Column Research Council

Guide to Design Criteria for Metal Compression Members

Second Edition

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Foreword

The major objectives of Column Research Council, since its founding in 1944, have been to foster research on the behavior of compressive elements in metal structures and to assist in the development of improved procedures for design. The Council attempts to offer guidance to practicing engineers and to specification writers. An effort is made to present both refined and simplified procedures and to assess their limitations; it is left to the user to choose among them as he sees fit.

The first edition of the *Guide* was dedicated to the Council's first chairman in these words: "As first chairman of Column Research Council, Shortridge Hardesty gave freely for twelve years his time, devotion, and material assistance. His mind grasped both the practical problems of engineering application and the fundamental knowledge essential to research. His influence was a personal inspiration to all who worked in Column Research Council."

The initial outline of the *Guide* was prepared in 1956 by Lynn S. Beedle and the late Jonathan Jones. The first edition was published by the Council in 1960. It subsequently played an important supporting role in connection with revisions of both the AISC and CISC design specifications. Special financial support from the Engineering Foundation and the Association of American Railroads made the first edition possible. Costs for the second edition have been borne jointly by the American Institute of Steel Construction and by Column Research Council.

For the preparation and editing of both the first and the second edition of the *Guide*, the Council was fortunate in securing the services of Bruce G. Johnston, and is indebted to him for the amount of time he has devoted to this work. He has had the assistance of a number of people who have contributed portions of the manuscript and reviewed the drafts at various stages of completion.

In the second edition, the original five chapters have been completely revised and brought up to date, with new material on the effects of residual stress and initial curvature, and on the strength of tubular columns. A chapter on plate girders and one on compression chords of pony trusses have been added. Many new references are given.

One of the important objectives of Column Research Council is to digest critically the world's literature in its field and to make the results of research widely available to the engineering profession. You, the reader of this book, are invited to share in the improvement of future editions. If you know of published papers or results of research that would enhance the value of the next edition, please communicate with the editor.

Committee on the *Guide*

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Contents

Nomenclature

Symbols

A	A coefficient. Area of cross section. Diameter of rivet or bolt head.
A–A	A reference line.
A_c	Area of compression flange.
A_e	Area of cross section remaining elastic.
A_f	Area of flange.
A_s	Area of stiffener cross section.
A_t	Area of tension flange.
A_w	Area of web.
a	Length of side of stiffened plate. Length of perforation in a perforated plate. Torsion bending constant for an I section.
B	A coefficient.
b	Width of rectangular cross section. Width of plate. Width of pony truss bridge, center to center of trusses. Length of short side of a box section, center to center of long sides. Length of side of stiffened plate. Transverse distance from edge of a perforation to nearest line of longitudinal fasteners.
b_c	Width of compression flange.
b_e	Effective plate width.
b_f	Half-width of flange.
b_t	Width of tension flange.
b_w	Width between centers of flanges in a wide flange column.
C	A coefficient. Transverse pony truss bridge frame spring constant, particularly the least one. Carry-over factor.
C_1, C_2, C_3, C_4	Coefficients for lateral-torsional buckling.
C_c	AISC column formula coefficient.
C_w	Torsional warping constant.
C_{req}	Required transverse pony truss bridge frame spring constant.
c	Distance to extreme fiber of beam or column section in bending. Distance center-to-center of perforations in a perforated plate. One-half distance between batten fasteners, measured longitudinally. Distance from middle plane of channel web to centroid of section.

D Flexural rigidity of a plate per unit width.
d Depth of a section. Diameter of circular cross section. Mean diameter of a tubular column. Long side of box section, center-to-center of short sides. Transverse distance between lines of longitudinal fasteners in a perforated plate. Stiffener spacing for a stiffened plate.
d_e Diameter of elastic portion of a circular cross section.
d_f Distance between flange centroids in a plate girder.

E Stress-strain modulus of elasticity.
E_{st} Strain-hardening modulus (initial).
E_t Tangent modulus.
e Distance from centroid of girder cross section to shear center (positive if shear center lies between centroid and compression flange, otherwise negative). Distance from shear center to the middle plane of a channel web. Width of end panel in a plate girder. Eccentricity of end load in a beam-column.
e_o Assumed equivalent eccentricity (representing defects, etc.).

F_a Allowable average compressive stress in axially loaded members.
F_{ao} Allowable stress for a column having zero slenderness ratio.
F_b Allowable compressive bending stress.
F_b^* Basic allowable design stress for light gage steel shapes.
F_f Normal force in flange of plate girder.
F_w Normal force in web of plate girder.
f_a Average compressive stress due to axial load.
f_b Compressive stress due to bending moment.

G Elastic shear modulus. A gage length.
G, G_A, G_B Joint bending stiffness ratio (subscripts apply to respective ends of the column).
g Distance from shear center of girder to point of application of transverse load (positive when load is below shear center, otherwise negative).

h Depth of a rectangular cross section. Clear depth of plate girder web between flange components. Depth of pony truss at truss vertical, measured from center of floorbeam to center of top chord. Long side of box section, center-to-center of short sides. Transverse distance between lines of fasteners in a battened column. Distance between beam or girder flange centroids.
h_c Distance to compression-flange centroid from centroid of section.
h_t Distance to tension flange centroid from centroid of section.
h_w Depth of web.

I Moment-of-inertia of cross section.
I_b Moment-of-inertia of floorbeam in a pony truss.

I_c Moment-of-inertia of column cross section. Moment-of-inertia of compression flange about the y axis. Moment-of-inertia of truss vertical in a pony truss.
I_e Moment-of-inertia of cross section remaining elastic.
I_g Moment-of-inertia of girder cross section.
I_o Optimum moment-of-inertia of web stiffener in a plate girder.
I_p Polar moment-of-inertia of cross section.
I_s Moment-of-inertia of stiffener about web-face axis.
I_t Moment-of-inertia of tension flange about y axis.
I_x, I_y Moment-of-inertia of cross section, x and y denoting the coordinate axes.
$(I_y)_{eff}$ Effective moment-of-inertia about axis y.

J Torsion constant.
J_I Integral torsion constant.
j Lateral-torsional buckling constant. Number of panels in a stiffened plate.

K Effective or equivalent length factor.
K' Modified effective length factor for laced columns.
K_m Average effective length factor of all panel length compression chords in a pony truss.
k Coefficient of proportionality. Coefficient applied in plate buckling.
k_b Buckling coefficient for a plate-girder web in pure bending.
k_c Buckling coefficient for a plate-girder web in combined shear and bending.
k_h Local buckling parameter for box columns.
k_s Buckling coefficient for a plate girder web in pure shear.
k_w Local buckling parameter for wide-flange columns.

L, l Length of member, particularly a laterally unbraced length.
L_c Unbraced length of a column.
L_g Unbraced length of a girder.
L_o Sublength of laced column; distance between lacing-bar connections or distance between centers of batten plates.
l Panel length in a pony truss bridge.

M Bending moment.
M', M'' Rotational stiffness of near end of member with far end fixed or hinged, respectively, but with no end translation.
$\overline{M}', \overline{M}''$ Rotational stiffness of near end of member with far end fixed or hinged, respectively, but with near end translationally restrained by a linear spring.
M_a, M_b End moments acting on a beam-column at ends a and b, respectively.
M_c Critical bending moment.
M_e End moment for a framed column.
M_{eq} Equivalent uniform moment in a beam column.
M_f Flange moment in a plate girder.

M_{max} Maximum bending moment.
M_o Applied end moment.
M_o, $M_{o(x-x)}$, $M_{o(y-y)}$ Moment in a beam-column without regard to moment caused by deflection.
M_p Plastic bending moment.
M_u, $M_{u(x-x)}$, $M_{u(y-y)}$ Ultimate bending moment in the absence of axial load in a beam-column.
M_x, M_y, M_z Moment about coordinate axes x, y, and z, respectively.
M_y Yield moment.
m Width of a perforation in a perforated plate.

N Number of component plates in a built-up flange. Nominal axial load.
N, n A factor-of-safety.
n Number of perforated plates used in a column. Number of parallel planes of battens in a battened column. Number of panels in a pony truss. Number denoting an individual compression member as one of several meeting at a common joint.

0–0 Principal axis of an angle cross section.

P Column axial load.
P_c Chord stress in a truss at maximum load.
P_{cr} Critical load.
P_e, $P_{e(x-x)}$, $P_{e(y-y)}$ Euler buckling load.
P_{max} Maximum column load.
P_n Axial compressive force of nth member.
P_p Column load at proportional limit.
P_r Reduced-modulus column load.
P_t Tangent-modulus column load.
P_u Ultimate load of axially loaded column.
P_y Column axial load at full-yield condition.
P_1, P_2 Axial compression force in truss member. (Subscripts refer to first and second member, respectively.)
p Rivet or bolt pitch.
p' Critical rivet or bolt pitch to insure integral action in torsion.

Q Transverse shear in centrally loaded column.

R Mean radius of tubular column.
R_a, R_b Rotational restraint at end of member. (Subscripts denote ends a and b, respectively.)
r Radius-of-gyration of member.
r_f Radius-of-gyration of column flange.
r_o Polar radius-of-gyration of the cross section about its shear center. Radius-of-gyration of one chord in a battened column.
r_x Radius-of-gyration about the centroidal axis x–x (strong axis).

r_y Radius-of-gyration about the centroidal axis y–y (weak axis).

S', S'' Rotational-stiffness reduction factor.
S_c Section modulus for compression.
S_t Section modulus for tension.
S_x Section modulus about x–x axis.
s Width of tension field in a plate girder panel.

T Tensile residual stress designation. Total thickness of several plates.
T' Translational stiffness of a member.
T'_a Translational stiffness of a spring at member end a.
t A thickness.
t_b Thickness of side b of box-section column.
t_c Thickness of compression flange.
t_h Thickness of side h of box-section column.
t_t Thickness of tension flange.
t_w Thickness of web plates of box-section beam. Thickness of web.

u Displacement in the x direction.

V Transverse shear force in plate girder.
V_p Plastic shear strength of plate girder.
V_u Ultimate shear strength of plate girder.
V_σ Shear strength of plate girder due to tension-field action.
V_τ Shear strength of plate girder due to beam action.
v Displacement in the y direction.

W Uniformly distributed total lateral load in a beam-column.
w Uniform load intensity. Displacement in the z direction.

X_e Width of rectangular cross section remaining elastic.
X–X, x–x Coordinate axis.
x Coordinate axis, particularly a principal axis. A distance.
x_o Distance between the shear center and the centroid in the direction of the x axis.

Y_e Depth of rectangular cross section remaining elastic.
Y–Y, y–y Coordinate axis.
y Coordinate axis, particularly a principal axis.
y_c Distance from centroidal axis x–x to face of tee flange.
y_o Distance between the shear center and the centroid in the direction of the y axis.

Z Plastic modulus. Limiting constant for medium-length tubular columns.
z A coordinate axis.

α Aspect ratio a/b or a/h for stiffened plates. Load ratio P/P_e.

β	Constant for stiffened plates. Angle of twist of cross section. Parameter h/t_w for plate girder panels.
γ	Buckling parameter for a stiffened plate.
γ_o	Optimum relative stiffness of stiffener to web in a plate girder.
γ_{ol}	Optimum relative stiffness of longitudinal stiffeners.
γ_{ot}	Optimum relative stiffness of transverse stiffeners.
Δ	A deflection.
Δ_ϵ	An increment of strain.
Δ_σ	An increment of stress.
δ	Column deflection caused by bending moment due to an axial load *P*. Buckling parameter for a stiffened plate.
δ_o	Maximum initial out-of-straightness of a column. Deflection without regard to moment induced by axial load.
ϵ	Strain.
ϵ_f	Strain in plate-girder web at the flange due to bending.
ϵ_{st}	Strain at initial strain hardening.
ϵ_y	Elastic strain at yield stress.
η	Ratio of tangent modulus to elastic modulus, E_t/E.
Θ	An angle.
κ	Moment coefficient for lateral torsional buckling.
λ	Slenderness function $\sqrt{\sigma_y/\sigma_e}$, $\sqrt{\sigma_y/\sigma_c}$.
ν	Poisson's ratio.
σ	Normal stress.
σ_a, σ_{av}	Average normal stress.
σ_c	Critical stress.
$\sigma_{c(v)}$	Critical stress for a variable cross section.
σ_e	Average stress at Euler buckling load.
σ_{eb}	Elastic buckling stress for a beam.
σ_f	Normal stress in a plate-girder flange.
σ_m	Maximum stress at mid-length of column by the secant formula.
σ_{max}	Maximum combined stress due to column load and bending moment.
σ_n	Transverse normal stress in a plate-girder web.
σ_p	Proportional limit-stress.
σ_{RC}	Maximum residual compressive stress.
σ_r	Local residual stress.
σ_t	Tension-field stress in plate girder.
σ_{uy}	Upper yield point stress.
τ	Shear stress.
τ_c	Shear stress at buckling load for plate girder.
τ_y	Shear stress at tension yield in plate girder.

ϕ	Angle of rotation. Tension-field angle.
ϕ_o	Optimum tension-field angle.
ψ	Parameter used in beam-column formulas.
W	Rolled wide-flange structural shape.

Abbreviations

AASHO	American Association of State Highway Officials.
AISC	American Institute of Steel Construction.
AISE	Association of Iron and Steel Engineers.
AISI	American Iron and Steel Institute.
AREA	American Railway Engineering Association.
ASCE	American Society of Civil Engineers.
ASME	American Society of Mechanical Engineers.
ASTM	American Society for Testing and Materials.
CISC	Canadian Institute of Steel Construction.
CRC	Column Research Council.
CSA	Canadian Standards Association.
NACA	National Advisory Committee for Aeronautics.
WRC	Welding Research Council.
ksi	Kips per square inch.
psi	Pounds per square inch.

Chapter One

Introduction

1.1 Scope

The second edition of the Column Research Council's *Guide* now includes centrally loaded columns, laterally unsupported beams, the compression components of plate girders, beam-columns, laterally restrained compression chords of trusses (for example, pony trusses), and local elements that transmit compression in any structural member.

Criteria to be considered as a basis for compression-member design include the evaluation of buckling loads or, alternatively, the determination of the nonlinear relationship in both the elastic and inelastic stages between internal resistance and external load when imperfections exist. Depending on the type of structural element, the buckling load itself may be inadequate as a design criterion. The post-buckling strength may require consideration, or, conversely, failure may be reached at a load less than the buckling load. Generally speaking, the *Guide* tries to provide for the calculation of the maximum strength and leaves it to the specification writer or engineer to introduce a factor-of-safety by means of which a suitable design load can be determined.

Materials covered herein include structural steels, light-gage cold-formed steels, and structural aluminum alloys. Because of the different characteristics of the stress-strain relationships for these materials, the same approach cannot necessarily be applied to all. CRC research was the first to draw attention to the importance of the effect of residual stress on the buckling strength of steel columns. There is currently (1966) a proliferation of strength levels available in structural steels as produced for plates and shapes. In a listing prepared by the CRC Task Group on Classification of Steels for Structures, no fewer than twenty-three different yield strengths are found to be available, ranging from 32 to 115 ksi. The heat-treated steels represented by the upper level of strength have been used in increasing quantities in towers, long-span bridges, and in other applications including buildings for which weight reduction may offer inducement to use more costly materials. Steel column behavior will be evaluated herein in terms of representative yield-point levels in conjunction with effects of shape, residual stress, and initial imperfections.

The CRC *Guide* generally does not provide derivations of the formulas that are presented. Such derivations can be found in references listed in the bibliography of each chapter, and in the general references given in the Appendix.

1.2 Column Research Council

The Column Research Council was formally organized in 1944, with a membership composed of from one to four appointed representatives from each of twenty-eight organizations. During the past twenty-two years, six of the original groups have dropped out, but two new ones have been added, leaving the following twenty-four organizations on the membership list as of 1966.

Aluminum Company of America
American Association of State Highway Officials
American Institute of Architects
American Institute of Consulting Engineers
American Institute of Steel Construction
American Iron and Steel Institute
American Society of Civil Engineers
American Society of Mechanical Engineers
Association of American Railroads
Boston Society of Civil Engineers
Bureau of Public Roads
Bureau of Yards and Docks, U. S. Navy
Canadian Institute of Steel Construction
Chief of Engineers, U. S. Army
Engineering Institute of Canada
General Services Administration
International Conference of Building Officials
National Bureau of Standards
National Research Council
Society for Experimental Stress Analysis
Structural Engineers Association of Northern California
Structural Engineers Association of Southern California
Welding Research Council
Western Society of Engineers

The initial leadership and direction of CRC were supplied by Shortridge Hardesty, the first chairman, until ill health forced him to give up active participation in 1956. He was succeeded by Bruce G. Johnston, who

served from 1956 until 1962, when Edwin H. Gaylord was elected chairman.

The Engineering Foundation brought the Council into being, and support has been provided by contributions (either directly or to individual research projects) from the following organizations:

Aluminum Company of America*
American Bureau of Shipping
American Institute of Architects
American Institute of Steel Construction*
American Iron and Steel Institute*
Association of American Railroads*
Bethlehem Steel Corporation*
Boston Society of Civil Engineers
Bureau of Public Roads*
Canadian Institute of Steel Construction
David Taylor Model Basin*
Engineering Foundation*
Modjeski and Masters*
National Science Foundation*
Pennsylvania Department of Highways*
Research Corporation*
Rhode Island Department of Public Works
Society for Experimental Stress Analysis
Society of Naval Architects
Structural Engineers Association of Northern California
Structural Engineers Association of Southern California
United States Navy, Bureau of Yards and Docks*
United States Steel Corporation*
Welding Research Council*

Projects were sponsored at many universities. These universities contributed more in materials and personnel than they received in reimbursement. Contributing institutions were:

Brown University
Columbia University
Cornell University
University of Florida
University of Illinois
University of Iowa
Lehigh University

* Major support.

University of Michigan
New York University
North Carolina State University
Pennsylvania State University
Purdue University
Stanford University
University of Washington

Column Research Council prepared at its inception a statement of general objectives, which remain as follows:

(*a*) To organize, maintain, and administer a national forum in which problems relating to the design and behavior of columns and other compression elements in metal structures can be presented, and pertinent structural research problems can be proposed for investigation with the assurance of an evaluation of all problems proposed and of support for those projects adjudged important.
(*b*) To digest critically the world's literature involving compression elements and the properties of metallic materials available for their construction, and to make the results widely available to the engineering profession.
(*c*) To organize and administer cooperative research projects in the field of compression elements.
(*d*) To stimulate, aid, and guide column research projects on the foregoing problems in the engineering colleges and research laboratories.
(*e*) To study the application of the results of this research to the design of compression elements.
(*f*) To develop a comprehensive and consistent set of formulas or rules covering their design.
(*g*) To promote the widest possible adoption of such formulas by designers and specification-writing bodies.
(*h*) To publish and disseminate original research information within the field of the Council.

Column Research Council has had corresponding members on every continent. As a by-product of such contacts, the Japanese Column Research Council (not affiliated with CRC) was established, with headquarters in Tokyo. This group has published a very comprehensive monograph on elastic stability formulas (A2)* that is presently in its second edition.

Each chapter of this second edition of the CRC *Guide* was sent in first-draft form to members of an advisory committee on review of the *Guide*, and certain chapters were also sent to specialists who were not necessarily members of the Council. The following persons provided substantial critical reviews of one or more chapters: R. M. Barnoff, Konrad Basler, L. S. Beedle, J. W. Clark, E. L. Erickson, T. V. Galambos, E. H. Gaylord, T. R. Higgins, A. Hrennikoff, O. G. Julian, W. G. Kirkland, M. G. Lay, L. C. Maugh, James Michalos, F. R. Shanley, L. Tall, S. S. Thomaides, D. L. Tarlton, Bruno Thürlimann, and George Winter.

* General references, prefixed by "A," are found in Appendix A.

Going far beyond the usual critical review, Jackson L. Durkee, of the Bethlehem Steel Corporation, scrutinized every sentence of the entire manuscript and suggested changes in detail in order to improve the clarity of the text. Special acknowledgment is also owing to M. A. El-Gaaly of the University of Michigan, who acted as a research assistant to the editor and who prepared drafts of the new chapters on plate girders and pony trusses. His work included thorough research into the literature of these two areas, and Chapters 5 and 7 are substantial condensations of his first drafts. R. B. Harris provided a careful check of the completed *Guide* manuscript, and also prepared the index. Checking of proof and other publication matters were handled by the Publications Committee, consisting of R. B. Harris, Chairman, along with J. L. Durkee and B. G. Johnston.

1.3 The Guide to Design Criteria for Metal Compression Members, Objectives, and Summary

It is the purpose of this publication not to supplant but rather to serve as a guide to the improvement of existing design procedures and specifications. As described in some detail in Chapter 2, a major thesis of this design guide is the unification of all centrally-loaded column strength procedures on the basis of the modified Eulerian (tangent-modulus) theory. That this is practical is evident on consideration of the effective gradual transition from the elastic to plastic behavior that is caused by the presence of residual stresses and by the nonuniform strain hardening that results from the fabrication of cold-formed sections. Thus, the differences that exist between the inelastic stress-strain properties of steel and aluminum alloys as determined by small coupon tests are less pronounced when the behavior of a full column cross section is involved.

Chapter 3 is concerned with the local strength of component parts in compression. Information on tubular columns has been expanded in Sec. 3.5 of this chapter to include both manufactured tubes and larger fabricated cylindrical members wherein imperfections, as well as riveting or welding, have an adverse effect on strength.

Before considering the column that is eccentrically and/or laterally loaded (the beam-column), we must first consider, as a limiting condition, the beam. The buckling of the laterally unsupported beam is considered in Chapter 4, in which a general formula is simplified for design of beams and girders of various types. The limitations in these simplified approaches are discussed.

In Chapter 5, on plate-girder design, the strength and rigidity requirements of transverse and longitudinal stiffeners, which improve web buckling strength, are considered. In addition, the girder flange and web design problems are treated in connection with the utilization of post-buckling strength and tension-field action of very thin webs.

Chapter 6 treats the beam-column design problem. Design procedures of various degrees of complexity are discussed. The recommended approaches are of the simple "interaction" type involving, in effect, semirational "interpolation" between the capacity of the beam and that of the centrally loaded column.

Chapter 7 provides information on the design of the compression chord of the pony-truss bridge, and includes tabular design information developed through CRC-sponsored projects.

1.4 The Factor-of-Safety

A most important and difficult problem in relating any compression-member strength evaluation to an economical and safe design load is the choice of the factor-of-safety. Ideally the safety of a structure would be predicted on the basis of known random variation of load, of material properties, of imperfections in fabrication, and so forth.

The choice of factor-of-safety in compression-member design is made more difficult by the multiplicity of factors which affect strength, and also because the effects of these factors vary with the equivalent slenderness ratio. The very short column varies in strength primarily in direct proportion to the yield strength of the metal. In the intermediate slenderness range the effects of eccentricity, initial curvature, residual stress, and so forth, are most marked. The strength of the very slender column is determined primarily by the bending rigidity, *EI*, of the section, a parameter that has relatively small variation. The effects of lateral force and end restraint are also important considerations in long-column design.

Until the day arrives (if ever) when the statistical variation and distribution of loads and material properties, and all other factors, are well established, there will always be a certain element of empiricism in the selection of the proper factor-of-safety. It is ultimately the responsibility of a specification-writing body to weigh what is known against the elements of risk, desired economy, anticipated life, and required behavior in service, in order to determine the factor (or factors) of safety that seems most appropriate to the particular class of construction.

1.5 Mechanical Properties of Structural Metals

A knowledge of the stress-strain relations that take place during the elastic and initial inelastic ranges of behavior is an essential requisite to compression-member analysis. In the elastic range there are accepted *average* values of the modulus of elasticity, and test values vary within reasonably small limits. *Minimum* specified values of the yield point or yield strength (depending on whether the initiation of yielding is a sudden or gradual process) are provided by the various specifications of the American Society for Testing and Materials (ASTM) and by product information provided by manufacturers. In this *Guide* the term "yield stress" will be used to denote either the yield point or yield strength, whichever is applicable.

The initial portions of the typical stress-strain curves for structural metals in compression and in tension are shown in Fig. 1.1. The strength of beams and columns is largely determined by stress-strain characteristics in the range shown. The complete curves plotted to the same scale would take up a horizontal space between twenty and thirty times that available on the drawing.

The stress-strain curve for a particular sample of lower-strength structural steel can be characterized by the following five items (see Fig. 1.1):

E = Young's Modulus = slope of stress-strain curve in elastic range,
σ_{uy} = upper yield point,
σ_y = yield-stress level (corresponding to the stress in the flat portion of the stress-strain curve after initial yield),
ϵ_{st} = strain at initial strain hardening, and

$$E_{st} = \left(\frac{d\sigma}{d\epsilon}\right)_{\epsilon=\epsilon_{st}} = \text{strain-hardening modulus (initial).}$$

The last four of these properties are essential to calculation of inelastic strength and deformation of the sample. For a particular sample of aluminum alloy, quenched-and-tempered steel, or cold-worked steel, two properties are of significance; namely, (1) the yield strength, σ_y, determined by the extension-under-load method or the offset method (ASTM Designation A370), and (2) the tangent modulus, $E_t = d\sigma/d\epsilon$, along the stress-strain curve. For both steels and aluminum alloys the ultimate or breaking strength, based on original area, is also a part of the record.

The yield stress of steel specimens will vary with temperature, rate of strain, and the surface characteristics of the test specimen, as well as with the testing machine and method of testing.

