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Flexural Stiffness of Rectangular Reinforced Concrete Columns

by S. A. Mirza

The ACI Building Code permits a moment magnifier approach for design of slender reinforced concrete (RC) columns. This approach is strongly influenced by the effective flexural stiffness EI *of the column which varies due to cracking, creep, and nonlinearity of the concrete stress-strain curve, among other factors. However, the* EI *expressions given in the ACI Building Code are quite approximate when compared with the values of* EI *derived from thrust-moment-curvature relationships. This study was undertaken to determine the influence of a full range of variables on* EI *of slender tied rectanguler RC columns bent in symmetrical single curvature under short-time loads. Approximately 9500 columns, each with a different combination of variables, were used to generate the stiffness data. The* EI *expressions were then statistically developed for use in slender column designs. Two sets of equations are proposed in this paper: (a) Eq. (21) through (23) for initial sizing and preliminary design of structures; and (b) Eq. (18) through (20) for use in final (more accurate) structural designs.*

Keywords: building codes; columns (supports); flexural strength; loads (forces); moments; **reinforced concrete; slenderness ratio;** statistical analysis; **stiffness;** structural design; **tied columns.**

The ACI Building Code[1] permits a moment magnifier approach for design of slender reinforced concrete columns. This approach uses the axial load obtained from a first-order elastic analysis and a magnified moment that includes the second-order effect caused by the lateral displacement of the column. The ACI approach is strongly influenced by the effective flexural stiffness *EI* of the column which varies due to cracking, creep, and nonlinearity of the concrete stress-strain curve, among other factors. The *EI* expressions given in the ACI Building Code [ACI 318-89 Eq. (10-10) and (10-11)] were based on the recommendations of the Design Subcommittee of ACI-ASCE Committee 441.[2] However, these expressions are quite approximate when compared with the values derived from load-moment-curvature relationships as indicated in the Commentary on Building Code Requirements for Reinforced Concrete.[1,3] A statistical analysis of the ratios of theoretical *EI* to ACI *EI* for all slender columns studied as part of this investigation confirmed that the variations in the ACI *EI* expressions are high.

The understanding of slender column behavior has been greatly enhanced during the past 15 to 20 years, and analytical procedures have become available to accurately model the reinforced concrete slender column strength and stiffness. However, these procedures are generally too complex to be efficiently used in normal calculations in design offices. As a result, several computer studies were conducted to develop *EI* design equations for slender columns.[4-6] However, these studies did not consider the full range of variables that affect the flexural stiffness of slender concrete columns.

The study reported herein was undertaken to determine the influence of a full range of variables on effective flexural stiffness of slender tied reinforced concrete columns. Approximately 9500 rectangular columns, each with a different combination of specified values of variables, were used to generate the stiffness data. The *EI* expressions were then statistically developed for use in slender column designs. The columns studied bent in symmetrical single curvature, in braced frames subjected to short-time loads. The moment magnifier approach specified in the ACI Building Code was developed for this type of column. The effects of different end restraints, loading conditions, and lateral supports are accounted for in the ACI Building Code through the use of effective length factor K, equivalent uniform moment diagram factor C_m, and sustained load factor β_d.

The columns studied are graphically represented in Fig. 1 and are similar to those investigated earlier by MacGregor, Breen, and Pfrang.[2] These columns were chosen because the errors in K, C_m, and β_d factors would not affect the accuracy of the *EI* expressions derived in the later part of this paper.

RESEARCH SIGNIFICANCE

Based on evaluations of the parameters that affect the flexural stiffness of slender reinforced concrete columns, *EI* design equations are proposed for initial sizing and for final (more accurate) designs. It is shown

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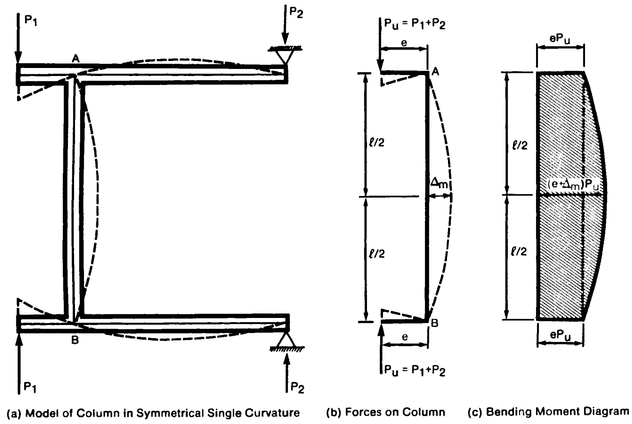

Fig. 1—Type of columns studied

that these equations are less likely to produce underdesigns than the current ACI expressions used for computing *EI* of slender rectangular columns.

METHOD USED FOR EVALUATING THEORETICAL FLEXURAL STIFFNESS

The equation for the effective flexural stiffness of slender reinforced concrete columns subjected to short-time loads is specified in the ACI Building Code[1]

$$EI = 0.2E_cI_g + E_sI_{se} \tag{1}$$

in which E_c, E_s = moduli of elasticity of concrete and reinforcing steel; and I_g, I_{se} = moments of inertia of gross concrete cross section and steel reinforcement taken about the centroidal axis of the column cross section. The inaccuracies in Eq. (1) result from the use of a constant value of the coefficient 0.2 used for computing the column *EI* regardless of different parameters that affect the stiffness. Hence, a modified version of this expression will take the form

$$EI = \alpha E_cI_g + E_sI_{se} \tag{2}$$

in which α is a dimensionless reduction factor (effective stiffness factor) which depends on a number of variables that affect the stiffness of slender columns. The value of α can be computed by rearranging Eq. (2)

$$\alpha = (EI - E_sI_{se})/E_cI_g \tag{3}$$

In Eq. (3), E_cI_g and E_sI_{se} are the stiffnesses of gross concrete cross section and steel reinforcement calculated in accordance with the ACI Building Code.[1] The

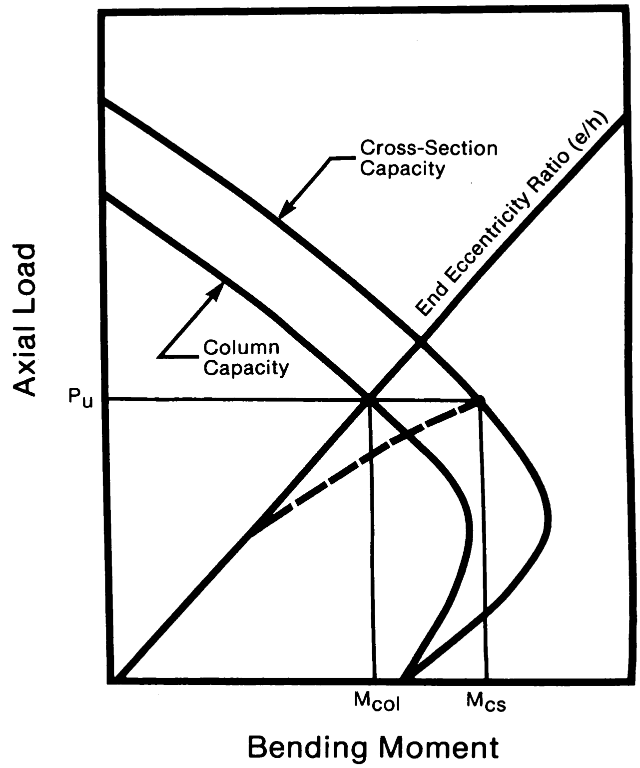

Fig. 2—Schematic cross section and column axial load-bending moment interaction diagrams

effective flexural stiffness *EI* used in Eq. (3) was computed using the procedure outlined in the following sections.

