sum_i F_i[(e_x\cos\alpha_i + e_y\sin\alpha_i)\sin\beta_i + e_2(1 + \mu)\sin 2\beta_i] \\ &+ \sum_r F_r^Y\,[(e_x\cos\alpha_r + e_y\sin\alpha_r)\sin\beta_r + e_2(1 + \mu)\sin 2\beta_r] \end{aligned} \quad \text{(B4)}$$

Eq. (B1b), (B2), and (B3) are now combined to give alternate forms of the equilibrium equations corresponding to different membrane-free bodies as discussed in the text. First, Eq. (B1b) is multiplied by $\mu\cos\theta$ and Eq. (B2) is multiplied by $\mu\sin\theta$. Their sum is then added to Eq. (B3) to result in Alternate Form 1

$$N_x\sin^2\theta + N_y\cos^2\theta - N_{xy}\sin 2\theta + F_c = \sum_i F_i\sin^2\beta_i + \sum_r F_r^y\sin^2\beta_r \quad \text{(B5a)}$$

Second, Eq. (B1b) is multiplied by $\cos\theta$ and Eq. (B2) by $\sin\theta$. Their sum is then added to Eq. (B5a) to give Alternate Form 2

$$N_x + N_y + F_c = \sum_i F_i + \sum_r F_r^y \quad \text{(B5b)}$$

Third, the expression for N_x from Eq. (B1b) and the expression for N_y from Eq. (B2) are inserted into Eq. (B5a) to give Alternate Form 3

$$F_c\sin 2\theta = 2N_{xy} - \sum_i F_i\sin 2\alpha_i - \sum_r F_r^y\sin 2\alpha_r \quad \text{(B5c)}$$

Equations in terms of strains

The equilibrium equations, obtained by using Eq. (9), are expressed in the following in terms of the strains. The equations include the overall strain ϵ_i, which is evaluated by Eq. (5a)

$$\epsilon_i = (e_x\cos\alpha_i + e_y\sin\alpha_i)\cos\beta_i + \mu e_2\cos^2\beta_i - e_2\sin^2\beta_i, \qquad \beta_i = \alpha_i - \theta \quad \text{(5a)}$$

From $\partial U_T/\partial e_x = 0$, the following is obtained

$$N_x\cos\theta + N_{xy}\sin\theta = \sum_i \rho_i h E_s \epsilon_i \cos\beta_i\cos\alpha_i + \sum_r \rho_r h f_r^y \cos\beta_r\cos\alpha_r \quad \text{(B6)}$$

From $\partial U_T/\partial e_y = 0$, the following is obtained

$$N_y\sin\theta + N_{xy}\cos\theta = \sum_i \rho_i h E_s \epsilon_i \cos\beta_i\sin\alpha_i + \sum_r \rho_r h f_r^y \cos\beta_r\sin\alpha_r \quad \text{(B7)}$$

From $\partial U_T/\partial e_2 = 0$, the following is obtained

$$N_x(\mu\cos^2\theta - \sin^2\theta) + N_y(\mu\sin^2\theta - \cos^2\theta) + N_{xy}(1 + \mu)\sin 2\theta - E_c h e_2 = \sum_i \rho_i h E_s \epsilon_i (\mu\cos^2\beta_i - \sin^2\beta_i) + \sum_r \rho_r h f_r^y (\mu\cos^2\beta_r - \sin^2\beta_r) \quad \text{(B8)}$$

From $\partial U_T/\partial\theta = 0$, the following is obtained

$$-N_x[e_x\sin\theta + e_2(1 + \mu)\sin 2\theta] + N_y[e_y\cos\theta + e_2(1 + \mu)\sin 2\theta] + N_{xy}[e_x\cos\theta - e_y\sin\theta + 2e_2(1 + \mu)\cos 2\theta] = \sum_i \rho_i h E_s \epsilon_i [(e_x\cos\alpha_i + e_y\sin\alpha_i)\sin\beta_i + e_2(1 + \mu)\sin 2\beta_i] + \sum_r \rho_r h f_r^y [(e_x\cos\alpha_r + e_y\sin\alpha_r)\sin\beta_r + e_2(1 + \mu)\sin 2\beta_r] \quad \text{(B9)}$$