For lower-strength steels the yield-stress level in a tension or compression test can be regarded as the level of stress, after initial yield, that

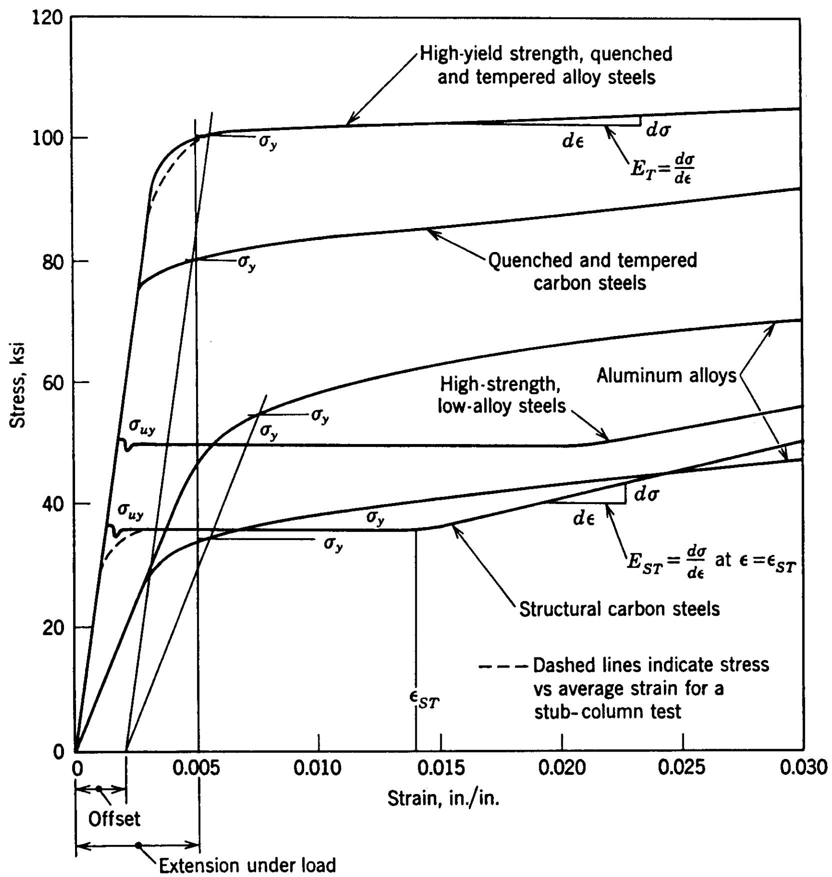


Fig. 1.1. Initial stress-strain relationships for structural metals in compression or tension.

is sufficient at a given temperature and rate-of-strain to develop successively new planes of slip in the portions of the test specimen that remain in the elastic state. After initial yielding has proceeded discontinuously from point to point throughout the specimen, strain hardening commences, and the stress rises with further increase in average strain. The yield-stress level is frequently of primary significance in determining the ultimate strength of a structure or structural member under either static or dynamic load conditions. The sharp yield point may disappear with cold work or heat treatment. The yield-stress level is structurally more significant than the upper yield point, and is a function of the speed of test, becoming lower with lower testing speeds. "Zero strain rate" defines a lower limit

for the yield-stress level. Mill tests conform to ASTM Specifications which specify a maximum strain rate. Although such tests are quite suitable for quality control, they do not give a true indication of the yield-stress level of steels under low rates of strain.

Of particular importance in column strength analysis are modifications in the shape of the *average*-stress-versus-strain curve as determined by a very short or stub-column test. Residual stresses cause modifications in this relationship (indicated qualitatively in Fig. 1.1) for structural carbon steel and for quenched and tempered alloy steel. Likewise, strain hardening caused by the forming processes used for cold-formed members (cold rolling, brake forming) results in increases in yield strength which vary in different parts of the cross section. These, too, result in a significant modification of the stress-strain curve, shifting it toward higher values and producing a more gradual yield process.

The typical stress-strain curves for structural steels shown in Fig. 1.1 are representative of the following four categories of steels:

1. *Structural carbon steels.* This category includes ASTM A7 and A36 carbon steel plates, shapes, and bars, and A245 carbon steel sheets (yield stresses ranging from 25 to 40 ksi).
2. *High-strength low-alloy steels.* This category includes ASTM A440, A441, and A242 high-strength low-alloy steel plates, shapes, and bars, A374 and A375 high-strength low-alloy steel sheets and strip, and a multitude of proprietary steels (yield stresses ranging from 40 to 70 ksi).
3. *Quenched-and-tempered carbon steels.* Steels in this category were introduced in 1964. These steels are currently produced as plates and are available as proprietary grades (yield stresses ranging from 70 to 80 ksi).
4. *High-yield-strength, quenched-and-tempered alloy steels.* Steel plates in this category are covered by ASTM A514. Structural shapes are available as proprietary grades (yield stresses ranging from 90 to 100 ksi).

A survey conducted by the CRC Task Group on Classification of Steels for Structures indicated the current (1966) availability of twenty-three different yield stresses for the great variety of steels in these main categories. However, a progression of preferred yield stresses is emerging that will simplify the designer's task of selecting the optimum steels for different parts of a structure. The progression consists of the following preferred yield stresses: 36, 42, 50, 60, 72, 85, and 100 ksi.

Fig. 1.1 also shows stress-strain curves for two structural aluminum alloys: 6061-T6, with a minimum yield strength of 55 ksi, and 2014-T6, having a minimum yield strength of 35 ksi. The various products made from these alloys are covered in ASTM Specifications B209, B210, B211, B221, B234, B241, B247, and B308. Another aluminum alloy that is frequently used in applications where relatively low strength is adequate is 6063-T6, for which the minimum yield strength is 25 ksi. The ASTM specifications just listed include alloy 6063-T6 as well as many other alloys that are used in structural applications.

Chapter Two

Centrally Loaded Columns

2.1 Introduction

The centrally loaded column transmits a compressive force whose resultant at each end is coincident with the longitudinal centroidal axis of the member. Such an ideal column can be at best approximated in practice; nevertheless, this concept forms a generally accepted basis for column strength analysis and design. Bending moments due to initial imperfections, accidental curvature, or unintentional end eccentricity will reduce the strength of a column that is intended to be centrally loaded, but these effects are taken care of in design formulas by an appropriate factor of safety, and/or by a modification of the basic column strength curve, i.e., the estimated functional relationship between the average column compressive stress at failure and the column slenderness ratio. When bending moment is caused by intentional end eccentricity or lateral load, or is induced by framing members, the problem is that of the "beam-column" (see Chapter 6).

The major advances in the understanding of column behavior in recent years have concerned the intermediate range between the short column, where the behavior is determined almost entirely by the *inelastic strength* properties of the material, and the long column, where the behavior is determined almost entirely by the *elastic flexural stiffness*, *EI*, of the member. Most columns are in this intermediate length range wherein secondary factors, such as residual stress, accidental crookedness, unintentional end eccentricity, and so forth, have the greatest effect on column strength and thereby introduce a confusing diversion of attention from the primary features of the column problem.

Column buckling theory was initiated by Euler (2.1), who more than 200 years ago developed the simple formula that bears his name. Euler's formula was initially limited to the column with a fixed base and no lateral support at the top. Lagrange (A4) extended the theory to higher modes with an analysis procedure that is still used. The "Euler load" is that load in the elastic range at which a slender axially loaded column of constant cross section may be in equilibrium in either a straight or a slightly deflected configuration. The subsequent development of inelastic modifications of the Euler formula will be reviewed by means of a chart (Fig. 2.1), adapted from a lengthier review by N. J. Hoff (2.5).*

For many years Euler's formula was not generally applied to actual design because proof tests of structures indicated that columns frequently failed below the Euler load. In 1889, Considère indicated why Euler's formula had not been more useful to engineers. He conducted a series of thirty-two column tests and suggested that if buckling occurred above the proportional limit the elastic modulus in the Euler formula should be replaced by an "effective" modulus which would lie between the elastic modulus E and the tangent modulus E_t.

Independently of Considère and during the same year (1889), Engesser (2.2) suggested that column strength in the inelastic range might be obtained by the substitution of E_t in place of E in the Euler formula. This is known today as the "tangent-modulus formula." However, in 1895, Jasinski suggested that there was an apparent mistake in Engesser's formula in that the nonreversible characteristic of the stress-strain diagram in the inelastic range was not considered, as had been done in a very general way by Considère. Engesser proceeded, within the same year, to produce a "corrected" general formula for a "reduced modulus," and he stated that this reduced modulus depended not only on E_t and E but also on the shape of the cross section as well (2.3).

In 1910, Theodor von Kármán derived explicit expressions for the "reduced modulus" for both rectangular columns and idealized H-section columns. From the classical instability concept the reduced-modulus theory was correct, because it indicated the load at which a perfectly straight and centrally loaded column could have neighboring equilibrium configurations with no change in load. This is identical in concept to the Euler load in the elastic buckling range. However, many experimenters found that columns tested in the laboratory with utmost care usually failed at loads just slightly above the tangent-modulus load.

In 1947, Shanley offered a new interpretation (2.4) of the tangent-modulus load. He showed that it was possible for a centrally loaded column to start to bend simultaneously with increasing axial load, without strain reversal, and that it was to be expected (because of inevitable imperfections, no matter how small) that such bending would start at the tangent-modulus load. Thus, the reduced-modulus load never could be reached, because it is based on the assumption of a perfect column that remains straight until it reaches the reduced-modulus load. In a letter published jointly with the 1947 Shanley paper, von Kármán (2.4) redefined

* See also Ref. 2.6.

the tangent-modulus load in a way that can be paraphrased as follows:

> The tangent-modulus load is the smallest value of the axial load at which bifurcation of the equilibrium positions can occur regardless of whether the transition to the bent position requires an increase of the axial load.

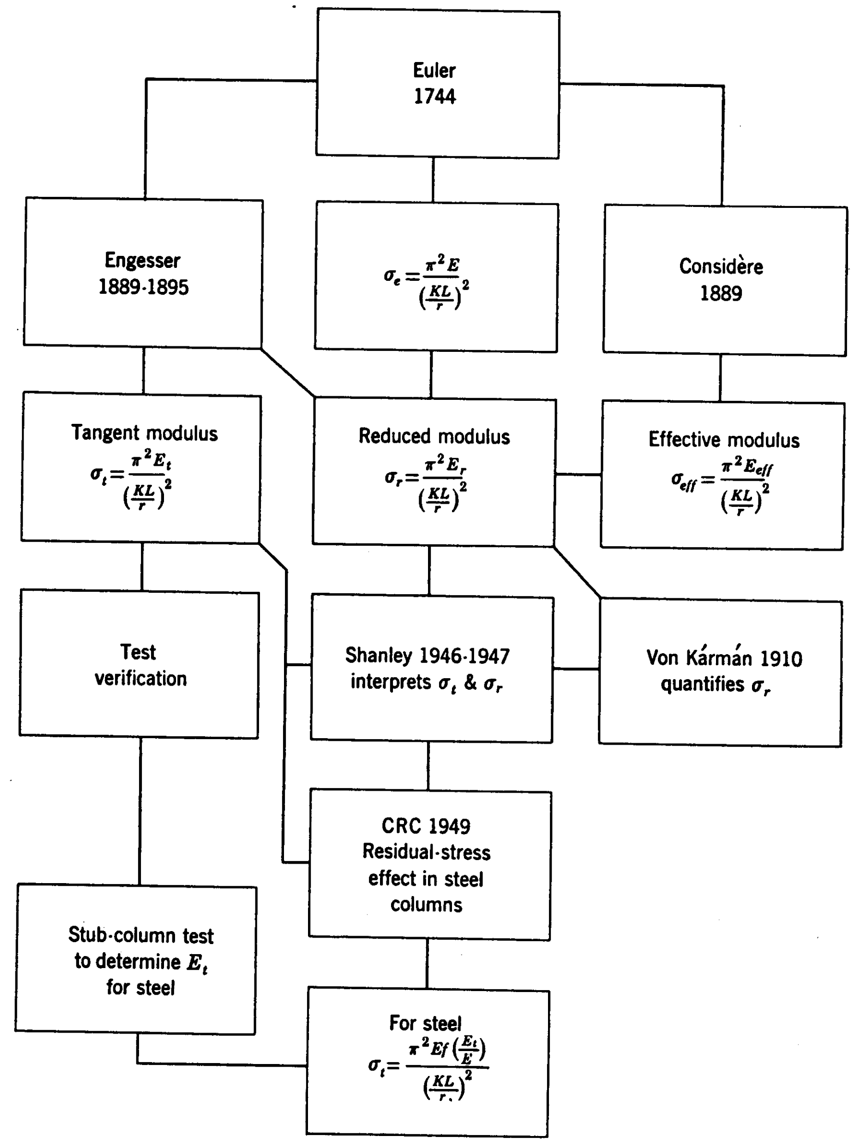


Fig. 2.1. Evolution of the column formula.

Shanley showed intuitively that after bending starts there must be some strain reversal to attain an equilibrium position for any finite load increment above the tangent-modulus load. Equilibrium in a bent configuration above the tangent-modulus load obviously is not possible if the stress-strain relationships everywhere across the section have been governed only by the tangent modulus. This is true even if the tangent modulus suffers no decrease within the cross section.

In 1950, Duberg and Wilder (2.7) extended the Shanley approach to the idealized H-section with flexibility over the full length of the column. The flanges were represented, as in the Shanley model, by two point concentrations of area with an intervening web of zero area which could, nevertheless, transmit shear. The result, an important contribution to column theory, shows how bending occurs under varying inelastic conditions all along the column. Duberg and Wilder confirm the Shanley concept with the following succinct statement:

> If the behavior of a perfectly straight column is regarded as the limiting behavior of a bent column as its initial imperfection vanishes, the tangent-modulus load is the critical load of the column—that is, the load at which bending starts.

Although the Euler and the associated tangent-modulus formulas found direct application in the airplane industry in the design of aluminum alloy struts, similar application to structural steel design lagged because of the discrepancy between tangent-modulus strength predictions, based on small coupon stress-strain curves, and the actual strengths of tested columns. As is well known, small coupon tests of steel usually show practically a linear stress-strain relationship up to loads very near the yield point. Thus, the tangent modulus is equal to the elastic modulus for stresses below the yield point and to zero at the yield point. On this basis, the stress-strain relationships determined by a coupon test will not provide enough information to explain the behavior of steel columns.

The key to the application of the tangent-modulus concept to structural steel columns is the determination of the tangent modulus of the *effective stress-strain diagram of the entire cross section.* The principal factors that cause differences between the effective stress-strain curve of the complete cross section and the individual stress-strain curve of a test coupon are the presence of residual stress and the variation in yield stress over the cross section.

In hot-rolled structural-grade steel columns, residual stresses approaching half the yield point are "locked" in the member, as a result of uneven cooling on the mill cooling beds. Residual stresses are present to a lesser

degree in angles, universal-mill plate, and channels, but in members built up by welding they may approach the yield point.

Although Salmon (A5) had suggested initial residual stress as an influencing factor, its importance in steel columns was discovered in 1949 by members of the Committee on Research of Column Research Council. This resulted indirectly from an investigation of residual-stress effects on the local buckling of box girders (2.8), and measurement of residual stresses in steel shapes as part of a project on the plastic bending strength of steel beams (2.9). Column Research Council, through its research committees, then initiated an extensive investigation of the influence of this variable. It was obvious that both theoretical and experimental work would be required to establish the validity of the concept. The initial CRC research assignment was to its Research Subcommittee on Mechanical Properties of Materials. W. R. Osgood, then chairman of this committee, developed a theoretical study of the residual-stress effect (2.10). Simultaneously, tests and theoretical studies were proceeding at Lehigh University (2.11), to be followed by additional investigations by A. W. Huber, R. L. Ketter, and L. S. Beedle (2.12, 2.13). The extension of residual-stress studies to riveted and welded built-up sections, high-strength steels, and circular sections has been carried out by Y. Fujita, L. Tall, A. Nitta, T. V. Galambos, G. C. Lee, and B. Thürlimann. A review of the relationships between stub-column behavior, variation of yield point, and the residual-stress effects is contained in a CRC Symposium paper by Beedle and Tall (2.14).

As a result of the early research on residual stress, it was possible for Column Research Council to make the following statement in its Technical Memorandum No. 1 on "The Basic Column Formula" (2.15), issued in May 1952:

> It is the considered opinion of the Column Research Council that the tangent-modulus formula affords a proper basis for the establishment of working-load formulas.

At first glance this pronouncement might seem rather obvious, since the tangent-modulus concept was already widely used, especially in aluminum-alloy applications. Its importance lay in the fact that extension of the concept to structural steels was now possible through introduction of the residual-stress modification to the tangent-modulus theory.

The past failure of the tangent-modulus concept to explain the behavior of steel columns had usually been attributed to the presence of accidental end eccentricities. Using the secant formula, a fictitious end eccentricity can always be calculated that will exactly account for the column strength determined by any given test, assuming that (1) the column fails when the maximum stress due to combined direct load and bending reaches the yield point, and (2) the material is perfectly elastic up to the yield point. Thus, the effects of factors such as initial curvature and residual stress can be included in design by means of the empirical determination of an equivalent end eccentricity.

Currently (1966), high-strength steels having minimum yield stresses up to 70 ksi, quenched-and-tempered carbon steels with yield stresses up to 80 ksi, and quenched-and-tempered alloy steels with stresses up to 100 ksi are coming into increased use in framed structures. There are indications (2.17) that in the use of these steels the role of residual stress in rolled shapes will be relatively less important than it is known to be for structural-grade steels. Thus, as the yield point increases, initial curvature (particularly in rolled shapes) takes on increasing importance in relation to residual stress. In addition, increased attention is being given to effects

Table 2.1. Factors that Cause Actual Column Strength To Be Different from the Euler Critical Load

Factors Related to Basic Properties of Material	Factors Introducing Accidental Bending Stress	Factors Related to Type or Shape of Column
1. Nonlinearity in actual stress-strain relationship in compression as obtained from a small coupon test	1. Accidental end eccentricity	1. Shear deformation, especially in built-up columns having lacing, batten plates, etc.
2. Variation in yield strength over column cross-section	2. Accidental curvature	2. Local buckling, especially when post-buckling strength of thin-walled plate components is a design factor
3. Residual stress (primary factor in structural steels)	3. Accidental lateral load, or lateral load unrelated to primary column load	3. Torsional buckling
4. Creep	4. Thermal effects	

of the nonuniform strain hardening that occurs in the fabrication of cold-formed sections (A12). However, for welded built-up shapes (irrespective of type of steel) the maximum tensile residual stress approaches the yield point and the adverse effects of the concurrent compressive residual stresses are large (2.18). Another factor of importance when light-gage

cold-formed sections are used as columns is the loss of effective section resulting from local buckling prior to the attainment of maximum column load.

It is exceedingly complex and unworkable to attempt to introduce into a single column formula all known factors that affect column strength. The major factors must be considered; however, the minor factors are most conveniently taken care of in design by an appropriate factor-of-safety.

Table 2.1 summarizes the major column-strength-determining factors which have been discussed and which will receive further consideration in this and later chapters of the *Guide*.

2.2 Basic Column Strength

The basic Euler formula (2.1), which gives the critical load at which a linearly elastic, initially straight, centrally loaded column will bend, is obtained from the beam equilibrium condition equating external to internal moment:

$$Py = -EI\frac{d^2y}{dx^2}$$

Upon integration and introduction of hinged-end boundary conditions, it can be shown that the load for the lowest mode of buckling is:

$$P_e = \frac{\pi^2 EI}{L^2} \tag{2.1}$$

where P_e is known as the Euler buckling load. Upon substitution of $I = Ar^2$ and division by A, the formula for the average stress at the Euler buckling load is obtained:

$$\sigma_e = \frac{\pi^2 E}{(KL/r)^2} \tag{2.2}$$

In Eq. 2.2, the factor K has been introduced to permit modification to other than the hinged-end condition. For purely flexural buckling, KL is the length between inflection points and is known as the *effective* or *equivalent* length. One or both the inflection points may be outside the column and the theoretical value of K ranges between 0.5 and ∞. For an ideal column with frictionless hinged ends permitting free end rotation, $K = 1$. Other end conditions will be discussed in Sec. 2.8.

For the lesser values of L such that σ_e from Eq. 2.2 would exceed the *effective* proportional limit of the material, the centrally loaded straight column will start to bend as the load increases, at an average stress obtained by substituting the tangent modulus E_t in place of the elastic modulus E in Eq. 2.2 (see Fig. 2.2c).

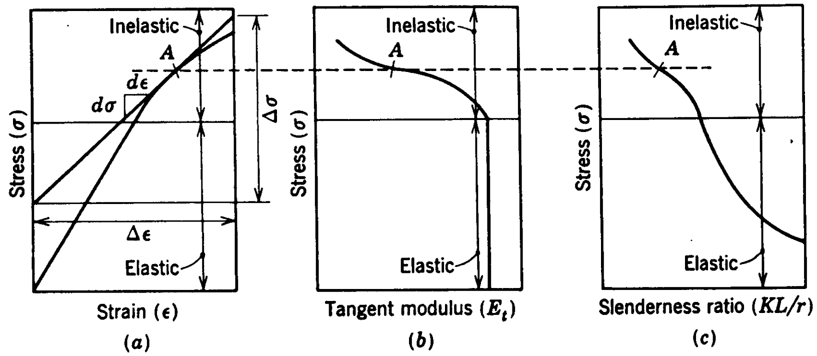


Fig. 2.2. Steps in the determination of a column strength curve from a stress-strain curve.

The tangent-modulus theory will now be reviewed. Initially, let it be assumed that the stress-strain properties of a small coupon in compression are identical with the average stress-strain properties of a short stub section of column manufactured of the same material. This assumption is equivalent to assuming that the material is homogeneous and free from residual stress.

Fig. 2.2 shows the two-stage process by which a stress-strain curve is used to develop the column strength curve by means of the tangent-modulus formula

$$\sigma_c = \frac{\pi^2 E_t}{(KL/r)^2} \tag{2.3a}$$

Alternatively, Eq. 2.3a can be written

$$\sigma_c = \frac{\pi^2 E}{(KL/r)^2}(\eta) \tag{2.3b}$$

where $\eta = E_t/E$.

Fig. 2.2a is a typical stress-strain curve for an aluminum alloy in compression. At any given point A in the inelastic range, the slope of the curve gives the tangent modulus for that particular stress:

$$E_t = \frac{\Delta\sigma}{\Delta\epsilon}$$

For accuracy in plotting, the inelastic portion of the curve should be drawn to an enlarged scale. Proper techniques for making compression tests and determining E_t are described in Technical Memorandum No. 2, Column Research Council (2.16), reprinted here in Appendix B.

The values of the tangent modulus determined in Fig. 2.2a may be plotted as in Fig. 2.2b, or they may simply be tabulated. Then, on the basis of the coupled values of σ and E_t at points A in Figs. 2.2a and 2.2b, the value of KL/r at A in Fig. 2.2c is determined by a transposition of Eq. 2.3a:

$$KL/r = \pi\sqrt{\frac{E_t}{\sigma_c}} \tag{2.4}$$

The complete column strength curve as illustrated in Fig. 2.2c is obtained by repeating the foregoing process for various levels of stress.

The procedure just described for the determination of the column strength curve in the inelastic range has been standard practice for aluminum and magnesium alloys and can also be applied to stainless steels if one neglects the effects of nonuniformity of yield strength across the section caused by cold forming. It also applies to other special steels that have a decidedly nonlinear stress-strain diagram and no sharp yield point above the proportional limit.

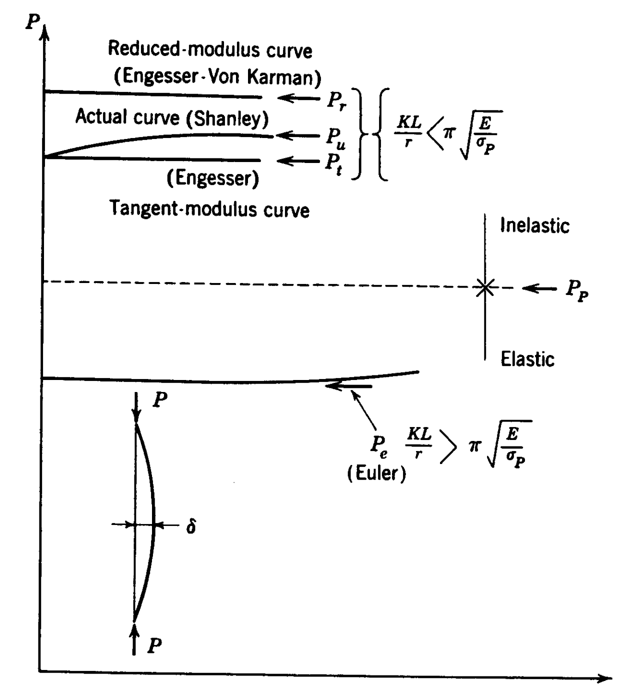


Fig. 2.3. Load-deflection relationships for a straight column above and below the proportional limit.

The two load-deflection relationships that are characteristic of elastic and inelastic column behavior are illustrated in Fig. 2.3. If buckling is elastic, the Euler load P_e is reached below the effective proportional limit load P_p, the lateral deflection can assume any value without increase in load, and its magnitude is indeterminate. If the more accurate large-deflection theory of the "elastica" is applied using the more precise expression for curvature in place of d^2y/dx^2, there will result a theoretical increase in load as the column deflects, and definite deflected equilibrium positions may be determined (Ref. A3, p. 429).

If the calculated Euler stress is above the proportional limit, inelastic buckling occurs and the initial bifurcation will be at the tangent-modulus load as predicted by Shanley (2.4), and definite deflections can be determined with increasing load. Since this involves (for incremental bending moments) a shifting neutral axis within the column cross section, with a nonlinear increase of stress on one side and a decrease of stress in proportion to the elastic modulus E on the other, the successive equilibrium positions above the tangent-modulus load and the corresponding load-deflection curve shown in Fig. 2.3 can only be determined by an incremental analysis. The initial slope of the curve, however, can be expressed explicitly by a simple formula (2.19). Procedures are available (2.4, 2.6, 2.7) for determining the maximum load, which is noted in Fig. 2.3 as the "Shanley" column load, P_u.

If the column were constrained to remain straight up to the reduced-modulus load, P_r, and then were released, it would buckle with no increase in load, as shown in Fig. 2.3. If the column were laterally supported up to a load between P_t and P_r and then released, it would remain in the straight position and would be in stable equilibrium. If the load were then increased, the column would start to deflect, at a rate of increase somewhat less than that at the tangent-modulus load, and it would reach a new maximum load having a value less than P_r (2.6).

2.3 Residual-Stress Effect on Steel Columns

Structural steel, tested by means of a small coupon, usually exhibits nearly linear stress-strain characteristics up to the yield stress. Yielding then spreads from point to point with no increase in average stress until average strains many times those at initial yield are reached. On such a basis, the tangent-modulus theory gives a simple but unrealistic result. Thus, if the stress-strain curve were linear up to the yield stress, a column would develop full yield stress for values of KL/r less than $\pi\sqrt{E/\sigma_y}$ and would buckle at the Euler load for greater values of KL/r.