Development of theoretical flexural stiffness equation *EI*

The bending moment relationship (secant formula) for a pin-ended slender column subjected to equal and opposite end moments is given by Timoshenko and Gere[7]

$$M_c = M_2 \sec\left(\frac{\pi}{2}\sqrt{P_u/P_c}\right) \tag{4}$$

in which M_c = design bending moment which includes second-order effects; M_2 = applied end moment calculated by a conventional elastic frame analysis; P_u = factored axial load acting on the column; and P_c = Euler's buckling strength. For the purpose of anlaysis, M_c and M_2 are respectively replaced by the cross-sectional bending moment capacity M_{cs} and the overall column bending moment capacity M_{col}, so that Eq. (4) becomes

$$M_{cs} = M_{col} \sec\left(\frac{\pi}{2}\sqrt{P_u/P_c}\right) \tag{5}$$

M_{cs} and M_{col} are defined in Fig. 2. Rearranging Eq. (5), solving for P_c, and simplifying yields

$$P_c = \frac{\pi^2 P_u}{4\,[\sec^{-1}(M_{cs}/M_{col})]^2} \tag{6}$$

Euler's buckling strength of a pin-ended compression member is also given by

$$P_c = \pi^2 EI/\ell^2 \tag{7}$$

in which EI = effective flexural stiffness; and ℓ = unsupported height of the column. Equating Eq. (6) and (7) and solving for EI gives the following expression

$$EI = \frac{P_u \ell^2}{4\,[\sec^{-1}(M_{cs}/M_{col})]^2} \tag{8}$$

Eq. (8) is the theoretical effective flexural stiffness of a pin-ended slender column subjected to single curvature bending with equal moments acting at both ends. The computations of the terms M_{cs} and M_{col} used in this expression were based on interaction diagrams as explained in the following sections.

Computation of cross section bending moment capacity M_{cs}

The strength of a reinforced concrete cross section was represented by an axial load-bending moment interaction curve, similar to the one shown in Fig. 2. A number of moment-curvature diagrams were generated for various levels of axial load and the maximum moment from the moment-curvature diagram for each axial load level defined one point on the cross-sectional capacity interaction curve. Sufficient points were generated to accurately define the entire interaction diagram. M_{cs} could then be calculated easily for a desired end eccentricity ratio e/h.

The theoretical analysis used for the cross-sectional capacity involved Hognestad's[8] stress-strain relationship for concrete in compression with maximum strength = 0.85 f'_c, where f'_c was the specified strength of concrete. A linear brittle stress-strain relationship was used for concrete in tension. The moduli of elasticity and rupture of concrete were taken as the functions of the maximum compressive strength. An elastic-plastic stress-strain curve was assumed for reinforcing steel. The specified values of reinforcing steel yield strength and modulus of elasticity were used for computing the cross-sectional capacities.

The theoretical model for strength of tied column cross sections was compared to results of 54 short column tests in which length effects were negligible. These tests were taken from Hognestad.[8] The test strength-to-theoretical strength ratio ranged from 0.85 to 1.18 with a mean value of 1.01 and a coefficient of variation that equaled 6.4 percent. These values indicate a reasonable accuracy of the theoretical model for the cross-sectional strength.

Computation of slender column bending moment capacity M_{col}

For a column bending in single curvature under equal eccentricities at both ends, a second-order parabola has been suggested to represent the shape function of the curvature line between the midheight and the ends of the slender column.[9] The lateral deflection at midheight of the column can then be computed from

$$\Delta_m = \ell^2 (\phi_m + 0.25\,\phi_e)/10 \tag{9}$$

in which ϕ_e = curvature at the column ends; ϕ_m = curvature at midheight of the column; and ℓ = height of the column.

For a given axial load and midheight curvature ϕ_m, the end curvature ϕ_e was obtained from Eq. (9) through a trial-and-error solution. Once ϕ_e was determined, the externally applied end moment was calculated using an extended Newton-Raphson technique.[10] The externally applied end moment was plotted against the curvature at the midheight. The maximum moment from this diagram and the corresponding axial load defined one point on the axial load-end moment M_{col} interaction curve for the slender column. A series of these points for different axial load levels defined the entire interaction curve that included the effect of slenderness in the column strength (Fig. 2). M_{col} could then be calculated easily for a desired end eccentricity ratio e/h.

To check the accuracy of the theoretical strength model for slender columns, the bias and variability were computed from the test data available in the literature. The ratios of test to theoretically calculated strengths for 20 slender column tests from Chang and Ferguson[11] and Mehmel et al.[12] ranged from 0.89 to 1.23 with the mean value of 1.03 and the coefficient of variation of 8.8 percent. These values indicate an acceptable level of accuracy for the slender column strength model.

SIMULATION OF THEORETICAL STIFFNESS DATA FOR COLUMNS STUDIED

Since the dimensional tolerances in reinforced concrete cross sections are independent of the size, the deviations in actual strength of a slender column tend to become more significant as the cross section size decreases. This makes the columns with smaller cross sections more critical.[13] A 12 x 12 in. (305 x 305 mm) cross section was chosen for study because this would represent about the smallest size of column cross section usually employed in building construction.[14]

Approximately 9500 columns were used, with each column having a different combination of the specified properties of variables. The specified concrete strengths f'_c, reinforcing steel yield strengths f_y, and longitudinal steel clear concrete covers C_c used in this study and listed in Table 1 represent the usual ranges of these variables employed by the construction industry.[14] The slenderness ratios ℓ/h selected (Table 1) were intended to approximate the range of ℓ/h for columns in braced frames designed according to ACI 318-89 Clause 10-11.[1] Eleven end eccentricity ratios e/h ranging from 0.05 to 1.0 were used, as indicated in Table 1. Note the usual e/h for columns in concrete buildings varies from 0.1 to 0.65.[15] Finally, the longitudinal reinforcement ratios ρ_g and steel arrangements for the column cross sections studied are shown in Fig. 3. The steel ratios used cover the range of ρ_g commonly employed for concrete buildings.[14]

Table 1 — Specified properties of columns studied*

Properties	Specified values	Number of specified values
f'_c, psi	3000; 4000; 5000; 6000	4
f_y, psi	40,000; 60,000	2
C_c, in.	1.5; 1.875; 2.5	3
ℓ/h ratio	10; 20; 30	3
e/h ratio	0.05; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 1.0	11
ρ_g and steel arrangement	See Fig. 3 for combinations of steel ratios and arrangements	12

*Total number of columns equals (4 x 2 x 3 x 3 x 11 x 12 =) 9504 with each column having a different combination of specified properties shown above. All columns had a cross section size of 12 x 12 in. with lateral ties conforming to ACI 318-89 Clause 7.10.5.
Note: 10 in. = 254 mm; 1000 psi = 6.895 MPa.

Steel Ratio ρ_g (Percent)						
No. of Bars / Bar Size	4	6	6	8	8	8
No. 5	0.86	1.29	1.29			
No. 6				2.44	2.44	2.44
No. 7				3.33	3.33	3.33
No. 8				4.39	4.39	4.39
Arrangement of Longitudinal Reinforcement						

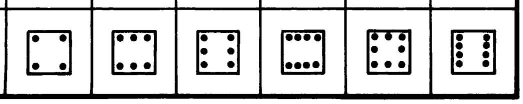

Fig. 3—Longitudinal reinforcement details of column cross sections (bar diameter: No. 5 = 16 mm; No. 6 = 19 mm; No. 7 = 22 mm; and No. 8 = 25 mm)

The short-time theoretical EI for each of the columns studied was computed from Eq. (8) using the interaction diagrams for the cross section and slender column capacities described earlier. The effective stiffness factor α was then computed for each column from Eq. (3) using the theoretical EI. Finally, the simulated column stiffness data were statistically analyzed for examining the current ACI column stiffness equations and for developing the design equations for EI proposed in the later part of this paper.

EXAMINATION OF ACI STIFFNESS EQUATIONS

The ACI Building Code[1] permits the use of the following design equations for calculating the stiffness of a slender reinforced concrete column

$$EI = (0.2E_cI_g + E_sI_{se})/(1 + \beta_d) \quad \text{[ACI 318-89 Eq. (10-10)]} \tag{10}$$

$$EI = 0.4E_cI_g/(1 + \beta_d) \quad \text{[ACI 318-89 Eq. (10-11)]} \tag{11}$$

in which β_d = ratio of maximum factored dead (or sustained) load to maximum total factored load and is always taken positive. For short-time loads, β_d equals zero and Eq. (10) and (11) are simplifed to Eq. (12) and (13), respectively

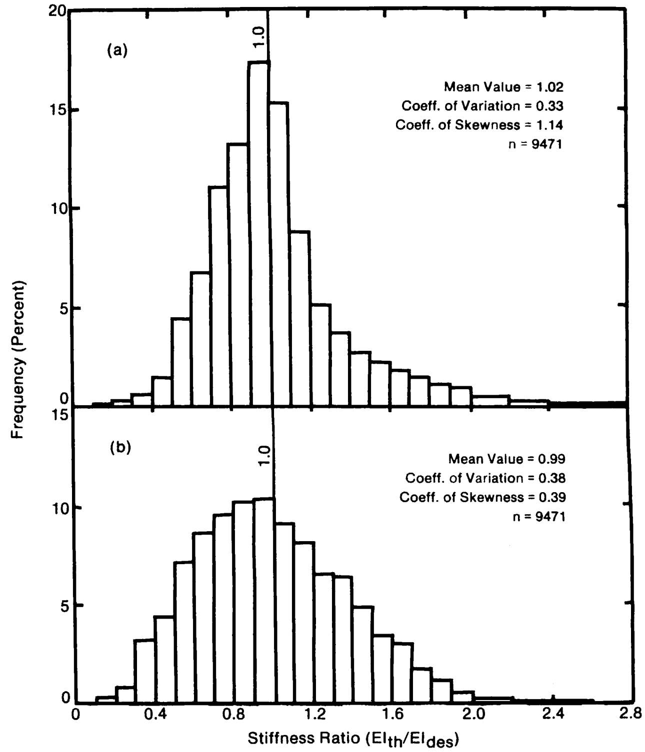

Fig. 4—Comparison of ACI stiffness equations with theoretical results: (a) Eq. (12) [ACI 318-89 Eq. (10-10)]; and (b) Eq. (13) [ACI 318-89 Eq. (10-11)]

$$EI = 0.2E_cI_g + E_sI_{se} \quad \text{[ACI 318-89 Eq. (10-10)]} \tag{12}$$

$$EI = 0.4E_cI_g \quad \text{[ACI 318-89 Eq. (10-11)]} \tag{13}$$

Eq. (12) and (13) were compared with the theoretical EI data generated for all the columns studied. The results of these comparisons are plotted in Fig. 4, which shows the histograms of the ratios of theoretical EI to ACI design EI (EI_{th}/EI_{des}). Note EI_{des} for Fig. 4(a) was computed from Eq. (12) and that for Fig. 4(b) from Eq. (13).

Fig. 4 indicates that although the mean stiffness ratios obtained from both ACI equations tend to be close to unity, the coefficients of variation V_R associated with these equations are quite high [V_R = 33 and 38 percent for Eq. (12) and (13), respectively]. This means the ACI equations on the average predicted EI values close to the theoretical values of EI. However, for a significant number of columns studied, ACI EI substantially deviated from the corresponding theoretically computed EI. This is because the ACI design equations do not include all the parameters that affect the stiffness of slender columns. It is evident from Fig. 4 that there is a need for modification in ACI EI design equations for the type of columns studied.

DEVELOPMENT OF PROPOSED DESIGN EQUATIONS FOR SHORT-TIME *EI*

The effective flexural stiffness of a slender column is strongly affected by cracking along its length and inelastic actions in the concrete and reinforcing steel. EI is, therefore, a complex function of a number of variables and does not lend itself to the derivation of a

Table 2 — Variable combinations used for regression analyses

Variable combination number	Variables: End eccentricity ratio e/h	Axial load index $1-\left(\frac{P_u}{P_o}\right)$	Axial load index $1-\left(\frac{P_u}{P_o}\right)^2$	Slenderness ratio ℓ/h	Stiffness ratio $\frac{E_s I_{se}}{E_c I_g}$	Stiffness ratio E_s/E_c	Stiffness ratio I_{se}/I_g	Steel index $\frac{\rho_g f_y}{f'_c}$	Steel index ρ_g	Steel index ρ_e/ρ_g	Concrete cover index γ	Standard error r_e	Multiple correlation coefficient r_c
(a) Series P													
P1	x	x		x	x			x			x	0.057	0.86
P2	x	x		x	x					x		0.057	0.86
P3	x	x		x	x				x			0.058	0.86
P4	x	x		x		x		x				0.058	0.86
P5	x	x		x			x	x				0.058	0.86
P6	x	x		x	x			x				0.058	0.86
P7	x		x	x	x			x				0.061	0.85
P8	x		x	x	x							0.061	0.85
P9	x	x		x	x							0.058	0.86
P10	x	x		x								0.058	0.86
(b) Series F													
F11	x			x								0.061	0.84
F12		x		x								0.067	0.81
F13	x											0.067	0.81
F14		x										0.088	0.64
F15				x								0.111	0.23

Note: r_e and r_c were computed for the effective stiffness factor α.

unique and simple analytical equation. In this study, multiple linear regression analysis of the simulated theoretical stiffness data was conducted to evaluate *EI* expressions. The linear regression was chosen as a method of analysis since the objective was to develop simple equations for *EI*.