If, instead of a small coupon, a stub column section is used to determine an *average* stress-strain curve, the tangent modulus determined therefrom

will reflect both the presence of residual stress and the variation in yield stress over the cross section (2.14).* The difference between the stress-strain curve for a steel coupon and the effective stress-strain curve for a stub column was illustrated in Fig. 1.1. If the material has uniform yield stress, the effective proportional limit of the stub column will be

$$\sigma_p = \sigma_y - \sigma_r$$

In the absence of actual stub column tests a predicted effective stress-strain curve for the column can be calculated on the basis of an assumed residual-stress distribution, a particular shape, and an assumed yield point. When at any point in the column cross section the sum of the average applied stress and the local residual stress exceeds the yield point, it will be assumed that the stress remains at the yield point, i.e., no strain hardening takes place. If A_e denotes the area of the cross section which has not reached the yield point, then the effective tangent modulus is

$$E_t = \frac{d\sigma_{\text{avg}}}{d\epsilon} = \frac{(dP/A)}{(dP/A_eE)} = \frac{EA_e}{A} = E\eta$$

In order to plot a predicted average stress-strain curve, it would be necessary to calculate the average stress as a function of the average strain. However, what is needed is simply the relationship between average stress and tangent modulus. This can be determined directly and the calculation of the average strain thus bypassed. After a portion of a stub column has yielded, as a result of the presence of residual stress, the applied load is

$$P = (A - A_e)\sigma_y + \int_{A_e} \sigma\, dA \tag{2.5}$$

and the average stress is

$$\sigma_a = \frac{P}{A} = \left(\frac{A - A_e}{A}\right)\sigma_y + \frac{1}{A}\int_{A_e} \sigma\, dA \tag{2.6}$$

Thus, by Eq. 2.6, the average stress depends on the initial distribution of residual stress and is a function also of A_e/A. Now, if an arbitrary set of values $A_e/A = \eta$ between zero and one is chosen, consistent with the cross-section shape and the residual-stress pattern, Eq. 2.6 can be used to determine the corresponding values of σ_a.

Buckling of the steel column with residual stress will be governed by the Shanley concept, with bending initiated at the critical load simultaneously with an infinitesimal increment of load and without strain reversal.

* Approved procedures for stub column tests are described in Appendix B, CRC Technical Memorandum No. 3.

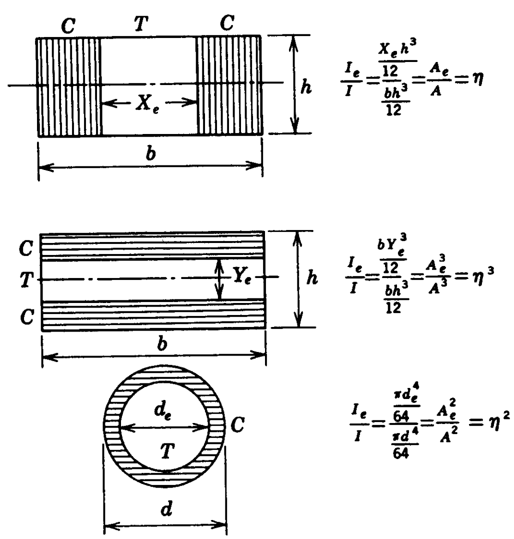


Fig. 2.4. $f(\eta)$ for various cross sections and patterns of residual stress.

If there is no strain reversal, none of the yielded portion of the column cross section can contribute to the internal resisting moment, which is a function of I_e alone. Thus, the strength of the column can be determined simply by replacing I in Eq. 2.1 by I_e, where I_e is the moment of inertia of that portion of the cross section that is still elastic. This very basic suggestion was first made by Yang (2.11). Thus, for a steel column, the critical buckling stress is

$$\sigma_c = \frac{\pi^2 EI_e}{A(KL)^2} = \frac{\pi^2 E}{(KL/r)^2}\left(\frac{I_e}{I}\right) = \frac{\pi^2 E}{(KL/r)^2} f(\eta) \qquad (2.7)$$

Actually, as soon as bending has started with a finite load increment, there will be a slight regression of strain at the most-strained fiber of the cross section, and there will be a reduction of stress at this location in proportion to $E\epsilon$. Only thus can the column develop the greater internal bending resistance requisite for equilibrium with the increased external moment caused by an increase in load above the buckling load. The load-deflection curve of a steel column with residual stress can then be calculated up to and beyond the maximum load. The initial slope can be calculated by an explicit formula (2.19). Initial work involving such calculations of maximum load is summarized by Tall and Estuar (2.20).

Examples of the calculation of I_e/I for Eq. 2.7 are shown in Fig. 2.4 for various cross sections and patterns of residual stress. The first two of these

calculations were given in the paper by Yang, et al. (2.11) and the third was developed by Nitta (2.21). The second calculation of Fig. 2.4 shows the approximate application to wide-flange shapes (web area neglected). Although $f(\eta)$ itself is independent of the initial distribution of residual

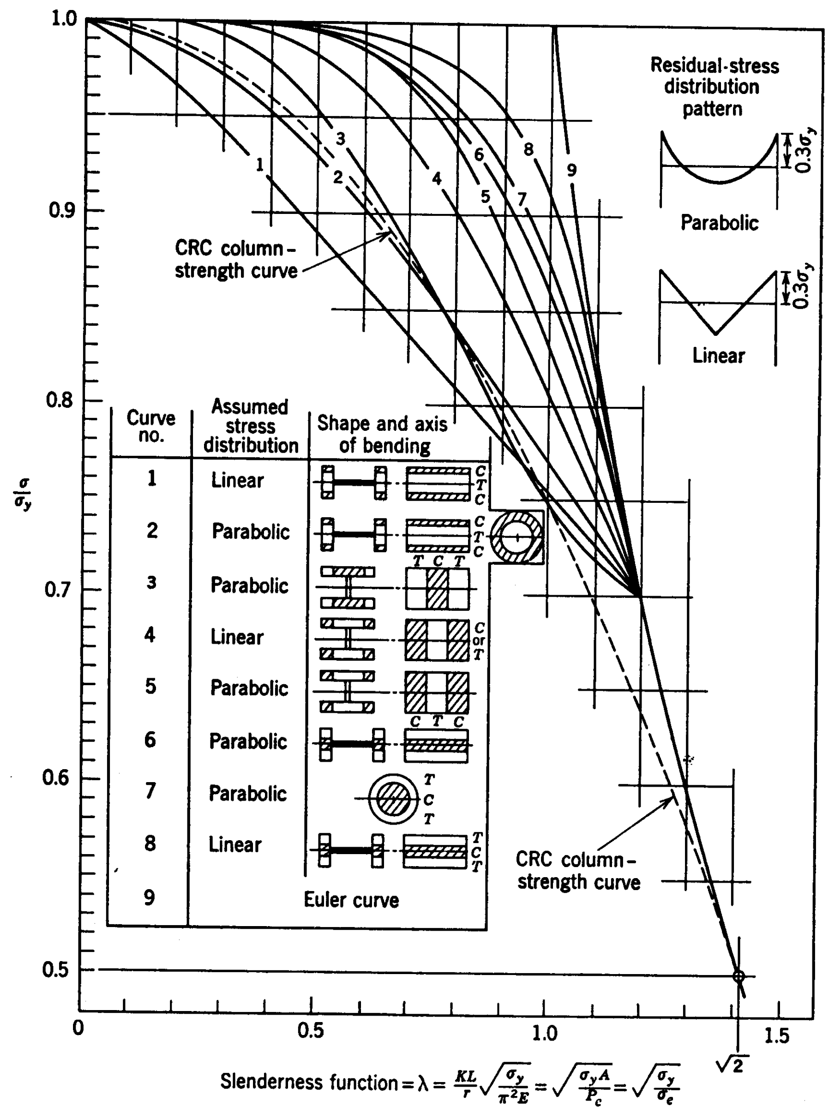


Fig. 2.5. Column strength curves for various patterns of residual stress in various cross sections, based on buckling load.

stress provided that it satisfies the general geometric requirements that will develop the yielded (shaded) areas as indicated in Fig. 2.4, the relationship between the critical stress σ_c and KL/r does depend on residual-stress distribution. This is because the average stress on the column cross section depends on the residual-stress distribution, as shown by Eq. 2.6.

A review of the various Lehigh investigations (2.12, 2.14) concludes that for structural carbon steels the average value of the maximum compressive residual stress σ_{RC} is approximately $0.3\sigma_y$. It should be noted, however, that the residual-stress levels for higher-strength steels do not increase in proportion with yield stress. The various column strength curves in Fig. 2.5 have been plotted on the basis of this average maximum compressive residual-stress level, and are given for two patterns of residual-stress distribution, i.e., parabolic and linear, as indicated. The actual distribution of residual stress in the flange of a wide-flange section, on the basis of many samplings, falls between the parabolic and the linear stress distribution (2.14). Thus, if the column strength of rolled wide-flange structural steel shapes is to be expressed by a single curve, irrespective of axis of bending, it should represent an averaging of curves 1, 2, 3, and 4 of Fig. 2.5. This is accomplished empirically by the dashed line labeled "CRC column strength curve."

Other curves in Fig. 2.5 are considerably above the CRC curve, and, of course, those reflecting residual tension at the most-strained fiber (curves 6, 7, and 8) show the highest strengths. For these improbable cases $f(\eta)$ is $3\eta - 3\eta^2 + \eta^3$ for the rectangular cross section (curves 6 and 8), and $2\eta - \eta^2$ for the circular cross section (curve 7). It should be noted, however, that residual stress *always* causes a *reduction* in column strength. Carefully made tests of straight annealed columns show strengths close to the yield point or to the Euler critical load, whichever is reached first (2.12).

2.4 Column Strength Curves

Bleich (A1) proposed the following parabolic column strength curve for steel in the inelastic range:

$$\sigma_c = \sigma_y - \frac{\sigma_p}{\pi^2 E}(\sigma_y - \sigma_p)\left(\frac{KL}{r}\right)^2 \tag{2.8}$$

This equation is similar to the "Johnson parabola," which was developed before 1890 on a curve-fitting basis (2.23). Eq. 2.8 is not suitable if $\sigma_p < 0.5\sigma_y$, in which case erroneous strength values greater than the Euler buckling stress would be predicted for a certain range of KL/r values.

Since the departure from linearity in the effective stress-strain curve for a steel column is explained by residual stress, the proportional limit should be replaced by

$$\sigma_p = \sigma_y - \sigma_{RC}$$

and, as pointed out in a special report to Column Research Council (2.14), the following result is then obtained for $\sigma_{RC} \leqslant 0.5\sigma_y$:

$$\sigma_c = \sigma_y - \frac{\sigma_{RC}}{\pi^2 E}(\sigma_y - \sigma_{RC})\left(\frac{KL}{r}\right)^2 \tag{2.9}$$

If σ_{RC} is arbitrarily taken as equal to $0.5\sigma_y$, the plot of Eq. 2.9 becomes tangent to the Euler curve at that stress and provides a suitable compromise between weak- and strong-axis buckling of wide-flange sections having an average maximum compressive residual stress of $0.3\sigma_y$ (see curves 1, 2, 3, and 4 of Fig. 2.5). Thus, for $\sigma_{RC} = 0.5\sigma_y$,

$$\sigma_c = \sigma_y - \frac{\sigma_y^2}{4\pi^2 E}\left(\frac{KL}{r}\right)^2 \tag{2.10}$$

Eq. 2.10 is plotted in Fig. 2.5 as "CRC column strength curve." Eq. 2.10 was recommended in the first edition of the *Guide* for the establishment of basic column strength curves for steels. It is particularly applicable to hot-rolled shapes, but further evaluation is needed as to its limitations when applied to shapes built up by welding. It is interesting to note that Eq. 2.10 can be reduced to be identical to Eq. 1.395 of MIL-HDBK5 *Metallic Materials and Elements for Flight Vehicle Structures*, August 1962 (2.25). Also, Winter (2.26) presented the same equation in 1946 and stated that it gave "a very close approximation of the values of the *secant formula* in the low and medium range of KL/r."

Thus, while Eq. 2.10 has been related to the effects of residual stress in wide-flange shapes, it is also consistent with the secant formula with an arbitrary end-eccentricity ratio determined so as to make it fit a large body of steel-column test results. The merit in using the tangent-modulus concept as the basis for the design curve lies in its generality and inherent correctness. There is no rational basis for the equivalent eccentricities that must be assumed to make the secant formula agree with test results. Eq. 2.10 has the further merit that when generalized to apply to the lateral buckling of steel beams, it agrees well with the lower limit of test results referred to in Chapter 4.

In a survey prepared for Column Research Council by Beedle and Huber (2.28) it was recommended that the minimum ASTM Specification yield point in tension be adopted as a suitable value for σ_y in compression as used in Eq. 2.10. This recommendation is hereby extended to all structural high-strength steels for which ASTM Specifications are provided.

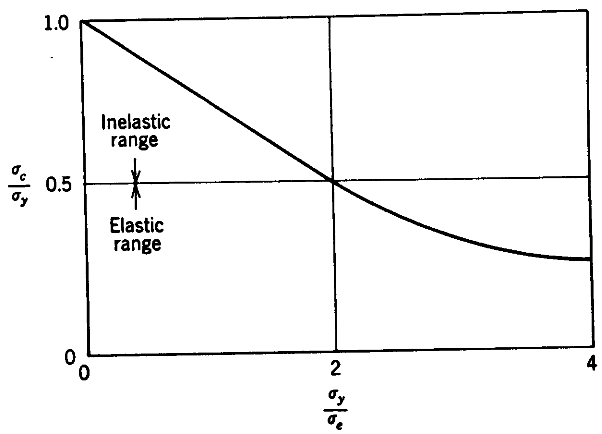


Fig. 2.6. CRC column-strength curve for steel.

In the inelastic range, $\frac{1}{2} < \sigma_c/\sigma_y < 1$, the column strength curve for steel, Eq. 2.10, can be written in a very simple form by introducing the Euler buckling stress as a parameter. Then, in the inelastic range,

$$\frac{\sigma_c}{\sigma_y} = 1 - 0.25\frac{\sigma_y}{\sigma_e} \tag{2.11}$$

For the elastic range, $0 < \sigma_c/\sigma_y < \frac{1}{2}$ and $\sigma_c = \sigma_e$, thus

$$\frac{\sigma_c}{\sigma_y} = \frac{1}{(\sigma_y/\sigma_e)} \tag{2.12}$$

Plotted in terms of these parameters, the CRC basic column strength curve takes the form shown in Fig. 2.6.

Research conducted by Estuar and Tall (2.18) at Lehigh University has shown that fabrication of H-shapes from universal plates by welding introduces tensile residual stresses that approach the yield point of the material. As a result, the effect of welding on the column strength of such *built-up* H-shapes is more adverse than that of residual stresses due to cooling on the column strength of *rolled* H-shapes. In fact, column strength for such welded shapes may be as much as 30% below the values predicted by the CRC basic column strength curve. A summary of test results for various welded column sections is given in Fig. 2.7, with the CRC curve shown by way of comparison. It may be noted here that for welded H-shapes, indications are that the use of flange plates having flame-cut edges reduces the adverse effect of welding. Current research (1966) is expected to provide information on residual stresses due to

welding that will permit specific design recommendations to be formulated for welded built-up columns.

The column strength curve in the elastic range is given for any metal by Eq. 2.2. The following values of elastic moduli are commonly used:

Structural steels	29,000 ksi
Aluminum alloys	
6061-T6,	
6063-T5, 6063-T6	10,000 ksi
Aluminum alloy	
2014-T6	10,600 ksi

Euler stress values are presented in Table 2.2. Although the Euler stress does not represent the actual column strength in the inelastic range, it enters into Eq. 2.11 and is useful in determining the bending amplification factor in problems of combined bending and direct stress (see Chapter 6).

In Table 2.3 are listed the critical average stresses as predicted by Eq. 2.10 for structural steels having yield stresses of 33, 36, 42, 50, 60, 70, 80, and 100 ksi. This progression includes the preferred yield stresses in

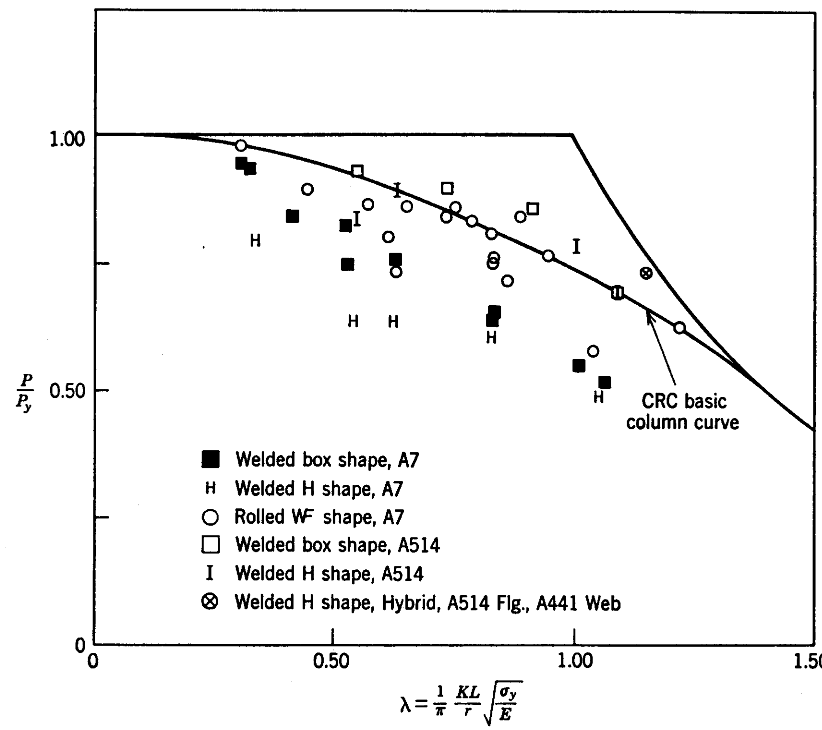


Fig. 2.7. Welded columns and the CRC column curve.

Table 2.2. Euler Formula Stress* for Constructional Metals in ksi

KL/r	Steel	Aluminum 6061-T6 6063-T5 6063-T6	Aluminum 2014-T6	KL/r	Steel	Aluminum 6061-T6 6063-T5 6063-T6	Aluminum 2014-T6	KL/r	Steel	Aluminum 6061-T6 6063-T5 6063-T6	Aluminum 2014-T6	KL/r	Steel	Aluminum 6061-T6 6063-T5 6063-T6	Aluminum 2014-T6
20	715.55	246.74	261.55	50	114.49	39.48	41.85	80	44.72	15.42	16.35	110	23.65	8.16	8.65
21	649.02	223.80	237.23	51	110.04	37.95	40.22	81	43.62	15.04	15.95	111	23.23	8.01	8.49
22	591.36	203.92	216.15	52	105.85	36.50	38.69	82	42.57	14.68	15.56	112	22.82	7.87	8.34
23	541.05	186.57	197.77	53	101.89	35.14	37.24	83	41.55	14.33	15.19	113	22.42	7.73	8.19
24	496.91	171.34	181.63	54	98.15	33.85	35.88	84	40.56	13.99	14.83	114	22.02	7.59	8.05
25	457.95	157.91	167.39	55	94.62	32.63	34.58	85	39.61	13.66	14.48	115	21.64	7.46	7.91
26	423.40	146.00	154.76	56	91.27	31.47	33.36	86	38.70	13.34	14.15	116	21.27	7.33	7.77
27	392.61	135.39	143.51	57	88.09	30.38	32.20	87	37.81	13.04	13.82	117	20.91	7.21	7.64
28	365.07	125.89	133.44	58	85.08	29.34	31.10	88	36.96	12.74	13.51	118	20.56	7.09	7.51
29	340.33	117.36	124.40	59	82.22	28.35	30.05	89	36.13	12.46	13.21	119	20.21	6.97	7.39
30	318.02	109.66	116.24	60	79.51	27.42	29.06	90	35.34	12.18	12.92	120	19.88	6.85	7.27
31	297.83	102.70	108.86	61	76.92	26.52	28.12	91	34.56	11.92	12.63	121	19.55	6.74	7.15
32	279.51	96.38	102.17	62	74.46	25.68	27.22	92	33.82	11.66	12.36	122	19.23	6.63	7.03
33	262.83	90.63	96.07	63	72.11	24.87	26.36	93	33.09	11.41	12.10	123	18.92	6.52	6.92
34	247.59	85.38	90.50	64	69.88	24.10	25.54	94	32.39	11.17	11.84	124	18.61	6.42	6.81
35	233.65	80.57	85.40	65	67.74	23.36	24.76	95	31.71	10.94	11.59	125	18.32	6.32	6.70
36	220.85	76.15	80.72	66	65.71	22.66	24.02	96	31.06	10.71	11.35	126	18.03	6.22	6.59
37	209.07	72.09	76.42	67	63.76	21.99	23.31	97	30.42	10.49	11.12	127	17.75	6.12	6.49
38	198.21	68.35	72.45	68	61.90	21.34	22.63	98	29.80	10.28	10.89	128	17.47	6.02	6.39
39	188.18	64.89	68.78	69	60.12	20.73	21.97	99	29.20	10.07	10.67	129	17.20	5.93	6.29
40	178.89	61.69	65.39	70	58.41	20.14	21.35	100	28.62	9.87	10.46	130	16.94	5.84	6.19
41	170.27	58.71	62.24	71	56.78	19.58	20.75	101	28.06	9.68	10.26	131	16.68	5.75	6.10
42	162.26	55.95	59.31	72	55.21	19.04	20.18	102	27.51	9.49	10.06	132	16.43	5.66	6.00
43	154.80	53.38	56.58	73	53.71	18.52	19.63	103	26.98	9.30	9.86	133	16.18	5.58	5.91
44	147.84	50.98	54.04	74	52.27	18.02	19.10	104	26.35	9.13	9.67	134	15.94	5.50	5.83
45	141.34	48.74	51.66	75	50.88	17.55	18.60	105	25.96	8.95	9.49	135	15.70	5.42	5.74
46	135.26	46.64	49.44	76	49.55	17.09	18.11	106	25.47	8.78	9.31	136	15.47	5.34	5.66
47	129.57	44.68	47.36	77	48.27	16.65	17.65	107	25.00	8.62	9.14	137	15.25	5.26	5.57
48	124.23	42.84	45.41	78	47.04	16.24	17.20	108	24.54	8.46	8.97	138	15.03	5.18	5.49
49	119.21	41.11	43.57	79	45.86	15.81	16.76	109	24.09	8.31	8.81	139	14.81	5.11	5.41

* Directly indicative of column strength only when stress is less than the proportional limit. For steel the effective proportional limit may be taken as one half of the yield point.

Table 2.3. Basic Column Strength in ksi for Structural Metals. Values below those underlined are Euler Stresses for Elastic Buckling.

KL/r (Material)	Steels with Yield Points of 33 ksi	36 ksi	42 ksi	50 ksi	60 ksi	70 ksi	80 ksi	100 ksi	Aluminum Alloy 6061-T6	Aluminum Alloy 2014-T6
0	33.00	36.00	42.00	50.00	60.00	70.00	80.00	100.00	35.0	53.0
5	32.98	35.97	41.96	49.95	59.92	69.89	79.86	99.78	35.0	53.0
10	32.90	35.89	41.84	49.78	59.69	69.57	79.44	99.13	35.0	53.0
15	32.79	35.75	41.65	49.55	59.29	69.04	78.74	98.03	35.0	53.0
20	32.62	35.55	41.38	49.13	58.74	68.29	77.76	96.51	34.0	53.0
25	32.41	35.29	41.04	48.64	58.03	67.33	76.51	94.54	33.0	51.7
30	32.15	34.98	40.61	48.04	57.17	66.15	74.97	92.14	32.0	50.1
35	31.84	34.61	40.11	47.33	56.15	64.76	73.15	89.30	31.0	48.3
40	31.48	34.19	39.53	46.51	54.97	63.15	71.06	86.02	30.0	46.3
45	31.07	33.71	38.88	45.58	53.63	61.33	68.68	82.32	29.1	43.7
50	30.62	33.17	38.15	44.54	52.14	59.30	66.02	78.16	28.1	40.0
55	30.12	32.58	37.34	43.40	50.49	57.05	63.09	73.58	26.9	34.58
60	29.57	31.93	36.45	42.14	48.68	54.59	59.88	68.56	25.4	29.06
65	28.98	31.22	35.49	40.78	46.71	51.92	56.38	63.10	22.8	24.76
70	28.34	30.45	34.45	39.30	44.59	49.03	52.61	57.20	20.14	21.35
75	27.65	29.63	33.33	37.72	42.31	45.93	48.56	50.87	17.55	18.60
80	26.91	28.76	32.14	36.03	39.88	42.61	44.22	44.72	15.42	16.35
85	26.13	27.82	30.87	34.23	37.28	39.08	39.61	39.61	13.66	14.48
90	25.29	26.83	29.52	32.32	34.53	35.33	35.34	35.34	12.18	12.92
95	24.42	25.78	28.09	30.30	31.62	31.71	31.71	31.71	10.94	11.59
100	23.49	24.68	26.59	28.17	28.62	28.62	28.62	28.62	9.87	10.46
105	22.51	23.52	25.01	25.93	25.96	25.96	25.96	25.96	8.95	9.49
110	21.49	22.30	23.36	23.65	23.65	23.65	23.65	23.65	8.16	8.64
115	20.42	21.03	21.62	21.64	21.64	21.64	21.64	21.64	7.46	7.91
120	19.30	19.70	19.88	19.88	19.88	19.88	19.88	19.88	6.85	7.27
125	18.14	18.31	18.32	18.32	18.32	18.32	18.32	18.32	6.32	6.70
130	16.93	16.94	16.94	16.94	16.94	16.94	16.94	16.94	5.84	6.19
135	15.70	15.70	15.70	15.70	15.70	15.70	15.70	15.70	5.41	5.74
140	14.60	14.60	14.60	14.60	14.60	14.60	14.60	14.60	5.03	5.34
145	13.61	13.61	13.61	13.61	13.61	13.61	13.61	13.61	4.69	4.97
150	12.72	12.72	12.72	12.72	12.72	12.72	12.72	12.72	4.39	4.65
155	11.91	11.91	11.91	11.91	11.91	11.91	11.91	11.91	4.11	4.35
160	11.18	11.18	11.18	11.18	11.18	11.18	11.18	11.18	3.86	4.09
165	10.51	10.51	10.51	10.51	10.51	10.51	10.51	10.51	3.62	3.84
170	9.90	9.90	9.90	9.90	9.90	9.90	9.90	9.90	3.41	3.62
175	9.35	9.35	9.35	9.35	9.35	9.35	9.35	9.35	3.22	3.42
180	8.83	8.83	8.83	8.83	8.83	8.83	8.83	8.83	3.05	3.23
185	8.36	8.36	8.36	8.36	8.36	8.36	8.36	8.36	2.88	3.06
190	7.93	7.93	7.93	7.93	7.93	7.93	7.93	7.93	2.73	2.90
195	7.53	7.53	7.53	7.53	7.53	7.53	7.53	7.53	2.60	2.75
200	7.16	7.16	7.16	7.16	7.16	7.16	7.16	7.16	2.47	2.62

addition to the yield stress of A7 steel, which was used in most of the basic research programs.