Variables used for regression analysis

The variables used in this study can be divided into the following six groups: 1) end eccentricity ratio e/h; 2) axial load index $[1 - (P_u/P_o)]$, or $[1 - (P_u/P_o)^2]$, in which P_u = factored axial load acting on the slender column and P_o = pure axial load capacity of the cross section; 3) slenderness ratio ℓ/h; 4) stiffness ratio E_sI_{se}/E_cI_g, or E_s/E_c, or I_{se}/I_g; 5) steel index $\rho_g f_y/f'_c$, or ρ_g, or ρ_e/ρ_g, in which ρ_e = ratio of the area of exterior layers of longitudinal reinforcement to gross area of the cross section; and 6) concrete cover index γ which is defined as the center-to-center distance between exterior layers of longitudinal reinforcement divided by the overall depth of the cross section.

The first and third groups of variables were considered important because a recent study established the effect of these variables on strength and behavior of slender columns.[13] Wood and Shaw[6] suggested the use of $1 - (P_u/P_o)^2$ as a variable for computing stiffness of slender columns. The other variable $[1 - (P_u/P_o)]$ in the second group was taken as a simplification of the variable suggested by Wood and Shaw. The fourth group of variables was intended to investigate the effect of relative stiffnesses of reinforcing steel and concrete in the cross section, while the fifth one took into consideration the influence of steel reinforcement. MacGregor, Oelhafen, and Hage[5] have suggested the use of ρ_g as a variable for *EI*. Finally, γ was included to study the effect of concrete cover on column stiffness (sixth group).

The variables within each group were considered dependent variables. Hence, a maximum of one variable from a chosen group was used for a particular regression analysis of the theoretical data. The combinations of variables employed for different regression analyses are given in Table 2.

Regression analysis of theoretical stiffness data

A multiple linear regression analysis of the simulated theoretical stiffness data (α values) was conducted and the resulting *EI* expression was developed for each combination of the variables listed in Table 2. The format used for regression equations was the same as that shown for Eq. (2). The prediction accuracy of a regression *EI* equation was based on the standard error r_e, a measure of sampling variability, and the multiple correlation coefficient r_c, which is an index of the relative strength of the relationship. A smaller value of r_e is associated with a smaller sampling variability of the regression equation, and vice versa. An r_c value of zero indicates no correlation, whereas $r_c = \pm 1.0$ represents 100 percent correlation. The r_c values smaller than -1.0 and greater than $+1.0$ are not possible. The computed values of r_e and r_c for each regression equation are given in Table 2.

Table 2(a) shows practically no change in r_e and r_c values computed for 10 different regression equations (Series P). This indicates that the variables other than those used for Combination P10 [e/h, $(1 - P_u/P_o)$, and

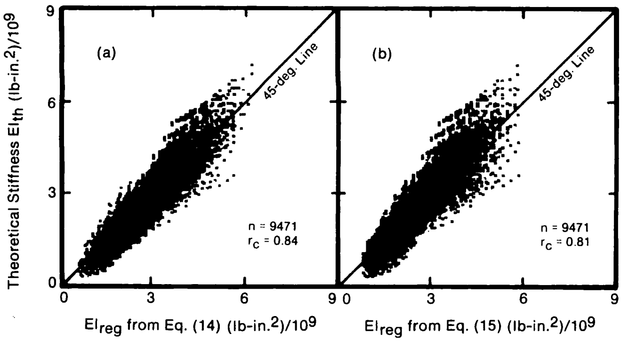

Fig. 5—Comparison of selected regression equations with theoretical data

ℓ/h] in Table 2(a) do not significantly influence the EI of slender columns. This is partly due to the fact that some of these variables were included explicitly or implicitly in the format of the regression EI equations [Eq. (2)]. Hence, further investigations involved the following variables: e/h, $(1 - P_u/P_o)$, and ℓ/h.

A correlation analysis of the variables used in Combination P10 [Table 2(a)] indicated: a) no correlation between e/h and ℓ/h ratios; b) some correlation between $(1 - P_u/P_o)$ and ℓ/h ratios; and c) strong correlation between $(1 - P_u/P_o)$ and e/h ratios. This means that e/h and ℓ/h are independent variables, whereas $(1 - P_u/P_o)$ is dependent on e/h. This seems reasonable because the axial load ratio P_u/P_o of a column depends on the load eccentricity. Hence, $(1 - P_u/P_o)$ was not grouped with e/h for further regression analyses, as indicated by variable combinations shown in Table 2(b) for Series F.

Table 2(b) shows that the variable combinations F11 and F13 produced the lowest r_e and the highest r_c values among the regression equations involving two variables and one variable, respectively. The corresponding regression expressions are

$$EI = (0.294 + 0.00323\,\ell/h - 0.299e/h)\,E_cI_g + E_sI_{se} \quad (14)$$

$$EI = (0.358 - 0.299\,e/h)\,E_cI_g + E_sI_{se} \quad (15)$$

Eq. (14) and (15) show that an increase in e/h ratio decreases EI of a column. This is expected because a larger e/h value is associated with more cracking of the column. Eq. (14) also indicates an increase in EI value for an increased ℓ/h ratio. This is because the cracks in a longer column are likely to be more widely spaced with more concrete in between the cracks contributing to the effective flexural stiffness of the column.

The EI values computed from Eq. (14) and (15) for all columns studied are plotted against the corresponding theoretical stiffnesses in Fig. 5. The lines of equality in the figure are labeled as 45-deg line. Note EI_{reg} in Fig. 5(a) was taken from Eq. (14), whereas that in Fig. 5(b) from Eq. (15). As expected, both equations produced reasonable correlation with the theoretical EI values, although Eq. (14) produced somewhat better results.

A statistical analysis of the ratio of theoretical EI to EI from Eq. (14) for all columns studied ($n = 9471$) produced the following results: mean value = 1.00, coefficient of variation = 0.16, and coefficient of skewness = 0.46. The respective values for Eq. (15) were 1.00, 0.18, and 0.44. Again, Eq. (15) produced a slightly higher variability than did Eq. (14). Even so, the coefficient of variation of Eq. (15) was about 50 percent of that associated with the current ACI Building Code[1] equations (Fig. 4).

Proposed design equations

For design purposes, Eq. (14) and (15) were simplified to Eq. (16) and (17)

$$EI = [(0.27 + 0.003\ell/h - 0.3e/h)\,E_cI_g + E_sI_{se}] \geqslant E_sI_{se} \quad (16)$$

$$EI = [(0.3 - 0.3e/h)\,E_cI_g + E_sI_{se}] \geqslant E_sI_{se} \quad (17)$$

At $\ell/h = 10$, both equations will give identical results. Eq. (17) is more conservative than Eq. (16) for $\ell/h > 10$, whereas Eq. (17) is less conservative than Eq. (16) for $\ell/h < 10$. The lower limit placed on Eq. (16) and (17) is to insure that the effective flexural stiffness of the column is at least equal to E_sI_{se}. This limit will control the design only for very large end eccentricities ($e/h > 1.0$).

ANALYSIS AND DISCUSSION OF RESULTS

Overview of stiffness ratio statistics

The coefficient of variation, mean, five-percentile, and one-percentile values of the stiffness ratios EI_{th}/EI_{des} were computed for different design equations. For computing the stiffness ratio of a column, EI_{th} was taken as the simulated theoretical stiffness while EI_{des} was computed from Eq. (12), (13), (16), or (17). Eq. (12) and (13) are the ACI design equations and Eq. (16) and (17) the proposed design expressions.

For the purpose of computing statistics, the data were divided into the following four groups: Group A included all columns studied; Group B considered all columns with low e/h ($e/h \leqslant 0.4$); Group C included all columns with high e/h ($e/h \geqslant 0.5$); and Group D took into consideration only the columns with usual e/h and ρ_g values ($0.1 \leqslant e/h \leqslant 0.7$ and $1.29 \leqslant \rho_g \leqslant 4.39$ percent). The results from the statistics computed for these groups can be summarized as follows:

1. The coefficients of variation for the proposed design equations were much lower than those for the ACI design equations. This was particularly valid for Group D columns with $e/h = 0.1$ to 0.7 and $\rho_g = 1.29$ to 4.39 percent.

2. The mean stiffness ratios for the ACI design equations were considerably smaller than 1.0 (0.73 to 0.87) for columns in Group C with $e/h = 0.5$ to 1.0. This means the ACI expressions on the average tend to overestimate the stiffness of columns with high e/h. Note the ACI equations were developed from data for columns with $e/h \leqslant 0.4$.[5]

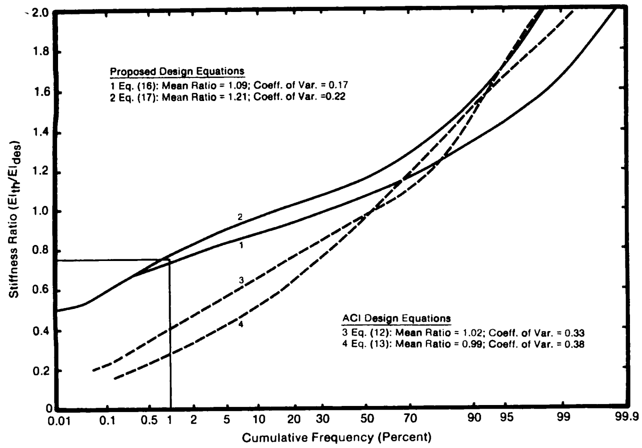

*Fig. 6—Probability distributions of stiffness ratios computed from data for all columns (*n *= 9471)*

3. The five-percentile and one-percentile stiffness ratios for the proposed design equations were subjected to much smaller variations than those for the ACI design expressions. This is expected because of the lower variability associated with the proposed design equations. In most cases, the proposed design equations gave five-percentile stiffness ratios that exceeded 0.8 and one-percentile stiffness ratios greater than or equal to 0.7. The ACI design expressions, on the other hand, produced far smaller corresponding values. This can be seen more clearly by comparing the cumulative frequency curves of stiffness ratios for different equations plotted on a normal probability net in Fig. 6. The curves in Fig. 6 were prepared from data or all columns studied and demonstrate the overall performance of the design expressions.

Effects of major variables

The effect of e/h on mean, five-percentile, and one-percentile values of stiffness ratios (EI_{th}/EI_{des}) obtained for ACI and proposed design equations [Eq. (12), (13), (16), and (17)] is shown in Fig. 7. The figure was plotted by using data for all the columns studied. Fig. 7 indicates the proposed equations [Eq. (16) and (17)] produced five-percentile and one-percentile stiffness ratios that are almost constant over the entire range of e/h studied. However, the mean stiffness ratios for the proposed equations tend to deviate slightly for $e/h \leqslant 0.8$ and significantly for $e/h > 0.8$. It should be pointed out that five-percentile and one-percentile values are more important than the mean value as indicators of safety in design expressions. The proposed design equations gave mean, five-percentile, and one-percentile stiffness ratios that respectively exceeded 1.0, 0.8, and 0.7 for most e/h ratios shown in Fig. 7. The ACI design equations [Eq. (12) and (13)], on the other hand, produced stiffness ratios that varied with e/h. This is expected because the ACI expressions do not use e/h as a variable. The ACI equations seem to be conservative for low e/h and unconservative for high e/h values, as indicated by Fig. 7.

The effect of ℓ/h ratio on mean, five-percentile, and one-percentile stiffness ratios obtained for different design equations is shown in Fig. 8 which includes data from all the columns used in this study. The proposed design equations produced mean stiffness ratios that were significantly greater than 1.0 for all values of ℓ/h ratio, whereas the mean stiffness ratios obtained for ACI design equations ranged from 0.9 for $\ell/h = 10$ to about 1.1 for $\ell/h = 30$.