In the case of aluminum alloys, the ASCE Structural Division Task Committee on Lightweight Alloys of the Metals Committee has evaluated the basic column strength (A20, A21, A22), including the effects of cross-sectional shape. Neither the secant formula nor Eq. 2.10 should be applied to structural aluminum alloys, for which the column strength curves can be determined by the tangent-modulus procedure for the minimum stress-strain curve of the alloy in question. Basic column strength curves so derived for structural aluminum alloys 6061-T6 and 2014-T6 are presented in Table 2.3, in which the upper limit of strength (for $KL/r = 0$) is arbitrarily taken as the specified minimum yield stress at 0.2% offset.

Column Research Council concurs in the column strength curves used as a basis for the allowable column stresses in *ASCE Proceedings Papers 971, 3341, and 3342*, covering specifications for structures of aluminum alloys (A20, A21, A22). Allowable stresses for alloy 2014-T6 (A20) were determined by applying a factor-of-safety directly to the tangent-modulus column curve. Allowable stresses for alloys 6061-T6 (A21) and for alloys 6063-T5 and 6063-T6 (A22) were determined from the tangent-modulus column curve, approximated by straight lines in the inelastic stress range.

2.5 Effect of Initial Curvature

In the case of higher-strength steels, the residual stresses due to the cooling of rolled shapes are lower in proportion to the yield point than for the structural carbon steels. Thus, for higher-strength steels, the effects of initial curvature and other geometric imperfections take on greater relative importance in comparison with the effect of residual stress.

In Fig. 2.8, δ_o is the maximum initial out-of-straightness of a column and δ is the additional deflection caused by bending moment due to the axial load P. If the initial shape of a hinged column were a half sine wave, it can be shown (A9) by elementary beam theory that the

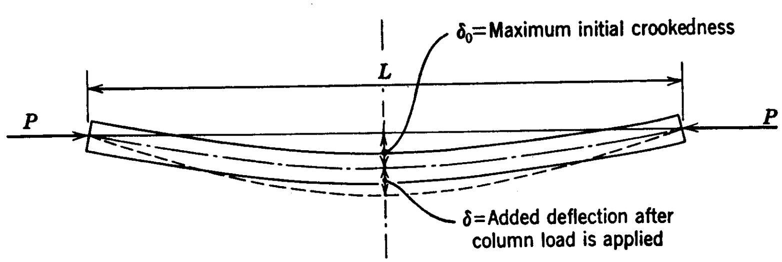


Fig. 2.8. Initially curved column.

total distance from the column thrust line to the centroid of the deflected column is

$$\delta_o + \delta = \frac{\delta_o}{1 - P/P_e} = \frac{\delta_o}{1 - [P(L/r)^2/\pi^2 EA]} \tag{2.13}$$

Eq. 2.13 may also be applied to columns with other end conditions if the initial shape is identical to the buckled configuration. The ratio $1/(1 - P/P_e)$ is sometimes called the "amplification factor."

The maximum bending moment in the initially-curved strut after load P is applied is

$$M_{max} = P(\delta + \delta_o) \tag{2.14}$$

and the maximum combined stress due to column load and bending moment is

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ar^2} \tag{2.15}$$

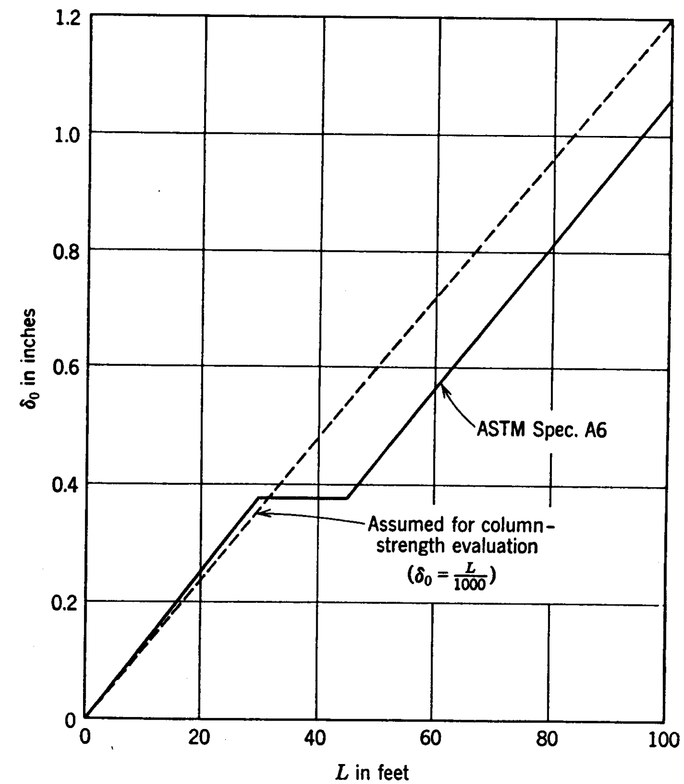


Fig. 2.9. Permissible variation in straightness for wide-flange shapes by ASTM A6.

Combining the foregoing relationships and letting $\sigma_{max} = \sigma_y$, the average stress for which the maximum stress in the column just reaches the yield stress is found as

$$\sigma_a = \frac{\sigma_y}{1 + [\delta_o c/r^2][1/(1 - \sigma_a/\sigma_e)]} \tag{2.16a}$$

or, alternatively, for direct solution,

$$\sigma_a = \frac{1}{2}\left[\left(\sigma_y + \sigma_e\left(1 + \frac{\delta_o c}{r^2}\right)\right) - \sqrt{\left(\sigma_y + \sigma_e\left(1 + \frac{\delta_o c}{r^2}\right)\right)^2 - 4\sigma_y\sigma_e}\right] \tag{2.16b}$$

Eq. 2.16b, known also as the Perry-Robertson formula (2.34), gives a conservative estimate of the strength of an initially-curved column free from residual stress.

In the case of rolled steel sections, the standard mill tolerances for out-of-straightness given in ASTM Specification A6 can be used as a basis for estimating δ_o. These standard tolerances are shown by the full line in Fig. 2.9. The dashed line shows a good straight-line approximation for these tolerances in the usual column length range and is over-conservative in the long column range.

Whereas the buckling load of a perfectly straight steel column in the inelastic range is affected markedly by residual stress, a realistic evaluation of ultimate column strength should include the combined effects of residual stress and initial curvature or crookedness. Ultimate strength can be evaluated by numerical procedures that determine, at successive increments of load and deflection, the column deflection curve up to and beyond maximum load. Such a procedure was used by Ketter and others (2.13) in determining the ultimate strength of eccentrically loaded beam-columns; and it was also used in a series of later investigations, summarized by Galambos (2.22, 2.33), on the ultimate strength of initially curved high-strength steel columns of circular cross section with both concentric and unsymmetrical distributions of residual stress.

More recently, a systematic evaluation of the effect of residual stress and initial curvature, in combination or separately, on the column strength of wide-flange sections of both steel and aluminum alloys has been made by Batterman and Johnston (2.24). Figs. 2.10 and 2.11 show column strength curves based on these studies, for weak-axis bending and strong-axis bending, respectively, of wide-flange steel shapes. Residual stress, when included, is held at a constant level of 10 ksi, and thus is very nearly equal to $0.3\sigma_y$ for structural carbon steel but only 10% of the yield stress for the high-strength steel. The effects of initial crookedness and residual stress may be compared with the idealized strength if both crookedness and residual stress were absent, i.e., with either the Euler stress or the yield point, whichever is less. Such a comparison shows that the maximum

effect of either residual stress or initial crookedness, alone or in combination, always occurs when the slenderness-ratio parameter λ equals unity. It will also be seen that the additive effect of initial curvature, for a given residual-stress magnitude, is greatest at the point where the curve for residual-stress alone meets the Euler curve. For values of λ greater than this Euler intersection point, the effects of initial curvature gradually diminish.

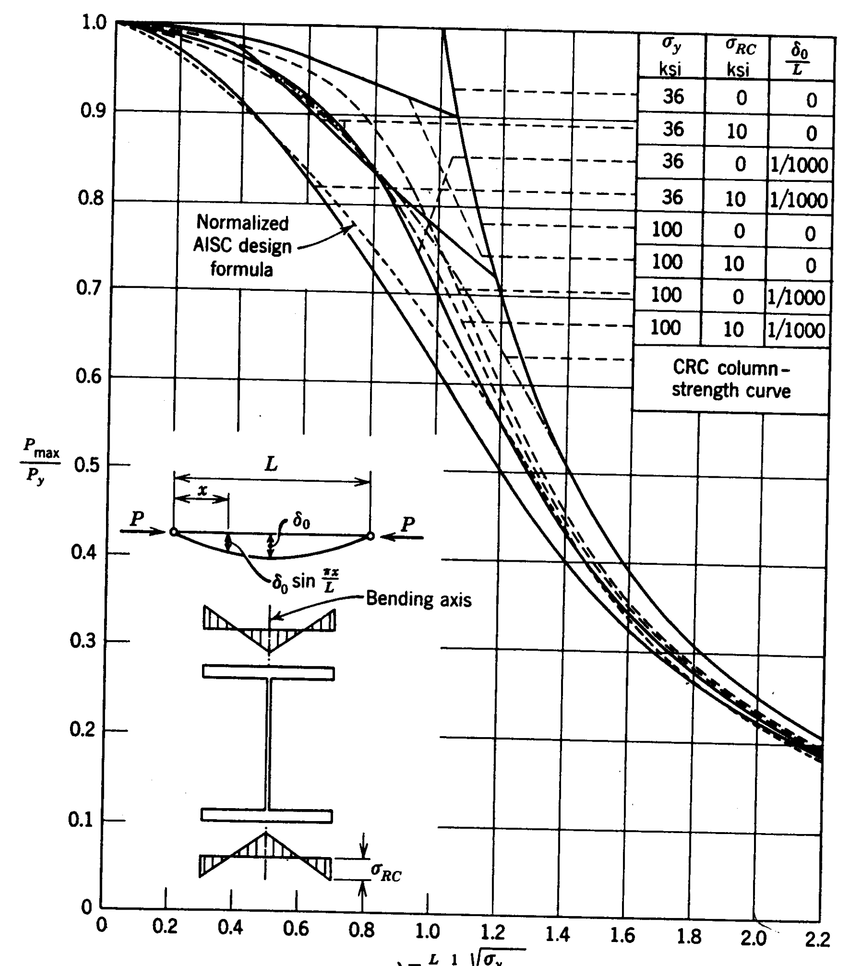


Fig. 2.10. Column strength curves for weak-axis bending of WF steel shapes, including effects of residual stresses and initial crookedness separately and in combination. (Based on Ref. 2.24.)

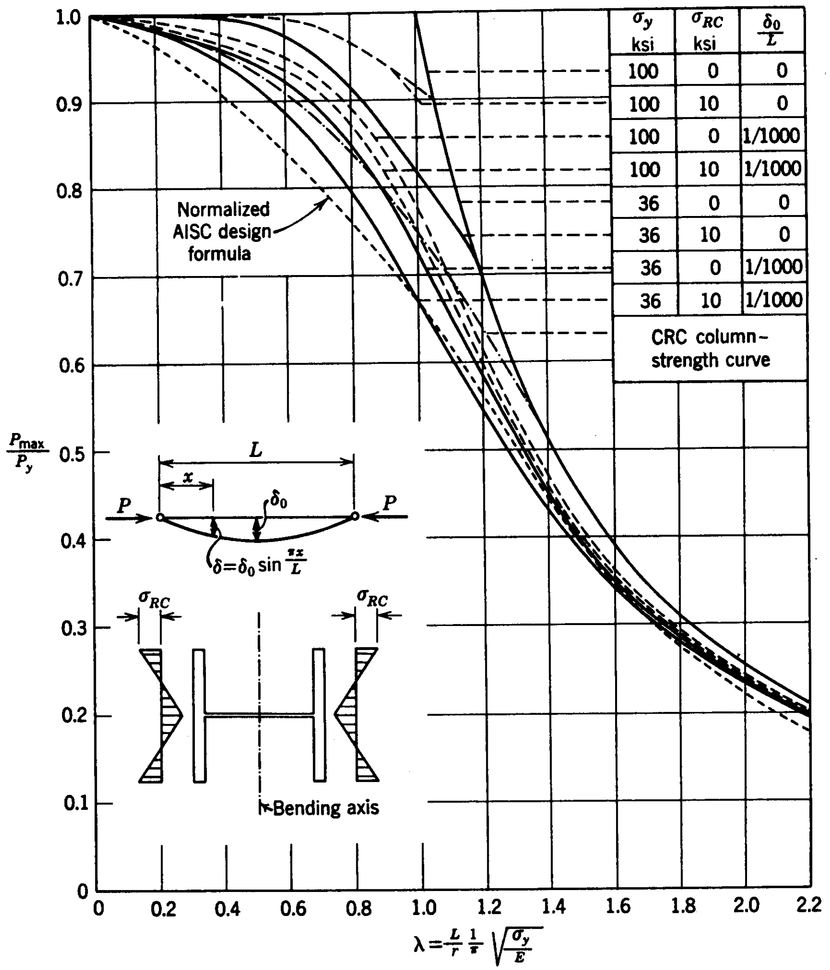


Fig. 2.11. Column strength curves for strong-axis bending of WF steel shapes, including effects of residual stresses and initial crookedness separately and in combination. (Based on Ref. 2.24.)

Figs. 2.10 and 2.11 show that, when the residual stress is held at a constant level, no single nondimensionalized curve is satisfactory for all levels of yield stress. They also show that the CRC curve is more satisfactory for higher-strength than for structural grade steels. Further, support is also given to the use of an increasing factor-of-safety with increasing KL/r, to compensate for effects of initial curvature. It is also shown that the relative effect of residual stress and initial curvature is

more a function of column slenderness than of yield stress or shape of cross section.

A comparison of Figs. 2.10 and 2.11 shows that a combination of a nominal amount of residual stress with a given initial crookedness produces relatively less difference between column strength curves for weak-axis bending and strong-axis bending than does residual stress alone. It should also be noted that the curves in Fig. 2.5 represent buckling loads, whereas Figs. 2.10 and 2.11 depict the maximum column strength. The buckling loads, as can be determined by comparison between Fig. 2.10 or Fig. 2.11 with Fig. 2.5, are somewhat less than the maximum strengths.

For a basic study of the effects of geometric imperfections, type of loading, dynamic disturbances, and other factors, the reader is referred to a paper by Drucker and Onat (2.32).

2.6 Allowable Stresses for Column Design

The question of factor-of-safety has been discussed briefly in Chapter 1. The differing character of the several uncertainties that affect columns makes it inevitable that no single design formula can satisfy all needs.

For a comprehensive review of world-wide column design practice as of 1962, reference may be made to a paper by Godfrey (2.34). Column design formulas of eleven different countries are discussed in this paper. Formulas for structural carbon steels of roughly the same yield stress are plotted in Fig. 2.12, which is adapted from the Godfrey paper.

On the subject of allowable stress in axially loaded compression members, the AISC Specification (A13) reads as follows:

1.5.1.3.1 On the cross section of axially loaded compression members when Kl/r, the largest effective slenderness ratio of any unbraced segment as defined in Sec. 1.8, is less than C_c:

$$F_a = \frac{1 - [(Kl/r)^2/2C_c^2]\sigma_y}{F.S.} \quad \text{Formula (1)} \tag{2.17}$$

where

$$F.S. = \text{factor-of-safety} = \frac{5}{3} + \frac{3(Kl/r)}{8C_c} - \frac{(Kl/r)^3}{8C_c^3} \tag{2.18}$$

and

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_y}} \tag{2.19}$$

1.5.1.3.2 On the cross section of axially loaded columns when Kl/r exceeds C_c:

$$F_a = \frac{149{,}000{,}000}{(Kl/r)^2} \quad \text{Formula (2)} \tag{2.20}$$

The appendix to the AISC Specification provides tables of numerical values for the various grades of steel used in building construction. The commentary (A14) to the Specification points out that "Formula (1) is founded upon the basic column strength estimate suggested by the Column Research Council."

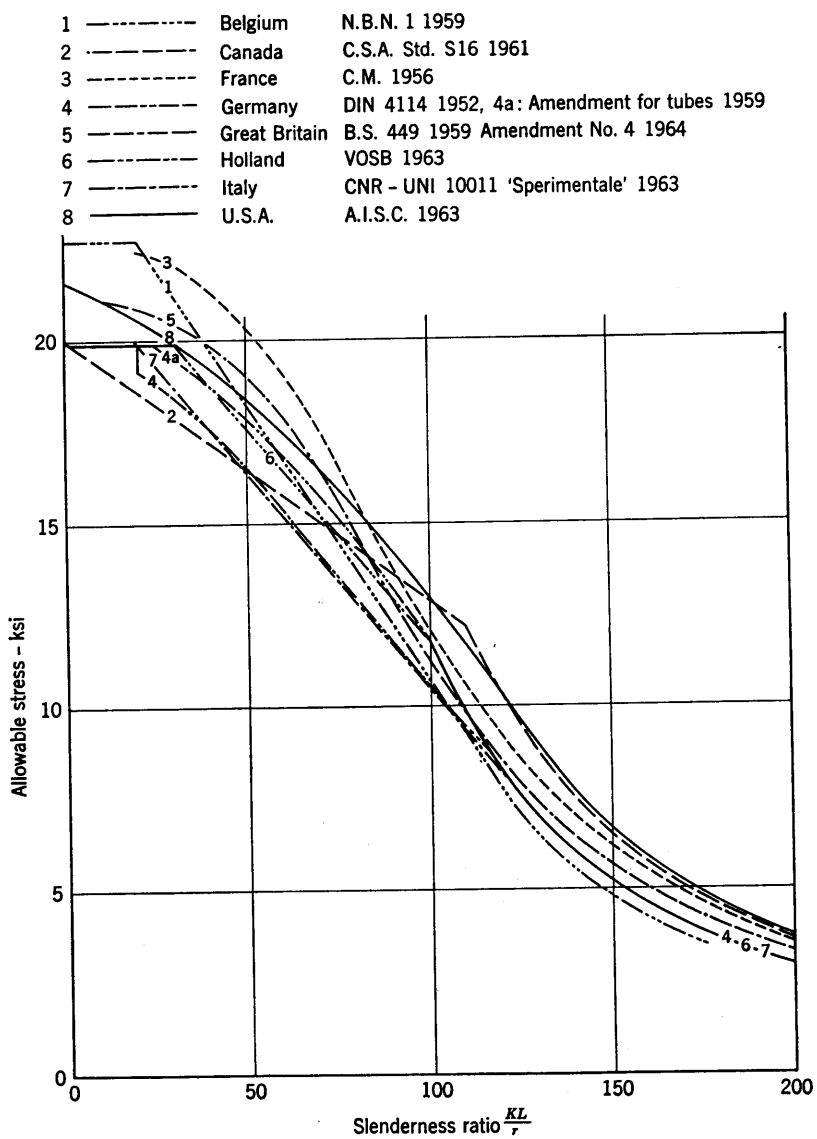


Fig. 2.12. Allowable stresses in axially loaded structural grade steel columns.

For very short columns the AISC factor-of-safety has been taken as equal to, or only slightly greater than, that required for members axially loaded in tension. Similar provisions have been included in the British and German design standards for some time and can be justified by the insensitivity of such members to accidental eccentricities. For longer columns, approaching the Euler slenderness range, the AISC factor-of-safety is gradually increased 15%, resulting in good agreement with column strength based on the combined effect of nominal crookedness and residual stress as shown by the normalized AISC design formulas drawn in Figs. 2.10 and 2.11. This confirms use of the maximum factor-of-safety in the intermediate slenderness range, in the vicinity of $\lambda = 1$. The factor-of-safety probably should not be reduced for values of $\lambda > 1$, because of the increasing sensitivity of long columns to variations in the effective-length factor K and the practical difficulty in determining the magnitude of K in an actual structure.

In further studies directed toward the goal of continued improvement of design criteria, attention should be given to the effect of end restraint and frame action in mitigating the adverse effect of initial crookedness.

The CISC Code incorporates the Canadian Standards Association specifications (A15), which are also based on recommendations of Column Research Council insofar as they pertain to compression members. However, the Canadian specification places more emphasis on the lowest column strength curve shown in Fig. 2.5 (i.e., curve 1), which is for weak-axis bending of a wide-flange section with a linear distribution of residual stress in each flange. This curve is approximately a straight line from $\sigma_c = \sigma_y$ to the Euler curve, and the CISC requirement is a straight-line formula in this region, as follows:

$$F_a = (20{,}000 - 70\ KL/r)\left(\frac{\sigma_y}{33{,}000}\right) \tag{2.21}$$

but not to exceed

$$\frac{145{,}000{,}000}{(KL/r)^2}$$

As in the case of the AISC formula, the factor-of-safety increases with increasing KL/r and reaches a value of about 1.97 when the slenderness ratio falls in the range governed by the Euler formula.

In general, bridge design formulas for columns (A16 and A17) are also of the parabolic type, but without a transition to an Euler-type formula for very slender members. Extremely slender members are not permitted in bridge construction.

The basis for both the AREA and the AASHO column formulas is the secant formula, along with an allowance for accidental end eccentricity of load (see Chapter 6). The effect of initial out-of-straightness is included, but residual stress and inelastic bending strength are not considered. These column formulas also include arbitrary effective-length coefficients K of 0.875 and 0.750 for members with pinned and riveted (or bolted) ends, respectively.

Column formulas specified for light-gage cold-formed steel construction (A11) are similar in form to Eq. 2.10 in the short-column range, and are identical with Eq. 2.20 for the long-column range. In the short-column range a reduction factor "Q" (A11) is introduced as a multiplier in order to account for loss in effective section owing to local buckling. Q factors for stiffened plate elements and for unstiffened plate elements are not the same.

The column formulas in the specifications for aluminum structures of the ASCE Task Committee on Lightweight Alloys (A21, A22) are based on the Euler equation and a straight-line approximation to the tangent-modulus curve in the inelastic range. A uniform factor-of-safety of 1.95 is used for building structures, while the factor-of-safety for bridge structures is 2.20. Effects of initial out-of-straightness are considered to be compensated for by use of a conservative estimate for K (2.31).

2.7 Torsional Buckling Strength

Short thin-walled columns of open cross section, and having the shear-center axis coincident with the centroidal axis, may have a lower critical load in torsion than in flexure. If the shear-center axis is not coincident with the centroidal axis, torsional buckling will always be accompanied by bending. Long columns of compact* cross section, as well as all box-section columns, need not be investigated for torsional buckling. However, single angles and tee sections are susceptible to torsional buckling. In the case of the single angle with equal legs, the local-buckling strength (see Chapter 3) gives an approximation of the torsional buckling load.

For point-symmetric sections in which the shear-center axis and the longitudinal centroidal axis coincide, simple expressions give the torsional buckling stress, which will govern if it is smaller than the Euler buckling stress. If the ends are free to warp but restrained against relative rotation about the longitudinal axis, the torsional buckling stress is (A1):

$$\sigma_c = \frac{JG}{I_p} + \frac{\pi^2 C_w E}{I_p L^2} \qquad (2.22)$$

The following description of the most general type of displacement possible during torsional-flexural buckling is taken from the recent paper by Chajes and Winter (2.29). The general displacements consist of bending

* Compact in the general sense of having *relatively* small width-thickness ratios for the component parts of the cross section.

about both principal axes and twisting about the shear-center axis. When a column buckles in this manner, its cross section undergoes translations u and v in the x and y directions and a rotation ϕ about the shear center (see Fig. 2.13). Equilibrium of a longitudinal element of a column deformed in this manner leads to the following differential equations:

$$EI_y u^{IV} + P(u'' + y_o\phi'') = 0 \qquad (2.23)$$

$$EI_x v^{IV} + P(v'' + x_o\phi'') = 0 \qquad (2.24)$$

$$C_w\phi^{IV} - (GJ - r_o^2P)\phi'' - Px_ov'' + Py_ou'' = 0 \qquad (2.25)$$

All derivatives are with respect to z, the direction along the axis of the member. Eqs. 2.23 and 2.24 express the equilibrium of the forces tending to bend an element of the column about the y and x axes respectively. I_y and I_x are the principal moments of inertia of the section and y_o and x_o are the distances between the shear center and the centroid in the two principal directions. Eq. 2.25 expresses the equilibrium of the forces

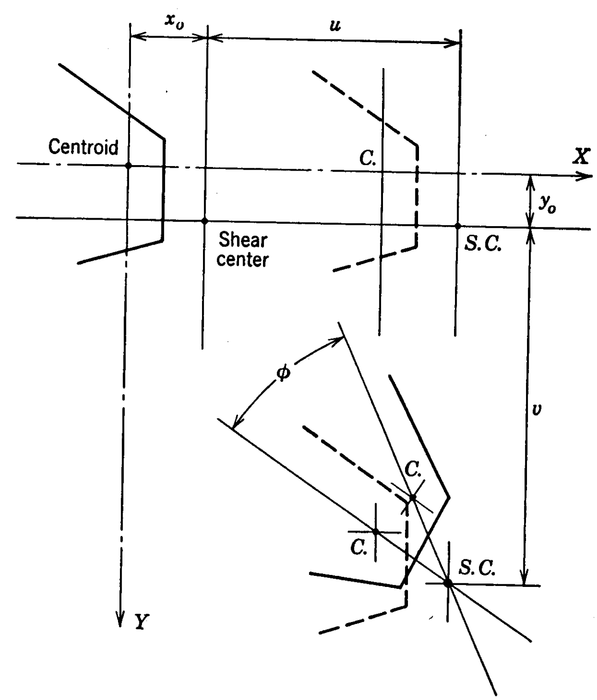


Fig. 2.13. Displacement of cross section during torsional-flexural buckling. (From Ref. 2.29.)

tending to twist an element of the member about the shear-center axis. In this equation r_o is the polar radius-of-gyration of the section about its shear center and C_w is the torsion warping constant.