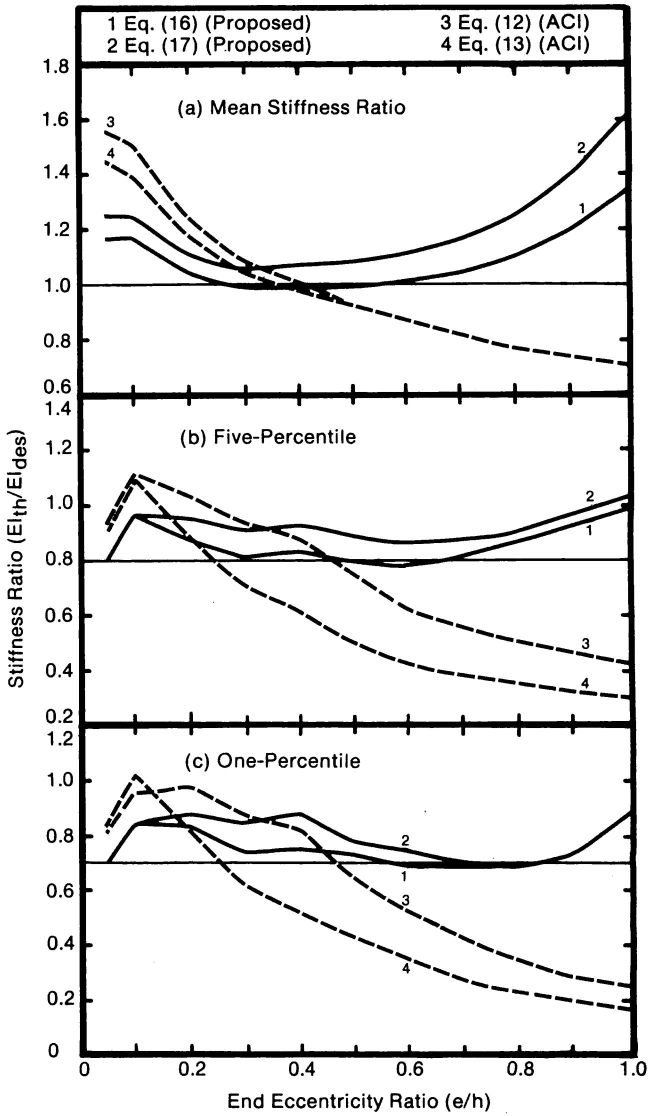

*Fig. 7—Effect of end eccentricity on stiffness ratio for different design equations (*n *= 861 for each of* e/h *ratio equal to 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0)*

The five-percentile and one-percentile stiffness ratios for the proposed EI equations were greater than 0.8 and 0.7, respectively, for almost all values of ℓ/h plotted in Fig. 8. However, much lower values of five-percentile and one-percentile stiffness ratios were obtained for ACI design equations, as indicated by Fig. 8. Note that ℓ/h is included as a variable only in Eq. (16); the remaining three design equations in Fig. 8 do not use ℓ/h as a variable.

Fig. 9 is plotted to illustrate the effect of ρ_g on mean, five-percentile, and one-percentile stiffness ratios obtained from the proposed and ACI EI design equations. Again, Fig. 9 included data from all the columns

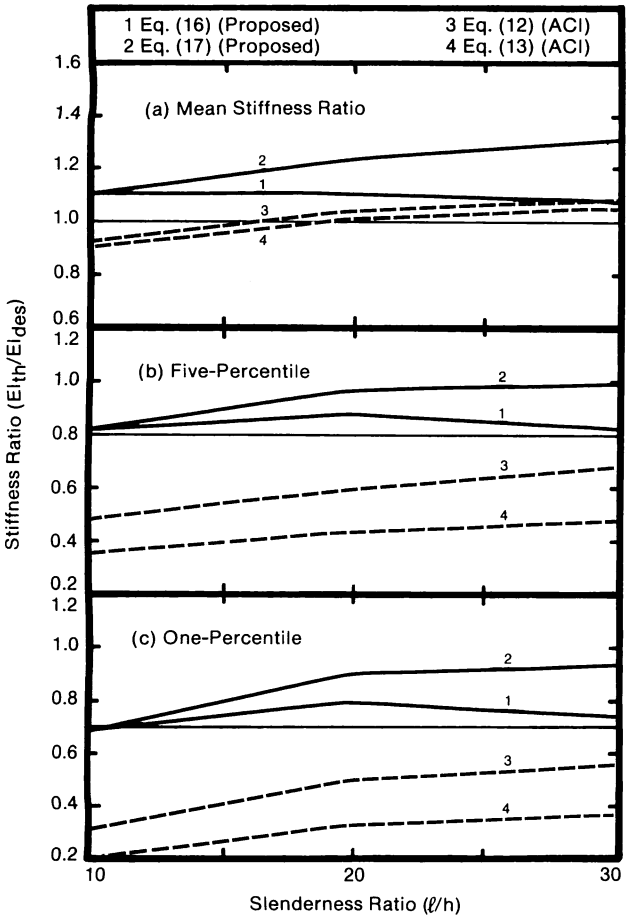

*Fig. 8—Effect of column slenderness on stiffness ratio for different design equations (*n = *3146 for* ℓ/h = *10;* n = *3157 for* ℓ/h = *20; and* n = *3168 for* ℓ/h = *30)*

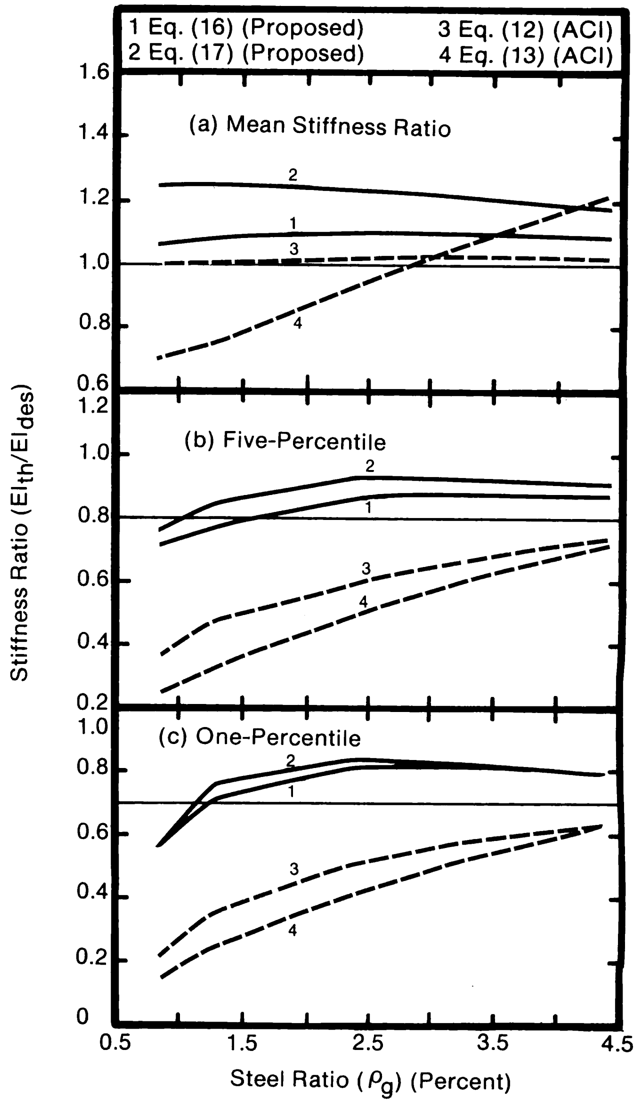

*Fig. 9—Effect of longitudinal steel reinforcement on stiffness ratio for different design equations (*n = *770 for* ρ_g = *0.86 percent;* n = *1573 for* ρ_g = *1.29 percent;* n = *2376 for each of* ρ_g *ratio* = *2.44, 3.33, and 4.39 percent)*

studied. In most cases, the mean, five-percentile, and one-percentile values obtained from the proposed design expressions significantly exceeded 1.0, 0.8, and 0.7, respectively. However, the values obtained from the ACI design equations were substantially smaller than the corresponding values from the proposed equations. This is particularly valid for five-percentile and one-percentile stiffness ratios shown in Fig. 9.