Eqs. 2.23 to 2.25 hold for an axially loaded column with arbitrary boundary conditions. They are linear equations based on the assumptions of small deformations and elasticity, and their solution gives the loads at which a state of neutral equilibrium is possible. For a member with completely fixed ends,

$$\left.\begin{aligned} u = v = \phi = 0 \\ u' = v' = \phi' = 0 \end{aligned}\right\} \quad \text{at } z = 0, z = L \tag{2.26}$$

and for a member with so-called hinged boundary conditions,

$$\left.\begin{aligned} u = v = \phi = 0 \\ u'' = v'' = \phi'' = 0 \end{aligned}\right\} \quad \text{at } z = 0, z = L \tag{2.27}$$

Eqs. 2.23 to 2.25 lead to the following characteristic equation:

$$r_o^2(P_{cr} - P_y)(P_{cr} - P_x)(P_{cr} - P_\phi) - P_{cr}^2 y_o^2(P_{cr} - P_x) - P_{cr}^2 x_o^2(P_{cr} - P_y) = 0 \tag{2.28}$$

whose roots P_{cr} are the three possible buckling loads of the member.

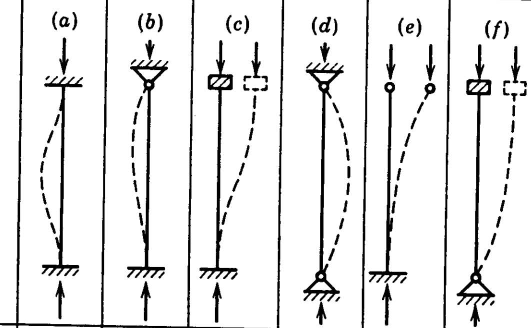
For thin-walled sections of constant thickness that are also singly symmetric, such as a channel or equal-legged angle, the Chajes-Winter procedure greatly simplifies evaluation of buckling loads. Charts are presented to indicate whether the section fails in bending alone, in twisting alone, or in combined bending and twisting. With this determined, simple formulas together with charts permit rapid determination of the buckling load. A wide variety of shapes, representing most of the commonly used cold-formed sections, is covered.

A very complete review of the torsional buckling problem is provided by Kollbrunner and Meister (2.39), and the subject is also treated by Timoshenko and Gere (A9) and by Bleich (A1).

2.8 Effective Length of Framed Columns

The determination of the effective-length factor K in Eqs. 2.1 through 2.3 and 2.8 through 2.10 will now be considered. Although applicable to both trusses and continuous frames, the coverage of this section is limited to those framed columns with no intentional bending moment at or between the ends as a result of frame action or intermediate lateral loads. (The beam-column with framed ends is treated in Chapter 6.)

Fig. 2.14 gives theoretical K values for idealized conditions in which the rotational and/or translational restraints at the ends of the column are either fully realized or are nonexistent. At the base, shown fixed under



	(a)	(b)	(c)	(d)	(e)	(f)
Buckled shape of column is shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended K value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.10	2.0

End condition code		
	Rotation fixed	Translation fixed
	Rotation free	Translation fixed
	Rotation fixed	Translation free
	Rotation free	Translation free

Fig. 2.14. Effective-length factors K for centrally loaded columns with various idealized end conditions.

conditions (a), (b), (c), and (e) in Fig. 2.14, the condition of full fixity can be approached only when the column is anchored securely to a footing for which the rotation is negligible. Column conditions (a), (c), and (f) are approached when the top of the column is integrally framed to a girder many times more rigid than the column. Column condition (c) is the same as (a) except that translational restraint is either absent or minimal at the top. Condition (f) is the same as (c) except that there is no rotation restraint at the bottom. The recommended design values of K are modifications of the ideal values, taking into account the fact that neither perfect fixity nor perfect flexibility can be attained in practice.

The K of 2.0 for condition (f) is not by any means the upper limit. In Fig. 2.15, for example, in which a column hinged at the base is attached to a flexible beam at the top, with sidesway not prevented, the value of K exceeds 2.0 and approaches infinity as the beam stiffness approaches zero. Obviously, the use of a very flexible beam in this situation would not be acceptable.

The more general determination of K for a compression member as part of any framework requires the application of methods of indeterminate

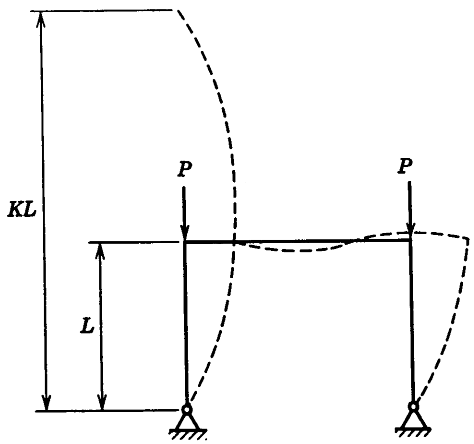


Fig. 2.15.

structural analysis, modified to take account of the effects of axial load and inelastic behavior on the rigidity of the members. Gusset-plate effects can be included; for this case Refs. 2.41 and 2.42 provide extensive charts for modified slope-deflection equations, and for moment-distribution stiffness and carry-over factors, respectively. These procedures are not directly applicable to routine design, but they can be used to determine end restraints and resultant modified effective lengths (KL) of the component members of a framework.

The effective-length factor K (defined in connection with Eq. 2.2) can be determined from the solution of an equation for the buckling of a compression member with end restraints, provided that the magnitudes of these restraints are known or can be approximated. Two conditions, both for columns of constant cross section, will be considered:

(*a*) Buckling of a compression member with known rotational (flexural) restraints at its ends, but with no translation at either end (see Fig. 2.16).

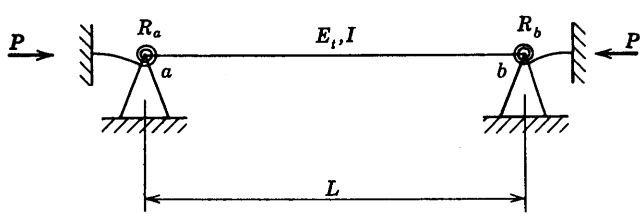


Fig. 2.16. Rotational end restraint (no end translation).

The following equation applies:

$$(R_a + R_b) + \frac{R_a R_b}{M'} + M'' = 0 \qquad (2.29)$$

where

R_a and R_b = rotational stiffness (in units such as lb-ft per radian) of the restraints at ends a and b,

M' = rotational stiffness of the near end of the member with the far end fixed and no translation at either end:

$$M' = 4S'\left(\frac{E_t I}{L}\right) \qquad (2.30)$$

M'' = rotational stiffness of the near end of the member with the far end pinned and no translation at either end:

$$M'' = 4S''\left(\frac{E_t I}{L}\right) \qquad (2.31)$$

S' and S'' are trigonometric functions of E_t, I, L, and the applied load P, and are tabulated in many references (A1, A9, 2.41, 2.42).

To design a column for a given load and end restraints using Eq. 2.29, one selects a trial cross section and determines its area and moment of inertia. The average stress P/A enables determination of E_t, and M' and M'' can then be found. At this stage, Eq. 2.29 would be applied to test the trial section. If a check is obtained, the section is satisfactory. If not, a new trial section is selected and the procedure is repeated.

(*b*) Buckling of a compression member with known rotational and translational restraints at both ends (the translational restraints may be replaced by an equivalent restraint at one end only) (see Fig. 2.17). For this case, Eq. 2.29 applies with the exception that M' and M'' are replaced by the more complex expressions $\overline{M}'$ and $\overline{M}''$ respectively, representing rotational stiffness of the near end of the member with the far end fixed

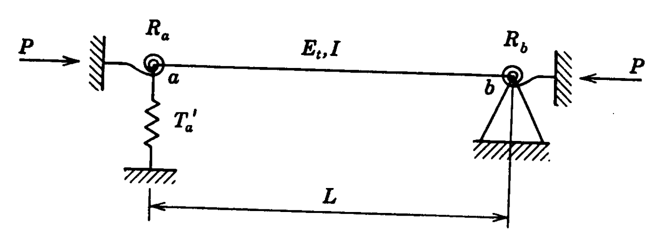


Fig. 2.17. Rotational and translational end restraint.

and pinned, respectively, but with the near end translationally restrained by a linear spring. Equations are as follows:

$$\overline{M}' = M' \left[\frac{(1 - C) + \tau(1 + C)}{2}\right] \qquad (2.32)$$

$$\overline{M}'' = M'' \left[\frac{2\tau}{\tau(1 + C) + (1 - C)}\right] \qquad (2.33)$$

where

$$\tau = \frac{T'_a - P/L}{T'_a + T'}$$

T'_a = translational stiffness of the spring at a (in units such as pounds per inch),

$$T' = \frac{E_t I}{L^3}\left[8S'(1 + C) - \frac{PL^2}{E_t I}\right] \qquad (2.34)$$

= translational stiffness of member ab under action of a transverse force at a, and

C = carry-over factor modified for effect of axial load (see, for example, Refs. 2.41 and 2.42).

The application to design would again be by trial and error, with repeated selection of trial cross sections until Eq. 2.29, modified as just noted, is satisfied.

The calculation of the effects of frame restraint on column behavior by the foregoing briefly sketched procedure is too complex for design use. Application to design can be simplified by the use of charts, derived by similar methods, which provide a direct approximation of the effective-length factor K (2.35, 2.36, 2.40).

In triangulated truss frameworks, loads are usually applied only at the joints, producing only axial loads in the members if the joints are hinged. Deflections of the joints are owing, then, to the axial deformations of the members under load and are, therefore, relatively small. On the other hand, if the joints are welded or heavily bolted or riveted, some secondary bending is induced. The effect of secondary distortions on the buckling strength of truss members is usually small and can be neglected in the buckling analysis.

If every member in a truss were designed to minimum weight, buckling stresses in compression members and yield stresses in tension members would be approached at the same level of live load. On this basis, no restraint would be supplied at the joints, and K would be unity for compression chords and the equivalent lengths would be equal to the full

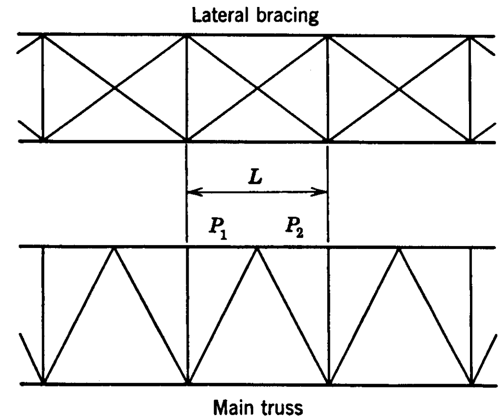


Fig. 2.18.

distance between panel points. In a roof truss of nearly constant depth, where a single compression chord of constant cross section is used for the full length of the truss, K may be taken as 0.9. In a continuous truss, K may be taken as 0.85 for the compression chord connecting to the joint where the chord stress changes from compression to tension.

When the magnitude of stress in the compression chord changes at a subpanel point that is not braced normal to the plane of the main truss (Fig. 2.18), the effective-length factor for chord buckling normal to the plane of the main truss can be approximated from the two compressive forces P_2 and P_1, as follows:

$$K = 0.75 + 0.25\frac{P_2}{P_1} \qquad (2.35)$$

where $P_2 < P_1$.

Web members in trusses which are designed for moving live-load systems may be designed with $K = 0.85$. This is because the position of live load which produces maximum stress in the web member being designed will result in less-than-maximum stresses in members framing into it, so that rotational restraints will be developed. K should be taken as unity for web members in a truss designed for a fixed load system, where maximum stress occurs in all members simultaneously.

The design of vertical web members, U_iL_i, of a K-braced truss (Fig. 2.19) should be based on the length KL. Web-member buckling occurs normal to the plane of the truss, and Eq. 2.35 again applies. P_2 is to be taken as negative in Eq. 2.35, since it is tension. When P_1 and P_2 are numerically equal, Eq. 2.35 yields a value of $K = 0.5$.

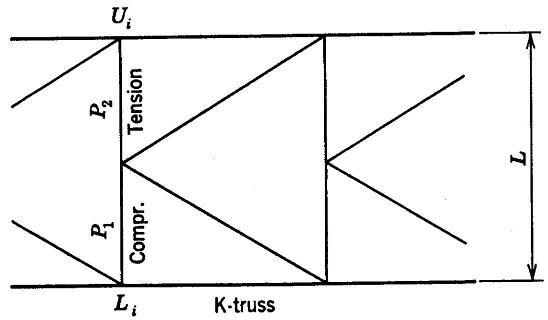


Fig. 2.19.

For buckling normal to the plane of a main truss, the web-compression members should be designed for $K = 1$ unless detailed knowledge of the make-up of the cross frames (perpendicular to the truss) is available. For example, with cross frames of Type 1 (Fig. 2.20) it is satisfactory to take $K = 0.8$, and for Type 2 it is satisfactory to use $K = 0.7$, provided that the top and bottom lateral bracing systems are adequate to prevent joint translation. Where translation of the cross frames is possible, a more exact analysis of web-member stability should be undertaken.

In the case of redundant trusses, there is a reserve strength above the load at initial buckling of any compression member. Masur (2.37) has reviewed developments on this subject and established upper and lower bounds for the ultimate load of buckled members of elastic redundant trusses.

For more accurate evaluation of the effective length of columns in nontriangulated continuous frames, two convenient alignment charts (2.30) were prepared by L. S. Lawrence for incorporation in the Boston Building Code. These charts (see Fig. 2.21) provide a rapid means of estimating effective column lengths in continuous frames in which side-sway is either prevented (Fig. 2.21*a*) or not prevented (Fig. 2.21*b*). The

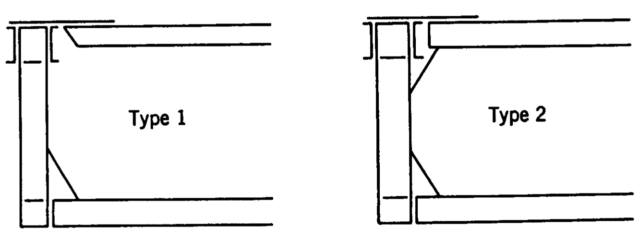


Fig. 2.20.

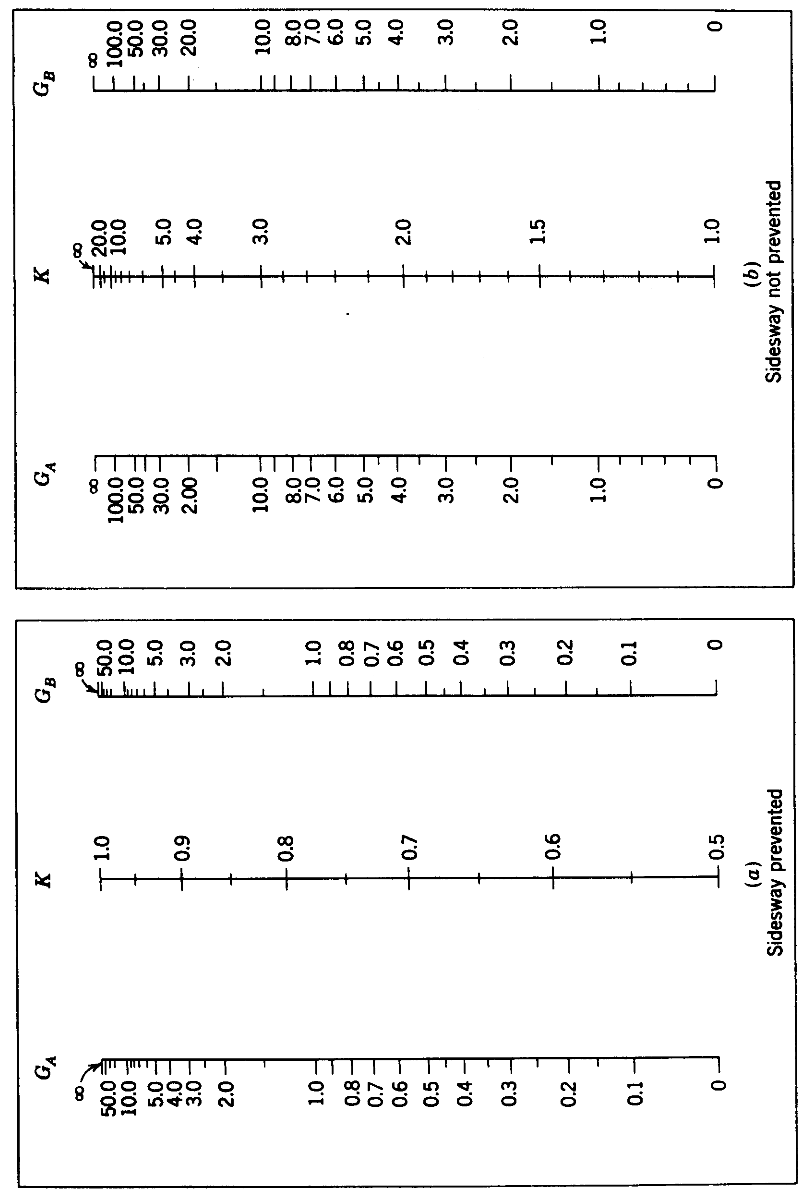


Fig. 2.21. Alignment charts for effective length of column in continuous frames. (*Courtesy of Jackson & Moreland Division of United Engineers & Constructors, Inc.*)

charts are based on the assumption that all columns in the framework reach their individual critical loads simultaneously. The equations upon which these alignment charts are based are as follows:

Sidesway prevented:

$$\frac{G_A G_B}{4}\left(\frac{\pi}{K}\right)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{2\tan(\pi/2K)}{\pi/K} = 1 \quad (2.36)$$

Sidesway not prevented:

$$\frac{G_A G_B(\pi/K)^2 - 36}{6(G_A + G_B)} = \frac{\pi/K}{\tan(\pi/K)} \quad (2.37)$$

These equations can be derived from Eq. 2.29, using Eqs. 2.30 and 2.31 when sidesway is prevented, and Eqs. 2.32 and 2.33 when sidesway is not prevented.

In Fig. 2.21 the subscripts A and B refer to the joints at the two ends of the column section being considered. G is defined as

$$G = \frac{\sum I_c/L_c}{\sum I_g/L_g} \quad (2.38)$$

in which $\sum$ indicates a summation for all members rigidly connected to that joint and lying in the plane in which buckling of the column is being considered, I_c is the moment-of-inertia and L_c is the corresponding unbraced length of the column section, and I_g is the moment-of-inertia and L_g the corresponding unbraced length of the girder or other restraining member. I_c and I_g are taken about axes perpendicular to the plane of buckling.

For a column base connected to a footing by a frictionless hinge, G is theoretically infinite but should be taken as 10 in design practice. If the column base is rigidly attached to a properly designed footing, G approaches a theoretical value of zero but should be taken as 1.0. Other values may be used if justified by analysis.

The girder stiffness I_g/L_g should be multiplied by a factor when certain conditions at the far end are known to exist. For the case with sidesway prevented (Fig. 2.21*a*), the appropriate multiplying factors are as follows:*

1.5 for far end of girder hinged, and
2.0 for far end of girder fixed against rotation

* These factors can easily be verified if it is kept in mind that the girder rigidity assumed in Eq. 2.36 is $2EI/L$ for symmetrical buckling (no sidesway) and $6EI/L$ (in Eq. 2.37) for antisymmetrical buckling (sidesway).

For the case with sidesway not prevented (Fig. 2.21*b*), the multiplying factors are:

0.5 for far end of girder hinged, and
0.67 for far end of girder fixed

Having determined G_A and G_B for a column section, K is obtained by constructing a straight line between the appropriate points on the scales for G_A and G_B. For example, in Fig. 2.21*a* if G_A is 0.5 and G_B is 1.0, K is found to be 0.73.

An example will now be presented to illustrate the application of the charts of Fig. 2.21. The columns of the frame shown in Fig. 2.22 will be analyzed. This frame presents a variety of column end conditions. Where hinges are shown, it is assumed that ordinary connections are involved that are not designed for moment resistance, but nevertheless provide a small rotational restraint that justifies the use of nominal values of G as previously noted. Also, it must be assumed that if sidesway is not prevented, there is a nearly complete absence of lateral restraining media, such as walls or elevator shafts, that would normally exist in conventional non-braced buildings. (When sidesway is prevented by such means, the columns can be considered as restrained against sidesway.)

Illustrative Example 2.1

All columns of Fig. 2.22 are oriented with their webs parallel to the plane of the drawing, and are sufficiently braced normal to their webs so that KL/r with respect to the strong axis controls.

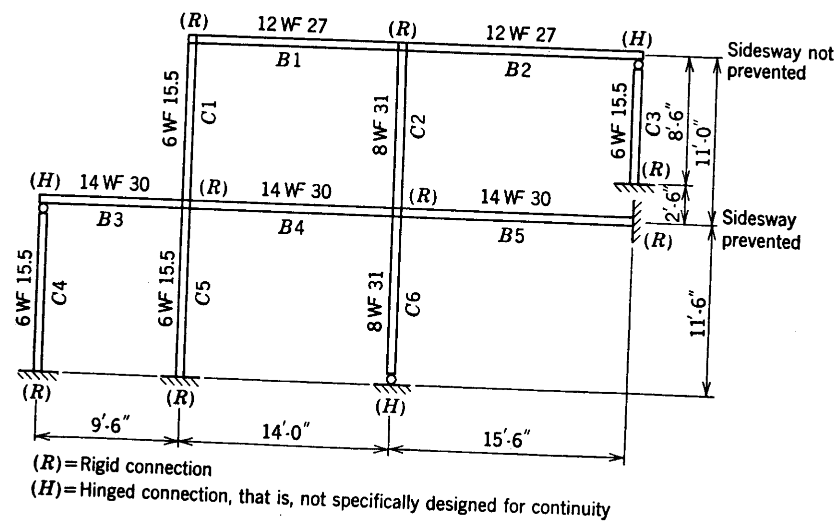


Fig. 2.22.

Beam I_g/L_g (in.^{3})	*Column* I_c/L_c (in.^{3})
B1: 204.1/168 = 1.215	C1: 30.3/132 = 0.230
B2: 204.1/186 = 1.097	C2: 109.7/132 = 0.831
B3: 289.6/114 = 2.540	C3: 30.3/102 = 0.297
B4: 289.6/168 = 1.724	C4: 30.3/138 = 0.220
B5: 289.6/186 = 1.557	C5: 30.3/138 = 0.220
	C6: 109.7/138 = 0.795

For any particular column, let $G_A = G$ for upper end, and $G_B = G$ for lower end. For 6W15.5, $r = 2.56$ in.; and for 8W31, $r = 3.47$ in.

Column C1: $$G_A = \frac{0.230}{1.215} = 0.189$$
$$G_B = \frac{0.230 + 0.220}{(0.5)(2.540) + 1.724} = 0.151$$

Column C2: $$G_A = \frac{0.831}{1.215 + (0.5)(1.097)} = 0.471$$
$$G_B = \frac{0.831 + 0.795}{1.724 + (0.67)(1.557)} = 0.589$$

Columns C3 and C4: $$G_A = 10.0$$
$$G_B = 1.0$$

Column C5: $$G_A = \frac{0.230 + 0.220}{(1.5)(2.540) + 1.724} = 0.081$$
$$G_B = 1.0$$

Column C6: $$G_A = \frac{0.831 + 0.795}{1.724 + (2.0)(1.557)} = 0.336$$
$$G_B = 10.0$$

Allowable Axial Stresses in ksi for Columns Made of Steels Having Yield Points as Indicated (Based on AISC Specification (A13))

Column	K	L/r	KL/r	Based on $K = 1$			K determined from Fig. 2.21		
				36 ksi	50 ksi	100 ksi*	36 ksi	50 ksi	100 ksi*
C1	1.06[1]	51.6	54.7	18.20	24.10	40.76	17.93	23.60	39.10
C2	1.18[1]	38.0	44.8	19.35	26.11	47.51	18.80	25.14	44.30
C3	1.89[1]	39.8	75.2	19.21	25.86	46.68	15.88	19.95	26.39
C4	0.86[2]	53.9	46.4	18.00	23.74	39.51	18.66	24.90	43.37
C5	0.65[2]	53.9	35.0	18.00	23.74	39.51	19.58	26.51	48.86
C6	0.78[2]	39.8	31.0	19.21	25.86	46.86	19.87	27.03	50.56

[1] From Fig. 2.21*b*. [2] From Fig. 2.21*a*.
* Not included in AISC Specification.

Note that for the first three columns (C1, C2, and C3), in which the effective length is greater than the actual length, it becomes increasingly more unsafe to disregard this difference as the yield point of the steel increases. Conversely, for the last three columns (C4, C5, and C6), where the effective length is less than the actual length, it becomes increasingly more uneconomical to disregard this difference as the yield point of the steel increases.

It has been shown in some applications that the matter of effective length is of little or no importance in the design of framed columns, while in other instances it may become a paramount factor. For structural-grade steel columns (σ_y in the vicinity of 36 ksi) with slenderness ratios less than about 60, the critical buckling load for a fixed-end column (type (a), Fig. 2.14) is within 10% of that of a pinned-end column (type (d), Fig. 2.14). Thus, the introduction of K-values has only a minor effect on building columns having small L/r ratios. The same 10% differential is found for columns of higher-strength steels (σ_y of 50 to 60 ksi) having L/r ratios less than about 50, and for columns of heat-treated high-strength alloy steels (σ_y in the neighborhood of 100 ksi) having L/r ratios less than 35.

For aluminum alloy columns the L/r value corresponding to a 10% differential between fixed- and hinged-end column strength varies from about 40 for the higher-strength alloys (σ_y about 50 ksi) down to 30 for those of lower strengths (σ_y about 30 ksi).

The difference in buckling loads for columns in the hinged- and fixed-end conditions increases steadily as the slenderness ratio rises. The strength of a fixed-end column may be 100% greater than that of the hinged-end column at an L/r of about 100 for steel columns and 70 for aluminum alloy columns. Thus, the use of K-values results in appreciable economies for columns having large slenderness ratios. The increasing sensitivity of columns to end restraint as L/r increases suggests the desirability of giving careful attention to end-restraint conditions when the members are slender.

2.9 Columns of Variable Cross Section

Important examples of variable cross section columns include tapered columns, such as derrick booms; and stepped columns, for which both the cross section and the load are variable, such as are found in mill buildings with crane runways. These are designed not by routine formulas, but on the basis of special structural analyses which may be aided by specially prepared charts or tables. A very complete compilation of such charts and tables is provided in Refs. 2.39 and A2. In certain cases, numerical procedures (2.43) may be applied. Charts giving the critical elastic buckling load for tapered columns of various cross section have also been developed (2.46).

Caution must be used in estimating allowable loads for variable-section columns if the proportional limit is exceeded at stresses produced by the allowable load multiplied by the design factor-of-safety. In such cases the tangent-modulus effect will vary along the column, resulting in a problem solvable by numerical methods (2.43, 2.44).

2.10 Lateral-Bracing Requirements

Structural bracing as discussed herein has as its purpose the maintenance of alignment of compression members, so as to permit them to develop maximum strength. Good initial alignment of structural components obtained by high-quality fabrication will minimize the forces in braces. Sporadic experimental and analytical studies tend to confirm the correctness of usual practice, which is to design transverse braces for 2% of the maximum compressive axial force in the compression element that is being braced. For a general study of the bracing problem, reference should be made to the report by Winter (2.47).

Bracing must have rigidity as well as strength. If strength requirements are met, rigidity will usually be sufficient, but braces must be securely anchored. For example, braces should not be attached to a structure equivalent to that being braced unless the whole assemblage is trussed in order to prevent concurrent buckling. The pony truss, treated in Chapter 7, illustrates procedures for the determination of requisite bracing rigidity.

2.11 Columns under Dynamic Loading

Columns may sustain loads in excess of critical values when the loads are impulsive and of short duration. This is a subject of increasing interest, although outside the scope of this *Guide*. A review of early work has been presented by Hoff (2.5); and an investigation by Housner and Tso (2.45) gives detailed response calculations for triangular-pulse loads and includes the effects of shear and rotatory inertia.

References

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Chapter Three

Compression Member Details

3.1 Introduction

This chapter takes up the design criteria needed in connection with local behavior of plate, bar, or tubular elements of columns, and the design of local elements to withstand shear forces developed as a result of eccentricity of load, lateral load, or column curvature.

The structural significance of local buckling may be quite different from the significance of general buckling of a column. Buckling, in the case of a centrally loaded column, is a departure from a perfectly straight configuration under constant load in the elastic range, or under increasing load in the inelastic range. Because of initial imperfections and/or residual stresses, actual column strength will be less than the theoretical buckling load; but this load is nevertheless a reasonable criterion for column design.

In the case of plates and hollow cylinders, the theoretical buckling load is not necessarily a satisfactory basis for design since this load may be either much too small or much too large, respectively. For example, a plate loaded in uni-axial compression, with both longitudinal edges supported, will develop transverse tensile forces after buckling that provide post-buckling support. Thus, additional load may be applied without serious structural damage. Initial imperfections in such a plate may cause bending to begin below the buckling load, and yet the plate, unlike an initially imperfect column, may sustain loads greater than the theoretical buckling load. For a plate loaded in shear, with tensile stress always a part of the stress condition at loads less than the buckling load, there is an initial restraint against buckling, and the post-buckling bonus in strength is even greater (see Chapter 5, Plate Girders). In the case of a thin-walled hollow cylinder, however, buckling is followed by the development of transverse compressive stresses which cause a precipitous decrease in strength from the buckling load. Thus, in this case, initial imperfections may reduce the maximum load much below the buckling load, a condition opposite to that of a plate with supported edges. Theoretical buckling

loads have been evaluated for plates and cylinders under a great variety of idealized conditions (see Refs. A1, A2, A9, A10, and A23).

Local strength analyses of the component parts of a column are important to the economical design of the complete column. To minimize the weight of a metal column, the effective L/r must be kept as small as possible in order that the material can be used at the greatest possible permissible stress. This is increasingly true as we turn to the use of higher-strength steels, as is obvious from the fact that all slender steel columns have the same Eulerian strength regardless of the yield strength of the material.

The length of any member is determined by the structural function, but the designer can select a cross section that will provide the largest possible radius-of-gyration without encroaching on clearance requirements or unduly increasing the cost of manufacture. The largest radius-of-gyration is obtained by placing material as far as possible from the centroid. For a given cross-sectional area, this means that the material will become thinner and thinner as the column size increases, for any particular type of cross section. This leads ultimately to such thin walls for any given column cross section that limiting width-thickness ratios must be invoked to keep the local buckling strength greater than the allowable stress. Alternatively, local post-buckling strength can be utilized at the expense of some loss in effective cross section available for general column strength. In some cases, in order to place the material as far as possible from the neutral axis (especially when only a small load is to be carried and the total area is small), angles, channels, or I-shaped sections are used, with lacing or batten plates to hold them straight. Such lacing bars and batten plates are not load-carrying elements, but function primarily to hold the load-carrying portions of the column in their correct relative positions and to provide points of intermediate support for each separate element of the built-up column. Thus, lacing bars and batten plates are economical only if the resulting increase in permissible stress for the load-carrying elements leads to a reduction in cost that exceeds the added expense of the lacing or battens.

3.2 Cross Section Types of Solid-Wall Columns

Closed-section columns such as those illustrated in Figs. 3.1*a*, *b*, *c*, and *d* are particularly efficient since they have approximately equal strength and rigidity in all directions, exactly so in the case of the hollow cylinder, *a*. Type *b* may be fabricated by welding four plates, and is also available in a range of sizes as a mill product. Closed column sections such as *a*, *b*, *c*, and *d* have great torsional rigidity, and the flat-plate elements of the latter three have effective longitudinal edge support. Towers of suspension

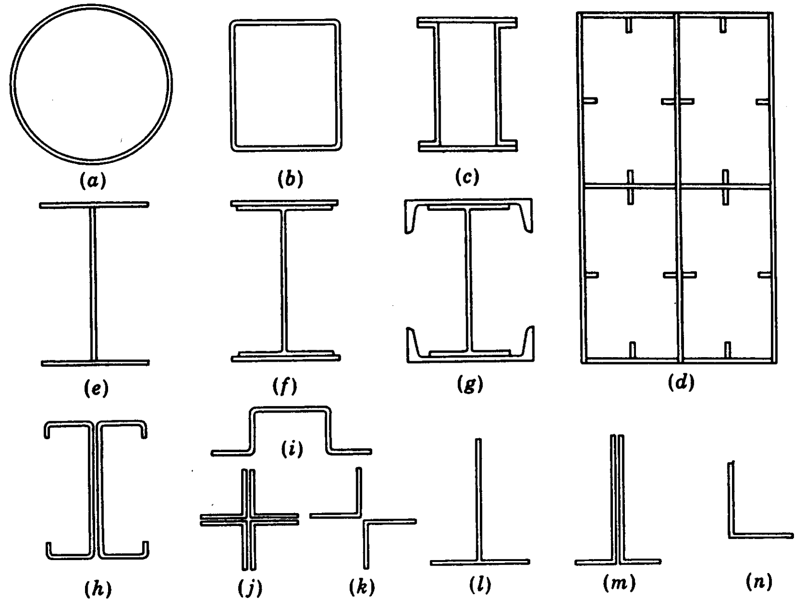


Fig. 3.1. Solid-wall column types.

bridges and other similar free-standing columns of monumental proportion frequently utilize multicell closed sections as illustrated in *d*, incorporating longitudinal stiffeners and requiring diaphragms to maintain cross-sectional shape. Wide-flange or similar type open sections are commonly used in buildings and in trusses (Figs. 3.1*e*, *f*, and *g*). Type *h* (two halves spot-welded together) is used (A11) for cold-formed steel sheet fabrication. The lipped edge stiffener increases the flange strength with respect to local buckling.

Another common cold-formed shape is the hat section, *i*. Types *j* and *k* illustrate the symmetric arrangement of structural angles to create a column section that can be assumed as centrally loaded provided the end connections have no more than nominal eccentricity.

Columns having appreciable end eccentricity should be designed as beam-columns using procedures suggested in Chapter 6. If, however, end eccentricity is only nominal, columns can be designed as centrally loaded. Generally speaking, all of the cross sections except *l*, *m*, and *n* may be designed on the basis of a single column formula for the centrally loaded condition, provided that local or torsional buckling does not occur at a lower load than the one given by Eq. 2.10.

Although increasing design use is being made of post-buckling strength,

the need for rigidity or the desire to avoid a wavy surface may make it preferable to take local buckling into consideration. In the design of light-gage metal columns, a local-buckling reduction factor should be applied to modify the otherwise-permissible column stress for those cases in which the ultimate stress of the column exceeds the local-buckling stress of one or more of the plate components. In the AISI Specification, this reduction factor is designated as "Q."* The problem of design for local buckling will be considered before discussing some of the special design problems pertinent to the various shapes.

3.3 Plate Thickness Requirements as Determined by Critical Stress

If increasing compression forces are applied to opposite edges of a flat plate, a critical stress will be reached at which it will buckle out of its plane. Such a critical stress may be in either the elastic or inelastic range, and is similar in concept to the critical stress (Eq. 2.2) for the perfect bar. For the plate as well as the bar, small imperfections and residual stresses may cause initial bending of the plate at a load less than the buckling load.

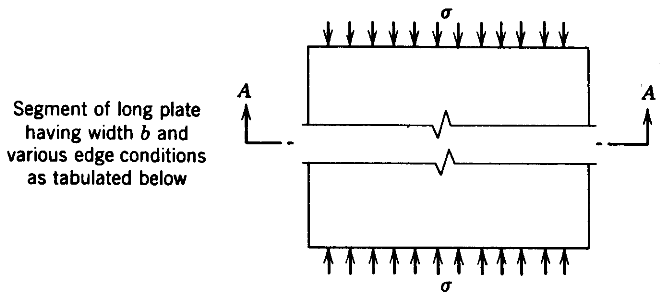
In 1891 Bryan (3.1) presented the analysis of the elastic buckling stress for a rectangular plate simply supported along all edges and subjected to a longitudinal compressive load. The elastic buckling strength of a long plate segment is primarily determined by the plate width-thickness ratio b/t, and by the restraint conditions along the longitudinal boundaries. The following equation from Bleich (A1) may be used to approximate the critical buckling stress of a flat-plate segment in a long column under uniform compressive stress (that is, under central load) in either the elastic or inelastic range:

$$\sigma_c = k\left[\frac{\pi^2 E\sqrt{\eta}}{12(1-\nu^2)(b/t)^2}\right] \tag{3.1}$$

In this equation, $\eta = E_t/E$, and b and t are as indicated in Fig. 3.2. The value of k in Eq. 3.1 is determined by the longitudinal boundary conditions, as shown in Fig. 3.2.

The introduction by Bleich (A1) of the factor $\sqrt{\eta}$ into Bryan's equation to adapt it to stresses above the proportional limit is a conservative approximation to the solution of a complicated problem. Reference to more elaborate theories, as well as more complete information on k factors, can be found in a number of references (3.2, 3.3, A1, A2, A9, A10, and A23). Recent research by Haaijer and Thürlimann (3.4, 3.5) has given special attention to the problem of inelastic local buckling of steel plates, with reference to the effect of the initial strain-hardening range of the material.

* See Ref. A11, Sec. 3.6.1.



Case	Description of edge support	k
1	Both edges simply supported	4.00
2	One edge simply supported, the other fixed	5.42
3	Both edges fixed	6.97
4	One edge simply supported, the other free	0.425
5	One edge fixed, the other free	1.277

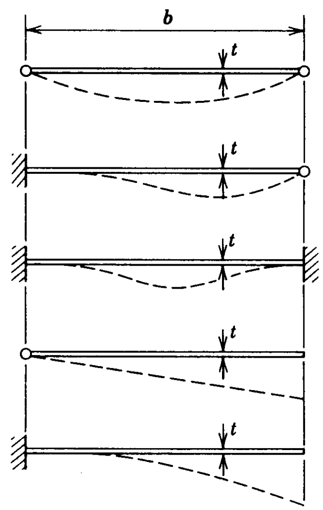


Fig. 3.2. Coefficients k for Eq. 3.1.

When the column cross section is composed of various connected elements (see Fig. 3.1) a lower bound of the critical stress can be determined by assuming, for each plate element, a simple support condition for each edge attached to another plate element, or a free condition for any edge not so attached. The smallest value of the critical stress for any particular plate element will usually be less than the actual strength, because of restraint supplied by the adjacent elements.

Critical stresses have played an important role in determining permissible width-thickness ratios for plate elements of columns and girders used in steel construction. In some bridge and building specifications (A6, A7, A13) the basic design requirement with respect to local buckling has been

that the yield point should be reached prior to elastic buckling. For example, consider an outstanding element of an A36 steel compression member. Assume (conservatively) that there is no rotational restraint along the supported edge ($k_{min} = 0.425$—see Fig. 3.2) and assume elastic behavior ($\eta = 1$). Then, equating the buckling stress found from Eq. 3.1 to the yield point of the steel,

$$36 = \frac{(0.425)\pi^2(29{,}000)}{(12)(1 - \overline{0.3}^2)(b/t)^2}$$

from which b/t is found as 17.6.* If this b/t ratio is exceeded, elastic buckling of the outstanding plate element will occur. However, this b/t value is not necessarily a conservative basis for design, since residual stresses and initial imperfections will have their greatest strength-reducing influence precisely at the b/t ratio found in this manner. AISC practice in steel design limits b/t for outstanding elements to $(b/t)_{max} = 3000/\sqrt{\sigma_y}$, and for elements supported along both edges to $(b/t)_{max} = 8000/\sqrt{\sigma_y}$.†

For centrally loaded columns, a different basis for the design of local plate elements has been described by Bleich (A1):

> To prevent premature failure of compression members by local buckling, the cross section should be selected so that the individual plates offer the *same or larger resistance to local buckling as the whole member presents to primary column buckling.*

This statement permits b/t to increase with increasing KL/r, a practice different from that used in the United States for the design of heavy steel columns. An example of the Bleich procedure is provided by the German Buckling Specification, which specifies, for an outstanding element of a column having $KL/r > 75$,

$$\left(b/t\right)_{max} = 0.2\left(\frac{KL}{r}\right)$$

while for $KL/r \leqslant 75$, the maximum permitted b/t is 15. Other criteria for design of plate-column sections can be found in this specification, which has been made available in English translation by Column Research Council (A18).

The inelastic buckling strength of a plate can be approximated conservatively by determining an equivalent column slenderness ratio, KL/r, which can then be substituted either in a column strength formula to obtain an estimated critical plate stress, or in an allowable column stress formula to obtain a local allowable plate stress. This equivalent KL/r is

* It may be noted that the largest b/t ratio of any commonly rolled steel angle is $(6 - 0.156)/0.312 = 18.7$, for the $6 \times 6 \times \frac{5}{16}$ size.

† See Ref. A13, Sec. 1.9.1 and Sec. 1.9.2.

obtained by equating the critical column stress (Eq. 2.3b) to the plate buckling stress (Eq. 3.1),

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \left(\frac{12(1-\nu^2)}{k}\right)^{1/2} (\eta)^{1/4} \left(\frac{b}{t}\right) \tag{3.2}$$

When η is very small, the critical stress is near the yield point and is insensitive to changes in KL/r; and when η approaches unity, it is satisfactory to assume that $\eta^{1/4} = 1$. Taking Poisson's Ratio as 0.30 (an approximate value applicable to any of the structural metals), and $\eta^{1/4} = 1$,

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \frac{3.3}{\sqrt{k}} \left(\frac{b}{t}\right) \tag{3.3}$$

Eq. 3.3 provides a conservative basis for calculating either the column strength or design stresses in accordance with buckling-stress limitations. It can be applied to columns made of high-strength steels or nonferrous alloys. Formulas which can be reduced to Eq. 3.3 are given in the Alcoa Structural Handbook (A19).

The application of Eq. 3.3 to a steel plate problem will now be illustrated. Determine the buckling stress of a long uniformly compressed plate 25 × ½ in. in cross section, with simple edge supports, made of steel having a yield strength of 50 ksi.

From Fig. 3.2, $k = 4.0$.

By Eq. 3.3, $$\left(\frac{KL}{r}\right)_{\text{equiv}} = \left(\frac{3.3}{\sqrt{4}}\right)\left(\frac{25}{\frac{1}{2}}\right) = 82.5$$

From Table 2.3, the buckling strength of this plate is estimated to be 35.1 ksi.

The possibility of interaction between local and general buckling has been considered by Bijlaard and Fisher (3.6). They conclude:

> The interaction effect is negligible for box sections, as indicated by both theory and experiment...the same conclusion applies to the common sizes of H and channel sections, but not to sections for which torsional instability is an important factor, such as the T and angle shapes.

This conclusion is based on the small-deflection theory of plate buckling and the fact that, for most structural columns fabricated from plates or shapes, the lengths between nodes for primary and local buckling are decidedly different. However, if columns are designed to fail in the post-buckling range, such interaction may be pronounced and must be considered (3.7).

In plastic design of steel columns, beam-columns, and beams, local buckling usually must be inhibited until the material passes through the plastic stress range and begins to strain harden. Haaijer and Thürlimann (3.4, 3.5, A25) have developed rules for proportioning steel plate elements in plastic design. More recently, Lay (3.8) has reported on the effect of local plastic buckling on beam-column and frame behavior. For a simplified analysis, we can assume that $\eta = E_{st}/E$ in Eq. 3.1, where E_{st} is the slope of the stress-strain curve at the onset of strain hardening (see Fig. 1.1). Then, solving Eq. 3.1 for b/t, and putting $\sigma_c = \sigma_y$, the following is obtained:

$$\left(\frac{b}{t}\right)_{\max} = \left(\frac{E_{st}}{E}\right)^{1/4} \sqrt{\frac{k\pi^2 E}{12(1-\nu^2)\sigma_y}}$$

or, introducing $E = 29{,}000$ ksi and $\nu = 0.30$,

$$\left(\frac{b}{t}\right)_{\max} \approx 13\sqrt{\frac{k\sqrt{E_{st}}}{\sigma_y}} \tag{3.4}$$

If E_{st} is assumed as 1300 ksi, Eq. 3.4 agrees well with b/t ratios currently (1966) specified for plastic design (A13). Experimentally determined values of E_{st} are usually less than 1300 ksi, but research and tests (A25) have proven the acceptability of the specified b/t ratios.

3.4 Effective Flat-Plate Width-Thickness Ratios Based on Post-Buckling Strength*

The economic use of material in any situation requires that permissible stresses be as large as possible. Since the permissible stress in a column decreases with increasing KL/r, it is obvious that the KL/r should be kept as small as possible. Since the length of a column is governed by the geometry of the structure, the column radius-of-gyration should be as large as possible. In the case of moderately heavy loads this is accomplished by concentrating the material at the periphery of the cross section and, where necessary, joining it by lacing bars or perforated cover plates. For light loads, however, such column cross sections are uneconomical and the use of thin material with only part of the column cross section considered effective may be desirable. This leads to the "effective-width" concept, which utilizes the post-buckling strength of the plate. The effective-width concept is advantageous not only for long columns carrying small loads, but also for columns that have the dual function of supporting loads and acting as walls, partitions, or bulkheads. Under the effective-width concept, only certain portions of the plate width are considered to be effective in carrying loads after the local-buckling stress has been exceeded. These effective plate regions are adjacent to stiffeners or at

* Sec. 3.4 is based in large part on a summary of existing information prepared especially for Column Research Council by Jombock and Clark (3.9).

corners where two or more joined plates stiffen one another. At such locations the plate yield strength will either be approached or actually will be reached before the member fails.

The effective-width concept is currently used in several specifications. The 1962 edition of the *Specification for the Design of Light-Gage Cold-Formed Steel Structural Members* (A11) and the suggested specifications for structures made of aluminum alloy 6061-T6 of the ASCE Task Committee on Lightweight Alloys (A21, A22) make use of effective-width concept. The AISC Specification (A13) tacitly allows the use of an effective width in the design of compression members having plate elements with b/t ratios greater than the stated permissible values.*

The effective-width concept seems to have had its origin in the design of ship plating (3.10). It had been found that longitudinal bending moments in ships caused greater deflections than those calculated using section properties based on the gross area of the longitudinal members. More accurate deflections could be calculated by considering only a strip of plate over each stiffener having a width of 40 or 50 plate thicknesses as effective in acting with the stiffeners in resisting longitudinal bending.

The advent of all-metal aircraft construction provided another opportunity for the use of the effective-width concept, since it was advantageous to consider some of the metal skin adjacent to stiffeners as being part of the stiffener in calculating the strength of aircraft components. Light-gage steel buildings also provide useful applications of stiffened-sheet construction. A discussion of the effective-width concept as applied to light-gage steel design has been prepared by Winter (A12).

Tests by Schuman and Back (3.12) of plates supported in V-notches along their unloaded edges demonstrated that, for plates of the same thickness, increasing the plate width beyond a certain value did not increase the ultimate load that the plate could develop. It was observed that wider plates acted as though narrow side portions or "effective load-carrying areas" took most of the load. Newell (3.13) and others were prompted by these tests to develop expressions for the ultimate strength of such plates. The first to use the effective-width concept in handling this problem was von Kármán (3.14). He derived an approximate formula for the effective width of simply-supported plates and, in an appendix to his paper, Sechler and Donnell derived another formula based on slightly different assumptions. Subsequently, many other effective-width formulas have been derived, some empirical, some based on approximate analyses, and some based on the large-deflection plate bending theory, employing varying degrees of rigor.

Von Kármán (3.14) developed the following approximate formula for

* See final paragraph, Sec. 1.9.1 of Ref. A13.

plate effective width, based on the assumption that two strips along the sides, each on the verge of buckling, carry the entire load:

$$b_e = \left[\frac{\pi}{\sqrt{3(1-\nu^2)}}\sqrt{\frac{E}{\sigma_e}}\right]t \tag{3.5}$$

Combining Eqs. 3.5 and 3.1, for $k = 4$ (simple edge supports), the formula suggested by Ramberg, McPherson, and Levy (3.15) is obtained (see Fig. 3.3 for notation):

$$\frac{b_e}{b} = \sqrt{\frac{\sigma_c}{\sigma_e}} \tag{3.6}$$

As a result of many tests and studies of post-buckling strength, Winter (3.16) suggested in 1947 the formula for effective width that has been adopted in the AISI Specifications for light-gage cold-formed steel (A11):

$$\frac{b_e}{t} = 1.9\sqrt{\frac{E}{\sigma_e}}\left[1 - 0.475\sqrt{\frac{E}{\sigma_e}}\left(\frac{t}{b}\right)\right] \tag{3.7}$$

or, alternatively, in the form of Eq. 3.6,

$$\frac{b_e}{b} = \sqrt{\frac{\sigma_c}{\sigma_e}}\left[1 - 0.25\sqrt{\frac{\sigma_c}{\sigma_e}}\right] \tag{3.8}$$

Eqs. 3.7 and 3.8 are basically the same as Eqs. 3.5 and 3.6, respectively, but include a correction coefficient determined from tests and reflecting

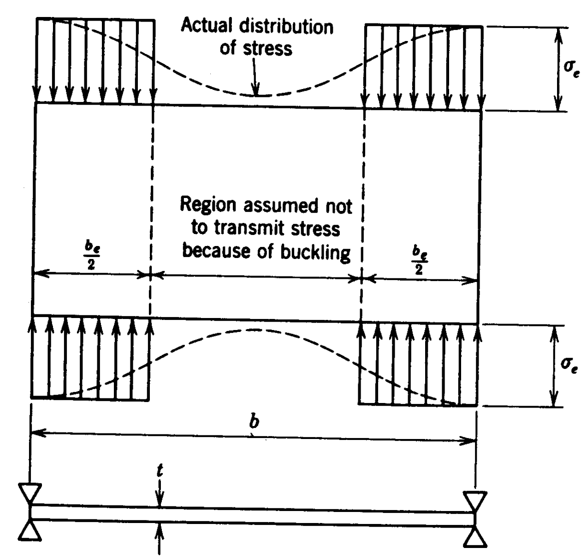


Fig. 3.3.

the total effect of various imperfections, including initial deviations from planeness.

In the calculation of the ultimate compression load for plates supported along the two unloaded edges, σ_e can be taken equal to the compressive yield point for steel; and for aluminum alloys, magnesium alloys, and stainless steel, σ_e is taken as 0.7 times the yield strength,* except that when the buckling stress σ_c exceeds 70% of the yield strength, the load capacity shall be taken as $bt\sigma_c$ and the effective width need not be calculated.

When a plate is supported along only one longitudinal edge, the post-buckling strength is only slightly greater than the critical stress. In this case, although an effective width is not explicitly used in the AISI Specification (A11), the post-buckling strength is nevertheless relied upon to justify a reduced factor-of-safety against initial local buckling. The practical consequence in thin-gage cold-formed metal construction has been the development of shapes having lipped stiffeners along the outer edges.

Since the effective-width concept has been well developed in current specifications and commentaries (A11, A21, A22, 3.11), it is suggested that reference be made thereto for further information on the subject. There appears to be no basic reason why the principle should not be extended into other specifications.

Jombock and Clark (3.9) list fourteen effective-width formulas, along with their sources, and discuss the assumptions upon which they are based. They suggest that the results of these comparisons can be summed up as has been done by Gerard (3.17): "Of all the theories shown, Equation (3.6, herein) apparently gives the best fit to the test data and is still predominantly conservative throughout the range of the data." For ease of application, there seems to be little advantage in any of the other formulas.

3.5 Circular Pipe or Tube Columns

The hollow cylinder provides the most efficient cross-sectional shape for columns having equal lateral restraint in all directions normal to the column axis. The diameter of such a column should be as large as possible, with the additional requirement that d/t (see Fig. 3.4) should be small enough to assure that premature failure by local buckling will not occur. A distinction between *manufactured* tubes, extruded or drawn to close tolerances, and welded or riveted *fabricated* tubes, will be made subsequently, along with the corresponding differences in recommended allowable stress.

Tubes may be classified as short, medium length, and long, with limits to be defined hereinafter. The local-buckling strength of very short

* Determined by the offset method.