The following conclusions can be summarized from the data plotted in Fig. 7 through 9 and the related discussions:

1. The proposed design equations [Eq. (16) and (17)] are not significantly affected by e/h, ℓ/h, and ρ_g ratios, whereas the ACI design expressions [Eq. (12) and (13)] demonstrate a pronounced effect of these variables.

2. The proposed design expressions predict stiffnesses closer to the theoretical values than do the ACI design equations. This is particularly valid for five-percentile and one-percentile values, indicating that the proposed equations are less likely to produce underdesign.

Stiffness ratios produced by proposed design equations for usual columns

The ACI Building Code[1] requires that the reinforcing steel placed in columns be equal to a minimum of 1 to a maximum of 8 percent of the gross area of cross section. The lap splicing of the reinforcing bars along the column height in multistory buildings decreases the maximum limit of ρ_g to 4 percent. The usual steel ratio for concrete columns is, therefore, expected to range from 1 to 4 percent. Similarly, the end eccentricity ratio for columns in reinforced concrete buildings usually ranges from 0.1 to 0.65.[15] Hence, the columns used in this study with e/h = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, or 0.7 and ρ_g = 1.29, 2.44, 3.33, or 4.39 percent can be considered as usual columns.

The mean, five-percentile, and minimum values of the stiffness ratios produced by Eq. (16) for usual columns are plotted against e/h in Fig. 10(a) for ℓ/h = 10; and in Fig. 10(b) for ℓ/h = 30. The one-percentile values were not plotted in these figures because the sample size for each point plotted ranged from 48 to 72 columns with minimum values representing 1.4 to 2.1 percentiles. Fig. 10(a) clearly indicates that, for almost all columns plotted (ℓ/h = 10), the mean, five-percentile, and minimum values exceeded 1.0, 0.8, and 0.7, respectively. For usual columns with ℓ/h = 30, however,

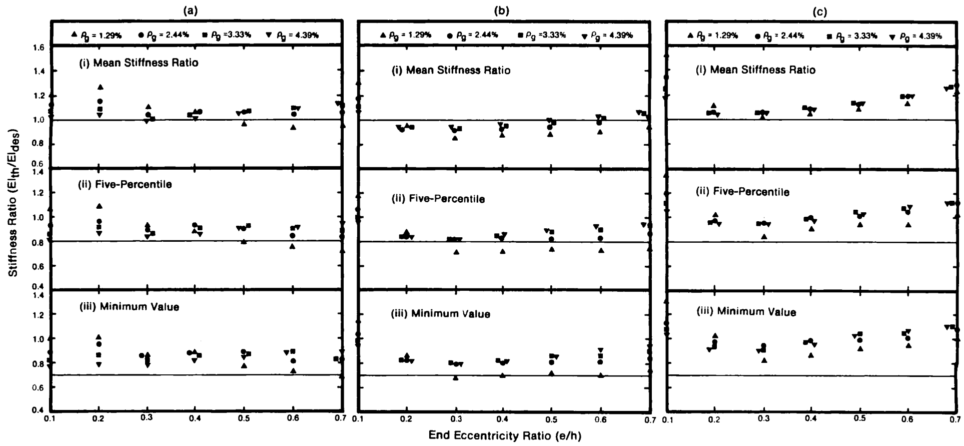

Fig. 10—Stiffness ratios obtained from proposed design equations: (a) using Eq. (16) or (17) for usual columns with ℓ/h = 10; (b) using Eq. (16) for usual columns with ℓ/h = 30; and (c) using Eq. (17) for usual columns with ℓ/h = 30 (for each combination of e/h and ρ_g ratios plotted, n = 48 when ρ_g = 1.29 percent; and n = 72 when ρ_g = 2.44, 3.33, or 4.39 percent)

the mean stiffness ratios fell somewhat lower than 1.0 for e/h = 0.2 to 0.6, but the most of the five-percentile and minimum values were again of the order of at least 0.8 and 0.7, respectively, as indicated by Fig. 10(b).

For ℓ/h = 10, Eq. (17) is identical to Eq. (16). Therefore, Fig. 10(a) also represents the stiffness ratios produced by Eq. (17) for usual columns with ℓ/h = 10. The stiffness ratios obtained from Eq. (17) for usual columns with ℓ/h = 30 are plotted in Fig. 10(c). As expected, Eq. (17) produced more conservative results than Eq. (16) for ℓ/h = 30. This can be seen by comparing Fig. 10(b) and (c).

Based on Fig. 10, the following conclusions seem to be valid for columns with e/h = 0.1 to 0.7, ℓ/h = 10 to 30, and ρ_g = 1.29 to 4.39 percent: a) the proposed design equations [Eq. (16) and (17)] are not subject to significant variations due to e/h, ℓ/h, and ρ_g ratios; and b) the mean, five-percentile, and minimum values of stiffness ratios for Eq. (16) or (17) can be taken as 1.0, 0.8, and 0.7, respectively.

Stability resistance factor for proposed design equations

The strength reduction factors ϕ specified in the ACI 318-89 Standard[1] were based on one-percentile strength ratios.[16] The one-percentile values obtained from Fig. 6 plotted for stiffness ratios of all columns studied are close to 0.75 [0.74 for Eq. (16) and 0.77 for Eq. (17)]. Similarly, Fig. 7 through 9 plotted to study the effects of e/h, ℓ/h, and ρ_g on the stiffness ratios obtained from Eq. (16) and (17) indicate that in almost all cases the one-percentile value exceeds 0.7. This value is the same as the ϕ factor used for computing the moment magnification of slender columns through the ACI *EI* design equations.[17]

Computing the ϕ factor associated with Eq. (16) and (17) from one-percentile stiffness ratios is not statistically justified, because ϕ depends on load statistics and safety criterion in addition to the probability distribution of *EI*. However, this computation provides a crude estimate of ϕ similar to the one that was used for computing the ϕ values specified in the current ACI Building Code. Although more complex calculations will be used in future evaluation of stability ϕ factor, the results from this simple analysis suggest that a ϕ factor of at least 0.7 can be used for computing ϕP_c based on the proposed *EI* expressions.

DESIGN APPLICATIONS

Columns in frames subjected to sustained loads

The effect of sustained loads on stiffness of reinforced concrete columns can be considered through the use of β_d factor. Applying β_d to Eq. (16) and (17) and rearranging produces the following expressions for columns subjected to sustained loads

$$EI = \frac{\alpha\, E_c I_g + E_s I_{se}}{(1 + \beta_d)} \tag{18}$$

in which

$$\alpha = [0.27 + 0.003\,(\ell/h) - 0.3\,(e/h)] \geqslant 0 \tag{19}$$

or alternatively

$$\alpha = [0.3 - 0.3\,(e/h)] \geqslant 0 \tag{20}$$

where β_d is taken greater than zero. For short-term loads (β_d = 0), Eq. (18) and (19) reduce to Eq. (16) and Eq. (18) and (20) to Eq. (17). For Eq. (19) and (20), ℓ is the unsupported height of the column, h is the overall depth of the cross section, and e is the larger end eccentricity = M_2/P_u, where M_2 = larger of the factored moments acting at the column ends. Eq. (19) and (20) are subject to the following limits: $f'_c \leqslant 6000$ psi (41.4 MPa), $\rho_g \geqslant 1$ percent, $\ell/h \leqslant 30$, and $e/h \geqslant 0.1$.

The effects of different end restraints, sway conditions, and moment gradients are accounted for in the ACI Building Code[1] through the use of K and C_m factors. The typical columns in nonsway frames would have K factors ranging from approximately 0.8 to almost 1.0 and C_m factors somewhere between 0.6 and 1.0. The columns investigated in this study were subjected to symmetrical single curvature bending in braced frames with $K = C_m = 1.0$ (Fig. 1) and represent the most critical condition for columns in frames without sidesway. Eq. (18) through (20) are, therefore, readily applicable to columns in nonsway frames with appropriate K and C_m factors taken from the ACI Building *Code*. For columns in frames subjected to considerable sidesway, C_m will be taken as 1.0 with K usually ranging from 1.2 to 2.0. Hence, Eq. (18) through (20) could also be used for such columns provided that K is not unreasonably high. It is suggested that for design of rectangular columns, the ACI EI equations [Eq. (10) and (11)] be replaced by Eq. (18) through (20). The values of K and C_m factors shall be taken as specified in the ACI Building Code.[1]

EI for use in preliminary designs

An approximate estimate of member sizes and steel reinforcement is normally required before the fine tuning of a structure can be achieved through final (more accurate) structural analysis and design. Hence, preliminary sizing of member cross sections is the first step in the design process. For initial designs, columns in reinforced concrete buildings can be divided into the following three categories:[15]

1. Columns in bottom stories supporting more than three floors plus the roof will typically be subjected to $e/h \leqslant 0.1$ and $\beta_d \simeq 0.7$. Substituting these values in Eq. (18) and (20) yields the approximate EI expression for columns in this category

$$EI = (0.27\,E_cI_g + E_sI_{se})/1.7 \qquad (21)$$

2. Columns in intermediate stories supporting one to three floors plus the roof will normally have e/h = 0.25 to 0.3 and $\beta_d \simeq 0.6$. Hence, the expression for preliminary design of these columns can be approximated by substituting e/h and β_d in Eq. (18) and (20)

$$EI = (0.21\,E_cI_g + E_sI_{se})/1.6 \qquad (22)$$

3. Columns in top stories supporting just the roof will usually be subjected to e/h = 0.65 to 0.7 and $\beta_d \simeq$ 0.5. Again, the equation for initial design of columns in this category can be obtained in a similar manner as for the other two categories

$$EI = (0.1\,E_cI_g + E_sI_{se})/1.5 \qquad (23)$$

For the first design run, the engineer would guess ρ_g and estimate EI from Eq. (21) through (23), making subsequent corrections in ρ_g if needed. The following suggestions are made for the first design run: (a) when the column cross section changes along the height of the building, assume $\rho_g \simeq$ 2 percent; and (b) when the column cross section remains constant over the building height, assume $\rho_g \simeq$ 3 to 4 percent for Eq. (21), 2 to 3 percent for Eq. (22), and 1 to 2 percent for Eq. (23). Eq. (21) through (23) could be used for initial sizing of members. For final (more accurate) designs, EI should be computed from Eq. (18) through (20).

CONCLUSIONS AND RECOMMENDATIONS

This paper presents a statistical evaluation of the parameters that affect the flexural stiffness of slender reinforced concrete columns subjected to short-time loads. Based on this evaluation, two sets of design equations are proposed: (a) Eq. (21) through (23) for initial sizing of columns; and (b) Eq. (18) through (20) for use in final (more accurate) designs. A value of ϕ factor equal to at least 0.7 is proposed for use when computing the critical buckling strength of a column based on the proposed EI equations.

The results presented in this paper indicate that the prediction variations of the ACI EI expressions are about twice as high as those of the proposed equations. Furthermore, the ACI 318-89 Eq. (10-11) is in most cases less conservative than the ACI 318-89 Eq. (10-10). This is contrary to what is stated in the ACI *Building Code*.

NOTATION

C_c	=	clear concrete cover to longitudinal reinforcing steel
C_m	=	equivalent uniform moment diagram factor
E_c, E_s	=	moduli of elasticity of concrete and steel
e	=	larger end eccentricity = M_2/P_u
EI	=	effective flexural stiffness of column
EI_{des}	=	computed short-time EI from design Eq. (12), (13), (16), or (17)
EI_{reg}	=	computed short-time EI from regression Eq. (14) or (15)
EI_{th}	=	simulated theoretical EI for short-time loads
f'_c, f_y	=	specified compressive strength of concrete and yield strength of steel
h	=	overall thickness of cross section
I_g, I_{se}	=	moment of inertia of gross concrete cross section and of steel reinforcement taken about centroidal axis of cross section
K	=	effective length factor
ℓ	=	unsupported height of column
M_{cs}, M_{col}	=	bending moment capacity of cross section and of member (column)
M_2	=	larger of factored moments applied at column ends
n	=	number of data points
P_c	=	Euler's buckling strength or critical load of column
P_o	=	pure axial load capacity of cross section
P_u	=	factored axial load acting on column
r_c, r_e	=	multiple correlation coefficient and standard error
α	=	effective stiffness factor = $(EI - E_sI_{se})/E_cI_g$
β_d	=	ratio of maximum factored axial dead load to maximum total factored axial load, where the load is due to gravity effects only in the calculation of P_c in ACI 318-89 Eq. (10-7); or the ratio of the maximum factored sustainted lateral load to the maximum total factored lateral load in that story in the calculation of P_c in ACI 318-89 Eq. (10-8)
γ	=	center-to-center distance between exterior layers of longitudinal reinforcement divided by h
Δ_m	=	lateral deflection at midheight of column
ρ_e, ρ_g	=	area of exterior layers of longitudinal reinforcement and total area of longitudinal reinforcement, both divided by gross area of cross section
ϕ	=	strength reduction factor used for computing ϕP_c
ϕ_e, ϕ_m	=	curvature at column ends and midheight

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