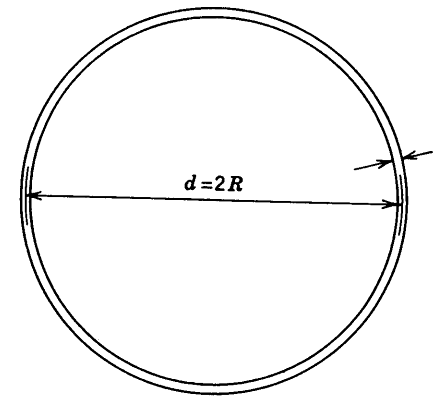


Fig. 3.4.

perfect tubes depends primarily on L/d. In the case of medium-length tubes, the local-buckling strength is primarily a function of d/t. For long tubes, local-buckling failure should be avoided so that general column buckling is the primary problem. Medium-length tubes are the most frequently used in practice; and for these the theoretical local-buckling stress, based on the small-deflection theory solution of Donnell's equation (3.18), is:

$$\sigma_c = \frac{CE}{(R/t)} \tag{3.9}$$

where

$$C = [3(1 - \nu^2)]^{-1/2} \approx 0.6 \quad \text{when} \quad \nu = 0.3$$

and R and t are defined in Fig. 3.4. The medium-length tube solution defined by Eq. 3.9 applies for values of Z greater than 2.85, where, as given by Batdorf (3.19),

$$Z = \frac{L^2}{Rt}\sqrt{1 - \nu^2} \tag{3.10}$$

Experiments show that as d/t increases, failure usually occurs at a stress increasingly lower than that obtained from Eq. 3.9. This is because residual stresses and/or small irregularities in shape have an extremely detrimental effect on the compressive strength of cylindrical tubes. Tubes are made of many different materials and by a variety of forming and/or welding processes. Therefore, the value to be used for the coefficient C in Eq. 3.9 will depend to some degree on the tube manufacturing process, and should be based on recommendations made by the particular manufacturer.

The AISI Specification (A11) is primarily applicable to slender manufactured steel tubes, and it provides restrictions on d/t to eliminate the possibility of local buckling. By this specification the ratio d/t for a cylindrical tubular member in compression or bending shall not exceed $3300/\sigma_y$, where σ_y is in ksi units. The allowable unit stress for a centrally loaded tubular column is then determined from the applicable column formula on the basis of KL/r, with the factor Q equal to unity.

For manufactured aluminum alloy tubes, the *Alcoa Structural Handbook* (A19) presents the following formula for equivalent slenderness ratio:

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = 4\sqrt{\frac{R}{t}}\left[1 + \frac{\sqrt{R/t}}{35}\right] \tag{3.11}$$

The equivalent KL/r value found from this formula is then substituted into the appropriate column design formula to obtain the allowable stress for the particular R/t. For more complete design information on aluminum tubes in compression, bending, or torsion, including the effects of welding, reference should be made to the report by Clark and Rolf (3.11).

To fill the need for design criteria for fabricated tubular steel compression members having large d/t ratios, Eqs. 3.12 and 3.13 for inelastic and elastic buckling, respectively, are presented here for the first time. These are based on about forty tests carried out by Wilson, Newmark, and others (3.20, 3.21) at the University of Illinois, in which the d/t ratios varied from 68 to 1980. In the inelastic range,

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \pi\sqrt{A\left(B + \sqrt{\frac{d}{t}}\right)} \tag{3.12}$$

where

$$A = \frac{639.2}{\sqrt{\sigma_y}}$$

and

$$B = -\frac{12.08}{\sqrt{\sigma_y}}$$

This KL/r value must be substituted into the appropriate column design or strength formula to obtain the allowable or critical stress respectively for the particular column d/t value. The constants A and B in Eq. 3.12 have been chosen to make the curve corresponding to this equation intersect the point ($d/t = 3300/\sigma_y$, $\sigma = \sigma_y$), where $d/t = 3300/\sigma_y$ will be recognized as the maximum AISI value mentioned previously. In the elastic range, the *critical* (*not* allowable) stress is

$$\sigma_c = \frac{8000}{d/t} \tag{3.13}$$

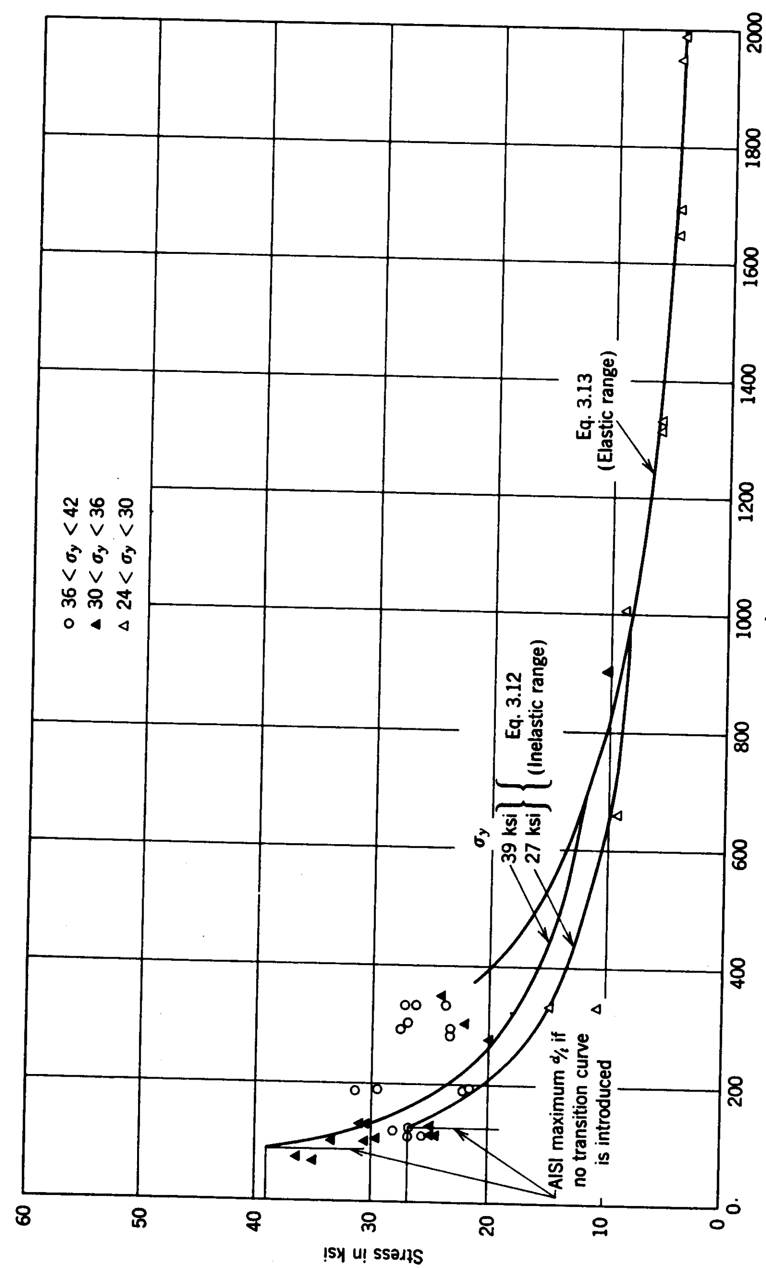


Fig. 3.5. Ultimate-strength formulas for fabricated cylindrical steel columns compared with Wilson tests (University of Illinois Bulletins 255 and 292).

The agreement between these two formulas and the Wilson test results is indicated in Fig. 3.5. The stresses shown for Eq. 3.12 were obtained by substituting the equivalent KL/r values into Eq. 2.2. In the inelastic range, the column strength is affected by the yield point of the material, which in the case of the Wilson tests ranged from 24 to 42 ksi. In Fig. 3.5, the test results are grouped under three band designations: 27, 33, and 39 ksi, all ± 3 ksi. Equation 3.12 is plotted for $\sigma_y = 27$ ksi and $\sigma_y = 39$ ksi. For large d/t ratios, values of σ_c from the Fig. 3.5 curves are approximately 75% greater than those obtained using Eq. 3.9 of the first edition of the CRC *Guide*. This latter equation is, therefore, overconservative.

Further inspection of the Fig. 3.5 curves shows them to be unconservative by about 30% with respect to one Wilson test point, and overconservative by from 20 to 40% with respect to a number of other points. It should also be noted that the curves of Fig. 3.5 would be, in general, overconservative for tubes which are manufactured without appreciable residual stress and to close manufacturing tolerances. Usually, however, manufactured tubes (as contrasted with fabricated tubes) will have d/t ratios less than the AISI maximum, in which case no reduction is needed for d/t and the full allowable column stress can be used. In the case of fabricated tubes, the allowable stress should be determined on the basis of either Eqs. 3.12 and 3.13, or of the allowable column stress, whichever yields the lowest value.

It should be noted once more that Eq. 3.13 provides an empirical estimate of the *maximum* or the *failure* stress. To obtain the *allowable* stress a factor-of-safety must be introduced.

If a tubular column has its ends sealed against moisture and air, no internal corrosion protection is necessary. Pipe columns are sometimes filled with concrete, providing composite construction, the design of which is covered by specifications for reinforced concrete. Kloppel and Goder (3.22) have reviewed a considerable number of tests of this type of column.

3.6 Box-Section Columns

The box section made of solid plates provides efficiency second only to that of the circular pipe or tube column discussed in Sec. 3.5. The box column is particularly well suited to fabrication by welding and is also available as a standard mill product. Its torsional rigidity is comparable to that of a circular tube of similar area and wall thickness, so that torsional buckling is not a problem. Diaphragms should be provided at ends, at points of load application or support, and at other intermediate points, to ensure preservation of the original cross-sectional shape. Where

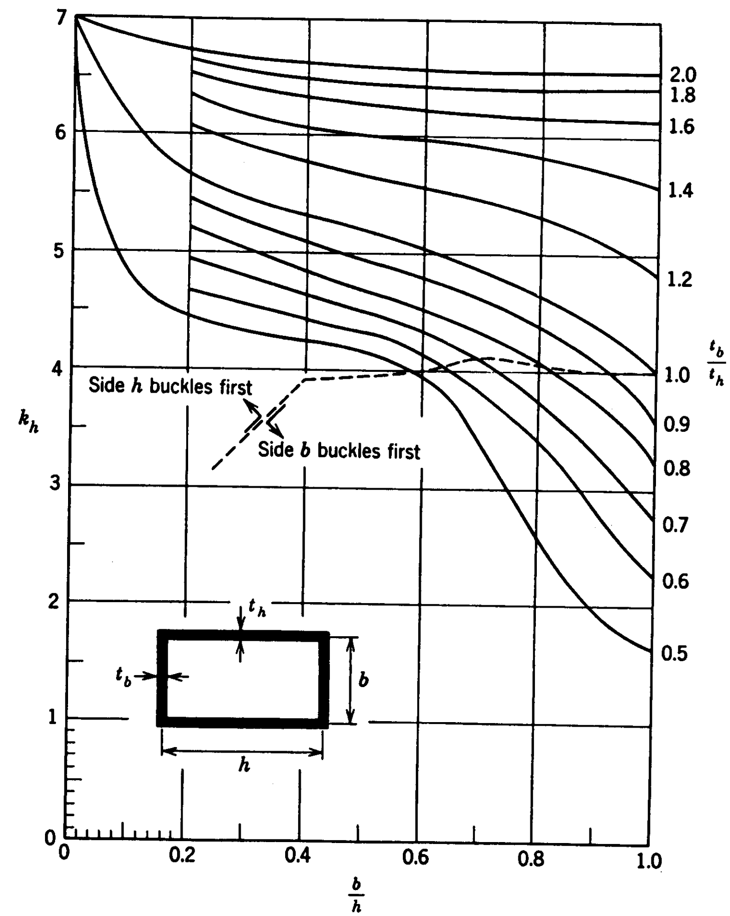


Fig. 3.6. Plate buckling coefficient k_h for side h of rectangular box column (from Ref. 3.23).

all-welded construction is used, the box section can be completely sealed so that no interior corrosion protection is needed.

Fig. 3.6 shows a chart developed by Kroll, Fisher, and Heimerl (3.23) for evaluation of k for rectangular box sections. The same chart is given in Fig. 7 of Ref. 3.2 and Fig. 5*c* of Ref. 3.3, Part II.

In Fig. 3.6, for $b/h \leqslant 1$ and $t_h/t_b \leqslant 1$, the buckling parameter k_h can be conservatively approximated by the following empirical relationship:

$$k_h = 7 - \frac{15}{7}\left(\frac{t_h}{t_b}\right)^2\left[0.4 + \frac{b}{h}\right] \tag{3.14}$$

The buckling parameter k_h, whether determined by Fig. 3.6 or by Eq. 3.14, may be introduced into Eq. 3.3, giving the equivalent KL/r for which the buckling stress of the column is approximately equal to that of the plate.

3.7 Wide-Flange Shapes

Wide-flange shapes are often used as columns because of their low fabrication cost and ease of framing to other members. These shapes are usually hot-rolled, but they can also be made up of two thin-wall channel shapes spot-welded together as shown in Fig. 3.1*h*. Heavier sections can also be built-up by welding or riveting various structural shapes and/or

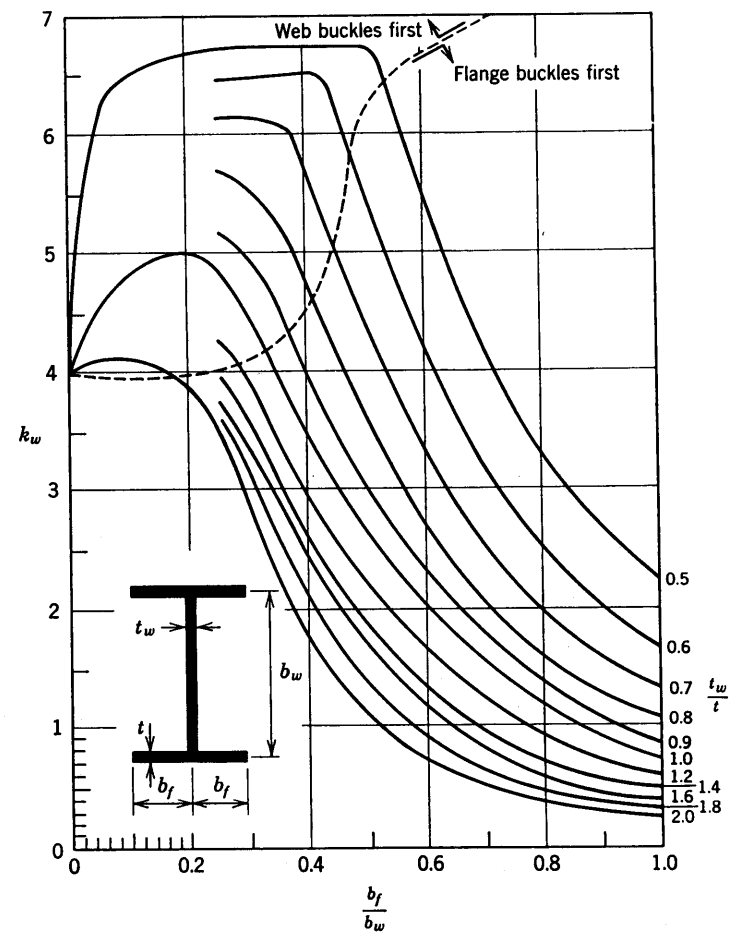


Fig. 3.7. Plate buckling coefficient k_w for wide-flange columns (from Ref. 3.23).

plates together, as shown in Figs. 3.1*f* and 3.1*g*. The wide-flange shape has been widely tested as a column and is the basis for most of the available data for basic column strength curves.

The lower bound for local buckling stress of a wide-flange section can be obtained by determining the critical stresses separately for the web and the flanges, assuming that the web is simply-supported along its two edges by the flanges (Fig. 3.2, Case 1) and that each flange-half has one edge simply-supported and the other free. The web or flange element with the lowest critical stress will tend to buckle first, but will be restrained by the other element. Conservative specifications can be written by using such a lower-bound critical stress for a member and the corresponding b/t ratios for its elements (see the procedure described on page 60). A more accurate analysis of the interaction between web and flange is provided by the chart of Fig. 3.7, in which the critical stress for the wide-flange section is given in terms of web thickness and width, for various ratios of b_f/b_w and t_w/t. This chart is given in Ref. 3.23. This chart, and many others that are similar, will also be found in Refs. 3.3 (Part II) and A23.

3.8 Tee Sections

If both the flange and the stem of a tee strut are adequately connected at the ends so that the load is substantially central, the designer can apply the full allowable stress for a centrally loaded column. If, however, the tee is attached by fasteners or welds along the outer face of its flange (line AA, Fig. 3.8), it must be considered as eccentrically loaded, and should be designed as a beam-column (see Chapter 6).

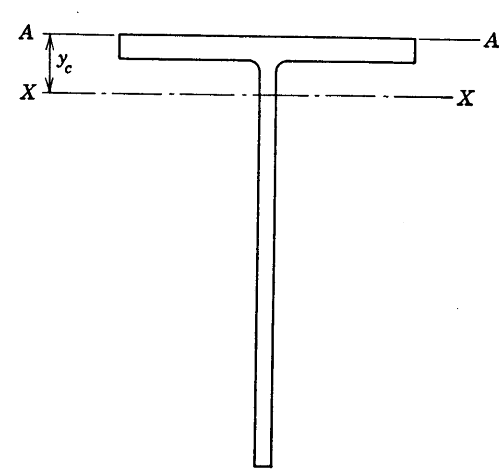


Fig. 3.8.

Tees used as columns are likely to be weak with respect to torsional buckling. The torsional buckling stress can be approximated conservatively by omitting the last term of Eq. 2.22, giving

$$\sigma_c = \frac{JG}{I_p} \tag{3.15}$$

If the tee center-of-twist is restricted to an axis not coincident with the shear center, and if the ends are restrained against rotation, the critical stress will be greater than that given by Eq. 3.15, in which case Ref. A1 may be consulted.

3.9 Stiffened Flat-Plate Elements

The compressive strength of a plate element can be increased by increasing the thickness of the plate, but a more economical procedure is to employ longitudinal and/or transverse stiffeners.

The buckling coefficient k for a stiffened plate (Fig. 3.9) depends on the following parameters:

$$\gamma = \frac{EI_s}{bD}$$

$$\delta = \frac{A_s}{bt}$$

$$\alpha = \frac{a}{b}$$

where I_s = moment-of-inertia of stiffener about web-face axis
A_s = cross-sectional area of stiffener
$D = Et^3/[12(1 - \nu^2)]$

Values of plate buckling coefficient k in terms of γ, δ, and α are given in Ref. 3.24 for the case of a plate stiffened by one, two, or three longitudinal stiffeners dividing the plate into two, three, or four equal panels, respectively. Seide and Stein (3.25) studied the case of a plate stiffened by an indefinitely large number of longitudinal stiffeners. Their solution can be applied without significant error to a plate having four or more stiffeners. For a stiffened plate where the stiffener spacing is uniform and equal to d (see Fig. 3.9), there is an optimum stiffening such that the elastic local-buckling strength will be the same as that of an unstiffened plate with a width-thickness ratio d/t. However, if the post-buckling strength of the subpanels is to be achieved, a stiffness exceeding that optimum is required.

Tables in Ref. A23 give the required value of γ for plates having one transverse stiffener and for plates having three equal and equidistant transverse stiffeners.

For a plate having $(j - 1)$ transverse stiffeners that divide the plate into j panels, the required stiffness for each stiffener varies. For this problem, Klitchieff (3.26) has given the following formula for the required value of γ for the stiffener of maximum stiffness:

$$\gamma = \frac{(4j^2 - 1)[(j^2 - 1)^2 - 2(j^2 + 1)\beta^2 + \beta^4]}{2j[5j^2 + 1 - \beta^2]\alpha^3} \tag{3.16}$$

where

$$\beta = \frac{\alpha^2}{j}$$

For a combination of longitudinal and transverse stiffeners, Ref. 3.3 gives figures showing the minimum value of γ as a function of α for various combinations of equally-stiff longitudinal and transverse stiffeners. When there is a large number of stiffeners in both directions having equal spacing and stiffness, the panel can be treated as an orthotropic plate. Formulas for design of longitudinal and transverse stiffeners for aluminum alloy plates in compression are given in Ref. A19.

Stiffeners of open cross section (the most common type) have negligible torsional stiffness and will, if they have optimum bending stiffness, provide a simple edge support for the subpanel plating. Closed-section stiffeners, on the other hand, provide partial or full fixation of subpanel plating, owing to their considerable torsional rigidity; and as a result, reduce the unsupported subpanel width.

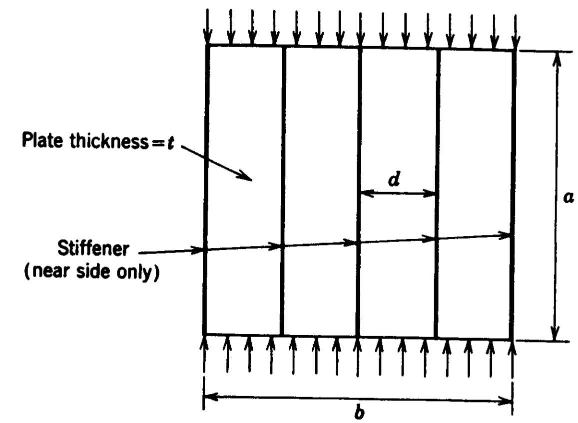


Fig. 3.9. Stiffened plate.

3.10 Angle Struts

Double-angle struts (Fig. 3.10) can be designed as centrally loaded columns, using the full allowable stress, when end connections are designed to produce central loading. To achieve this condition, the gusset plate for

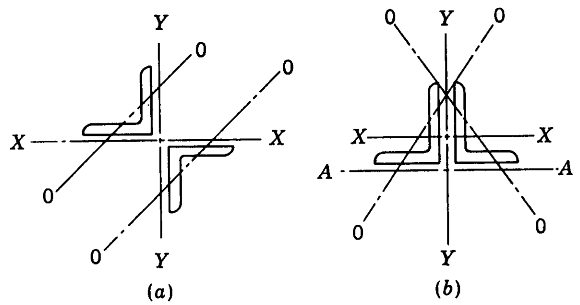


Fig. 3.10.

the strut of Fig. 3.10*b* must be in the *y–y* plane. Double angles must be adequately stitched together so that minimum slenderness ratios L_o/r_o for a single angle between stitching will not exceed a suitable proportion of the minimum slenderness ratio of the complete strut. Proportions of 2/3 or 3/4 have proven satisfactory in practice for this purpose.

As in the case of the tee strut, the double-angle strut of Fig. 3.10*b* should be designed as an eccentrically loaded column if it is attached on plane *A–A*. Such a strut should also, of course, be checked for buckling about the *y–y* axis under concentric loading.

Single-angle struts are used only for lightly loaded secondary members. The best guide for design of such members is probably found in present design practice. Their strength is somewhat dependent on the rotational stiffness of the end connections. If attached by rigid end connections to rigid members, the single-angle strut can be considered as concentrically loaded. For less adequate end conditions the effective end eccentricity is uncertain, and such struts should be designed for a conservative fraction of the allowable stress for a centrally loaded column (see, for example, Refs. A18 and A24).

3.11 Open-Web or Open-Flange Shapes

The use of open-web or open-flange columns (involving laced, battened, or perforated plates) requires consideration of local shear. Shear flexibility also affects the buckling strength of the column, but generally this is a very minor effect.

Shear in a column arises principally from the following three sources:

(*a*) Lateral load, resulting from wind, dead weight, or other causes.
(*b*) Slope, due either to accidental curvature or to that developing during the buckling process.

(*c*) End eccentricity of load, introduced either by the end connections or by fabrication imperfections.

The shear from (*a*) should always be calculated and added to the semi-empirical allowance for shear caused by (*b*) and (*c*). Cause (*b*) is increasingly important in slender columns, and cause (*c*) in short columns.

Typical allowances for column shear in specifications are plotted in Fig. 3.11. The AASHO-AREA shear allowances (A6, A7) for steel columns give rather large weight to shear caused by accidental end eccentricity in short columns, whereas the German Buckling Specifications (A18) and aluminum alloy specifications (A21, A22) emphasize the shear resulting from slope in the bent column. This difference accounts for the two broad groups of curves in Fig. 3.11. The increased shear allowance for aluminum columns results from the lower modulus of elasticity, which causes larger deformations and, therefore, larger transverse components of the axial load.

The failure of the first Quebec Bridge in 1907 pointed to the importance of transverse shear strength in compression chords, and bridge design practice in this country today reflects the lessons learned from that failure and the extensive research that followed. A descriptive review of past column failures has been presented by Wyly (3.27), who concludes

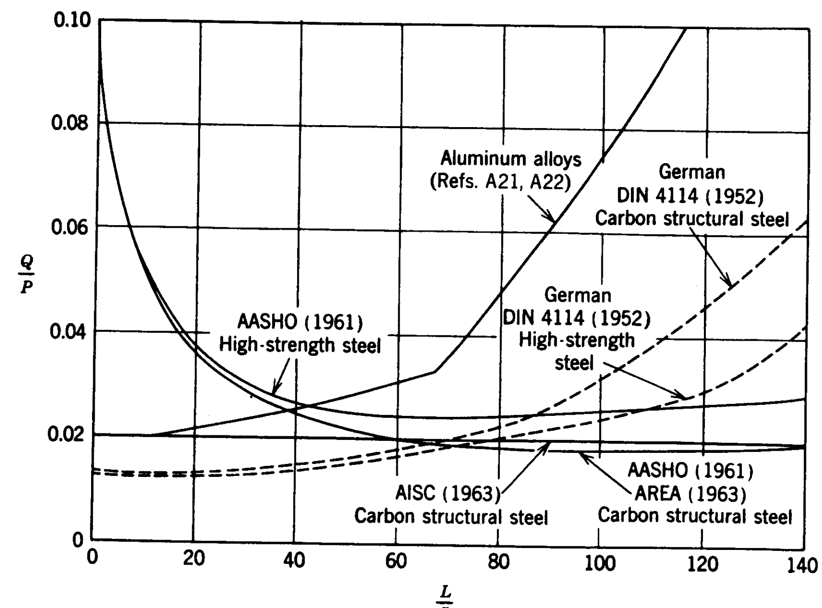
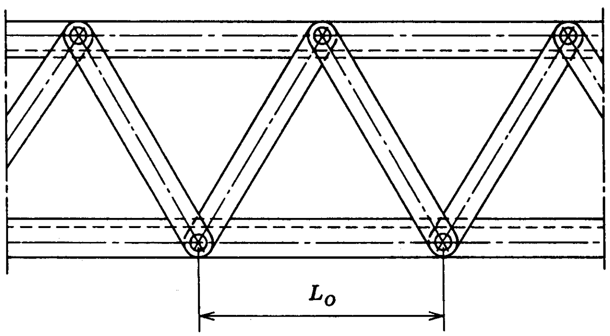


Fig. 3.11. Specification allowances for shear in centrally loaded columns.

that about three-fourths of recorded failures of laced structural columns have resulted from local weakness rather than from general buckling.

3.12 Laced Columns

Laced columns are open-web members in which one or more planes of the column consist of triangular truss frames. The diagonal lacing bars (acting either in tension or compression) must be designed for the specified shear. The effect of the lacing in increasing shear deflections and thereby reducing the general buckling strength of the column has been reviewed and evaluated in standard references (A1, A9). A conservative estimate of the influence of 60° or 45° lacing as generally specified in bridge design practice can be made by modifying the effective-length factor



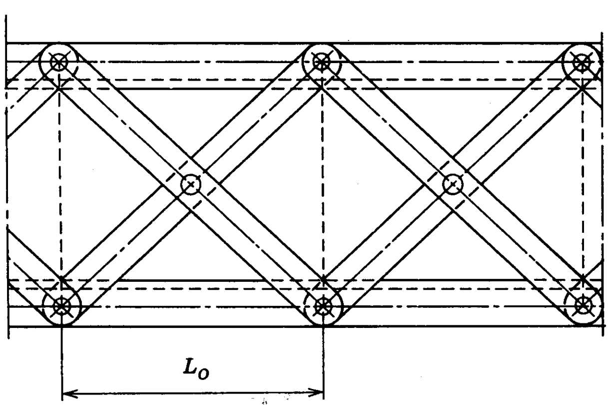


Fig. 3.12. Laced columns.

K (determined by end-restraint conditions) to a new factor K',* as follows:

$$\text{for } \frac{KL}{r} > 40: \qquad K' = K\sqrt{1 + \frac{300}{(KL/r)^2}}$$
$$\text{for } \frac{KL}{r} \leqslant 40: \qquad K' = 1.1K \tag{3.17}$$

Such a modification will have little effect on the estimated strength of short columns.

Conservative bridge design practice (A6, A7) requires that the slenderness ratio of the portion of the flange included between lacing-bar connections (sublength L_o of Fig. 3.12) be not more than 40 nor more than two-thirds of the slenderness ratio of the member. German Buckling Specifications (A18) simply require that the L/r of this portion of the flange shall not exceed 50.

The lacing bars and their connections must be designed to act either in tension or compression, and the rules for general column design are pertinent to them as well. In exceptional cases, such as very large members, double diagonals can be designed as tension members and the truss system completed by compression struts as indicated by the vertical dashed lines of Fig. 3.12*b*.

3.13 Columns with Perforated Plates

A study by White and Thürlimann (3.28) provides a digest of earlier investigations at the National Bureau of Standards (3.29, 3.31) and gives recommendations for design of columns with perforated cover plates.

The following design suggestions for columns with perforated plates are derived from both the White-Thürlimann study and from AASHO Specifications (A6).

1. The perforations may have the form of two semicircles connected with straight sides, or they may be elliptical or circular. For the first two cases, the long axis should be in the direction of the column axis.
2. The clear distance between perforations should be not less than the distance between the nearest lines of longitudinal fasteners, i.e., $(c - a) \geqslant d$ in Fig. 3.13*a*.
3. The net section of the column (defined as the section at the perforations) should be used in computing the axial rigidity AE and the column moments-of-inertia about the x and y axes.
4. If the slenderness ratio a/r_f (length of perforation divided by radius of gyration of flange—see Fig. 3.13) is 20 or less, and also no greater than

* Based on Eq. 339 of Ref. A1, p. 174.

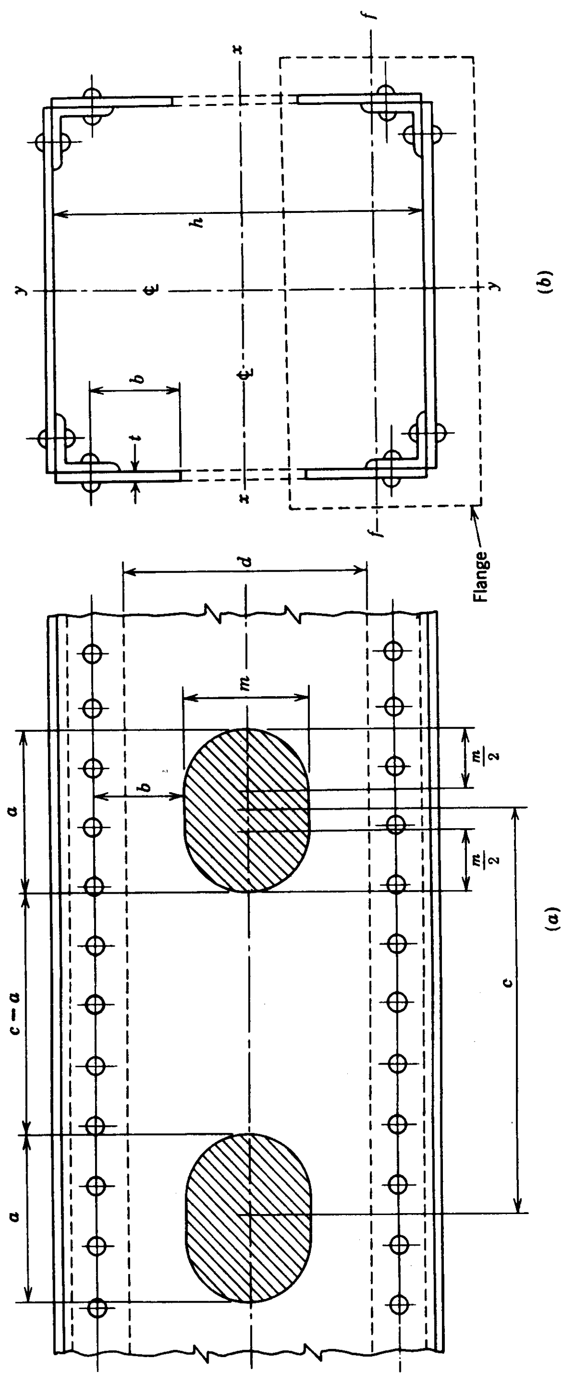


Fig. 3.13. Column with perforated web plates.

one-third of the column slenderness ratio L/r_x, the appropriate specification column stress, applied to the column net section, can be used to determine the permissible load.

5. For columns built-up of plates (Fig. 3.13), the net area of each web at the perforation should be sufficient to resist $1/n$ times the transverse shear force, where n is the number of perforated plates. Perforated plates designed in accordance with rule (2) need not be checked for shear introduced as a specified percentage of the column load.

6. The transverse distance from the edge of a perforation to the nearest line of longitudinal fasteners, divided by the plate thickness, i.e., the b/t ratio of the plate adjacent to a perforation (see Fig. 3.13), should conform to minimum specification requirements for plates in main compression members.

3.14 Columns with Batten Plates

The batten-plate column has greater shear flexibility than either the laced column or the column with perforated cover plates; hence, the effect of shear distortion must be taken into account in calculating the effective length of the battened column. Such columns are not permitted by current United States specifications for bridges and buildings. However, small television and radio towers are frequently made of battened columns, and some specifications permit such columns for secondary applications.

Bleich (A1) gives the following approximate formula for the effective length of a battened column:

$$\frac{KL}{r} = \sqrt{\left(\frac{L}{r}\right)^2 + \frac{\pi^2}{12}\left(\frac{L_o}{r_o}\right)^2} \tag{3.18}$$

where L/r is the slenderness ratio of the column as a whole and L_o/r_o is the slenderness ratio of one chord center-to-center of battens (see Fig. 3.14*a*). Bleich shows that the buckling strength of a steel column having an L/r of 110 is reduced by about 10% when $L_o/r_o = 40$, and by greater amounts for larger values of L_o/r_o. Most specifications place an upper limit on L_o/r_o; for example, a British specification (3.30) has the following requirements:

> In battened compression members in which the ratio of slenderness about the y–y axis (axis perpendicular to the battens) is not more than 0.8 times the ratio of slenderness about the x–x axis, the spacing of battens centre-to-centre of end fastenings shall be such that the ratio of slenderness l/r of the lesser main component over that distance shall be not greater than 50 or greater than 0.7 times the ratio of slenderness of the member as a whole, about its x–x axis (axis parallel to the battens).
>
> In battened compression members in which the ratio of slenderness about the y–y axis is more than 0.8 times the ratio of slenderness about the x–x

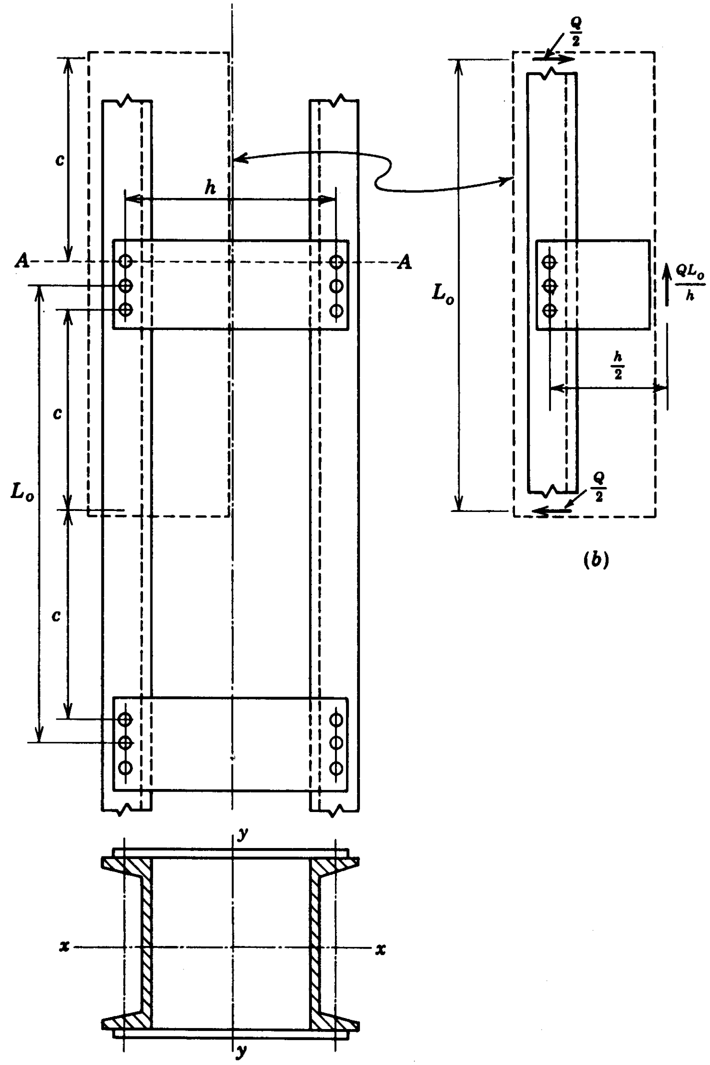


Fig. 3.14. Battened column.

axis, the spacing of battens centre-to-centre of end fastenings shall be such that the ratio of slenderness l/r of the lesser main component over that distance shall be not greater than 40 or greater than 0.6 times the ratio of slenderness of the member as a whole about its weaker axis.

The design of both the individual chords and the batten connections of a battened-plate column should take account of the local bending resulting

from specified shear forces, shown acting longitudinally and transversely on the free-body portion of the column shown in the dashed rectangle (Fig. 3.14b).

Each group of fasteners between the batten plates and the chords should be designed to resist the following moment:

$$M_b = \frac{QL_o}{2n} \tag{3.19}$$

where Q = shear as indicated in Fig. 3.11 plus shear due to any transverse loading

L_o = distance center-to-center of battens

n = number of parallel planes of battens (two in Fig. 3.14)

The maximum combined bending and direct compression stress in each individual chord at line AA of the column shown in Fig. 3.14 should not exceed the maximum permissible stress for a zero-length column. For this calculation, the chord bending moment should be taken as

$$M = \frac{Qc}{2}$$

These combined stresses are not secondary stresses and therefore cannot be neglected.

3.15 Miscellaneous Details

Because of the complexity of the subject, many aspects of good column design practice are too complicated to be covered herein. Important topics not covered include spacing of fasteners in built-up columns, design of column splices, and design of column anchorages. United States specifications (A6, A7, A11, A13) give requirements for such column details.

Some examples of poor practice in design of column support details which, in some cases, have contributed to column failures, are shown in Fig. 3.15. In Fig. 3.15*a*, the cross-framed angles between the two bar joists are intended to prevent rotation of the upper ends of the columns, thus adding to their capacity to resist wind loads; but the moment connection between the top of the column and the structural tee is inadequate, since slotted holes are provided for a field-bolted connection. In Fig. 3.15*b*, the column acts simply as a restraining spring to the rocker which may become unstable at a fraction of the load that the supporting column will carry. In Fig. 3.15*c*, Section B-B shows a column inadequately

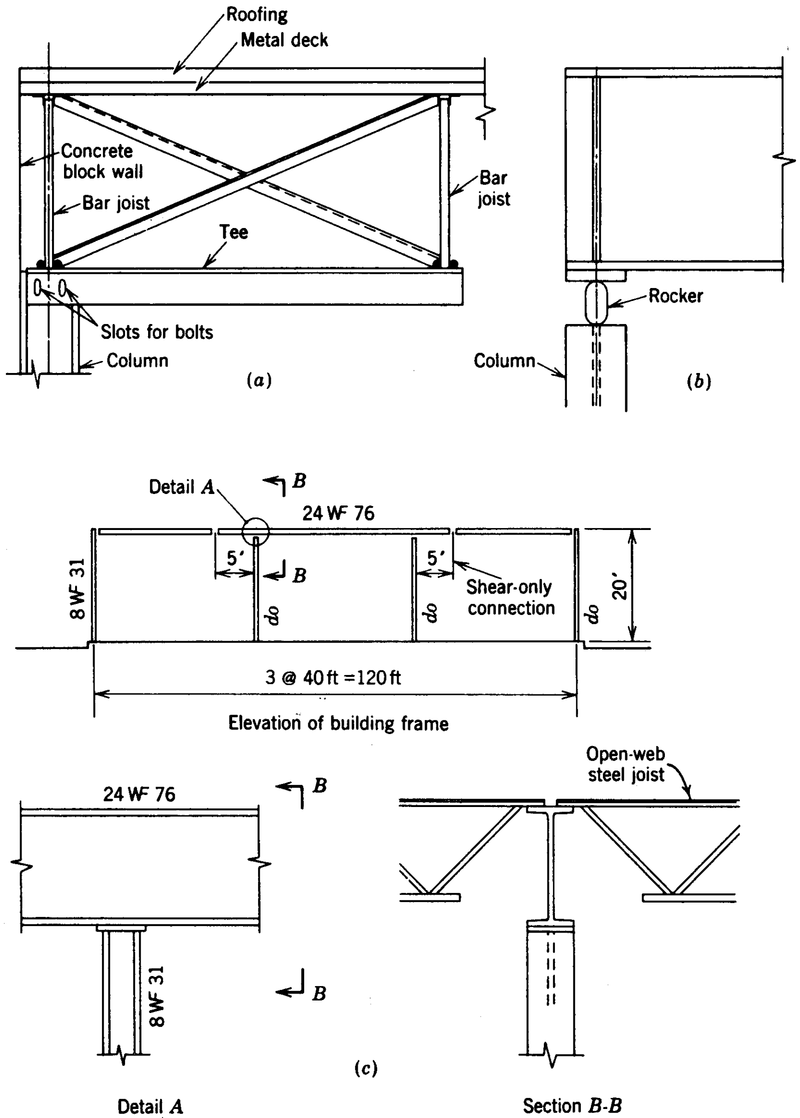


Fig. 3.15. Examples of faulty design of column-support details.

supported with respect to weak-axis bending. The effective length of this column is more than twice the distance from the top of the beam to the base of the column. Many other examples could be given.

References

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Chapter Four

Laterally Unsupported Beams

4.1 Introduction

The compression flange of a beam or girder may have (1) continuous lateral support, (2) bracing at intermediate points, or (3) less frequently, no intermediate lateral supports. In the latter two cases, the beam may fail in combined twist and lateral bending of the cross section in a phenomenon known as "lateral-torsional buckling." It is also important to check the lateral-torsional buckling of members under their own dead load during handling, erection, and immediately after installation, when braces are either absent or different in type from the permanent ones.

A review of early tests involving lateral buckling has been prepared by Procter (4.1). The earliest paper to which he refers was published in 1854 by Fairbairn, who made beam tests and suggested correctly that improved buckling strength would result if the compression flange were rolled both thicker and wider than the tension flange. Later tests of steel beams by Burr (1884), Marburg (1909), and Moore (1910) provided experimental evidence that led to design formulas wherein the allowable stress was a function of the L/r ratio of the compression flange.

The first theoretical solution for elastic buckling of a beam of rectangular cross section was presented by Prandtl (4.2) in 1899 for a number of load and support conditions. An independent solution at about the same time was made by Michell (4.3) for the case of the simply supported beam under constant bending moment. The earliest solution for lateral buckling of an I-beam was made by Timoshenko, who published papers on this subject in Russian and German between the years 1906 and 1910. These and other early developments are reviewed by Timoshenko (A4, A9) and by Bleich (A1), whose readily available works provide a complete account before 1951, when Column Research Council initiated the first of a number of investigations made in connection with the development of this *Guide*. These investigations have been reviewed for CRC by Clark and Hill (4.6), and much of the following material is abstracted from their review.

Elastic buckling theory for several types of loading and beam cross sections has been confirmed by tests of both steel and aluminum alloy members. The aluminum beams tested include symmetrical I-beams under unequal end moments (4.7) and under a concentrated load at the center of the span (4.8), and channels (4.8), zees (4.9), and unsymmetrical I-sections (4.10) under uniform bending moment. Tests of carbon structural steel beams subjected to concentrated loads on the top flange at the two quarter points have also demonstrated the validity of elastic buckling theory (4.12).

The general problem of lateral buckling of thin-walled open cross sections has been studied by Goodier (4.13). In a CRC-affiliated project, Austin and associates (4.14) have studied the effect of varying degrees of end restraint on the I-beam buckling problem.

A qualitative insight into the nature of the elementary beam buckling problem can be gained by considering Fig. 4.1, where a straight rectangular beam is shown loaded by end couples in the plane of maximum resistance. The beam is supported vertically and held against twisting at each end. A segment at the left end is shown in a hypothetical bent and twisted configuration, with vectorial representation of the equilibrating moment that must exist at the cut section a distance z from the left end. It is seen that *if the beam twists*, there is induced a lateral bending moment $M_x\beta$ (approximate for small deflections); and concurrently, *if the beam bends laterally*, there is induced a torsional moment $M_x\theta$. Thus, at a critical beam buckling load, when the laterally deflected beam is in equilibrium under loads or moments that cause bending in the major principal plane, the buckled configuration involves both lateral bending and twisting, since either of these configurations induces the other.

Rolled I, wide-flange, and channel sections are very efficient and strong

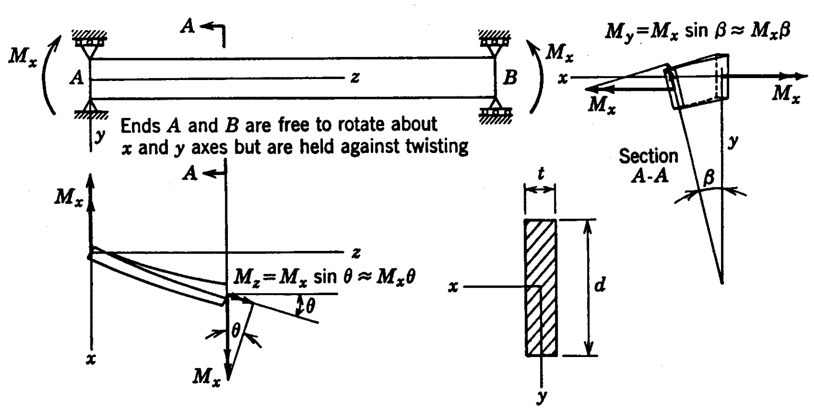


Fig. 4.1. Torsion (M_z) and lateral bending moment (M_y) induced during lateral-torsional buckling by uniform bending moment M_x.

when loaded through the shear center in a plane parallel with the web. These same sections are weak in lateral bending and in torsion, and these weaknesses accentuate the design problem that arises when they are not supported laterally. Combined bending and torsion may also be a problem if the sections are not loaded through the shear center; and biaxial bending is introduced if the load is not parallel with the web. In this *Guide* only the problem of lateral-torsional buckling will be considered. However, steps taken by the designer to improve resistance to lateral-torsional buckling will also improve the resistance of the beam to the other two conditions.

The strength of a laterally unsupported beam of relatively short length, like that of a corresponding column, will be determined by inelastic rather than by elastic behavior. Like the short column, the short beam may be expected to develop the full yield strength of the material. Bleich (A1) has pointed out that it is possible to obtain a lower limit to the theoretical buckling stress in the inelastic range by substituting the tangent modulus E_t (corresponding to the maximum stress in the beam) for the elastic modulus E in the elastic buckling formula. Tests on aluminum alloy beams have shown that this substitution gives a close approximation to the experimental buckling stress when the bending moment is constant along the length (4.6, 4.7). Tests of aluminum alloy beams (4.7) subjected to unequal end moments, with the ratio of the end moments varying from 1.0 to -1.0, showed critical stresses varying from 8% below to 39% above the values calculated by the method suggested by Bleich.

An approximate method of estimating the effect of plastic action on the buckling strength of beams and girders is to assume that the relationship between elastic and inelastic buckling strength is the same for beams as it is for columns. The inelastic buckling strength of beams can then be estimated from a column curve. This procedure is applicable to both steel and aluminum alloy members, for which the tangent-modulus buckling curve has been verified by tests on both columns and beams (4.15). In the case of steel, the tests by Hechtman, Hattrup, Styer, and Tiedemann (4.12) are in good agreement (generally conservatively so) with the predictions of the basic column-strength curve given in Fig. 2.6.

This hypothesis amounts to the extension of Table 2.3 to cover beam as well as column buckling. In the first edition of the *Guide*, this approach was compared graphically with test results. Galambos (4.20) has evaluated this procedure by comparison with his theoretical solution, as shown in Fig. 4.2, which applies to I-shaped sections.

The widespread adoption of plastic design has created the need for criteria for lateral buckling and lateral bracing when plastic hinges are developed within a beam. Theoretical and experimental work on lateral bracing requirements in plastic design, with consideration of such effects as moment gradient, partial yielding, end fixity, and the effect of yielding on St. Venant torsion, is summarized on pages 51 to 62 of Ref. A25. Lee (4.18) has reviewed the literature, from the study by Neal (4.17) of the partially yielded rectangular bar to the experimental and theoretical work of Lee and Galambos (4.19, 4.20) on the wide-flange section. Baker, Horne, and Heyman (see pages 236 to 248 of Ref. A7) have reviewed the problem of inelastic lateral instability in relation to plastic design, covering primarily British research.

In the design of a beam without lateral support, various alternatives should be considered:

1. *Use of a box-girder section.* This will usually eliminate entirely the problem of lateral buckling.
2. *The use of boxed flanges* (see Fig. 4.3). This will improve not only the lateral-buckling resistance, but also the local-buckling strength of both the flange and web.
3. *If an open section is used, concrete encasement.* This will increase the beam's torsional resistance by several hundred percent and will greatly improve the lateral-buckling characteristics.

Basic to any of the foregoing is the study of the behavior of the rectangular cross section, to which attention will first be given. Special attention will also be given to wide-flange sections, and to design recommendations in

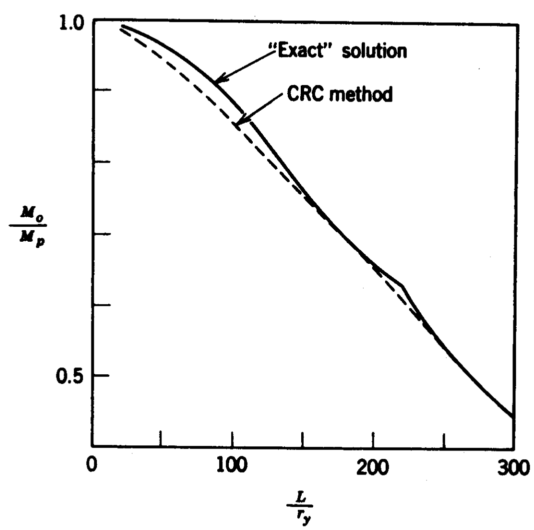


Fig. 4.2. Ultimate moment capacity of 8WF31 beam (4.20).

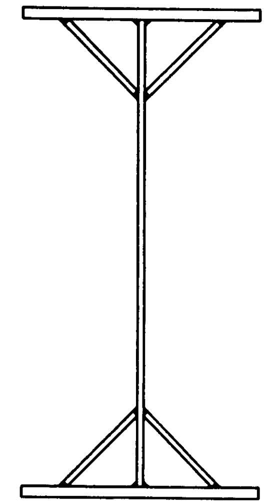


Fig. 4.3. Girder section with boxed flanges.

the first edition of the *Guide* that led to the 1961 and 1963 revisions of the AISC Specification (A13).

Simplified formulas will be presented for buckling of wide-flange sections that are not symmetrical about the major axis. Nylander (4.16) has provided an extensive treatment of this problem and also has considered the instability of continuous beams. Solvey (4.23) has summarized solutions for a great variety of end-restraint and load-distribution conditions.

4.2 Rectangular and Box-Girder Sections

The solution of the lateral-buckling problem for the rectangular beam deserves first attention because it is simple and can be used directly in the design of box girders. Because of their inherent lateral stability, box girders should be used where long and/or heavily loaded spans must be designed without lateral bracing.

For a rectangular section subjected to uniform bending moment as shown in Fig. 4.1, the critical moment is

$$M_c = \frac{\pi}{L}\sqrt{JGEI_y} \tag{4.1}$$

For a solid rectangular cross section of usual structural proportions, an accurate formula for the torsion constant J is

$$J = \frac{dt^3}{3} - 0.21t^4 \tag{4.2}$$

The greatest error in Eq. 4.2 is for the square cross section (for which no lateral-buckling investigation would be needed). The error in J for the square is about 12%, on the conservative side. For a rectangular section having a d/t ratio of 1.5, the error drops to 1.5% and rapidly approaches zero as d/t increases further.

Eq. 4.1 can be converted to an equation for critical stress by substituting it into the bending-stress formula. The result is

$$\sigma_{eb} = \frac{M_c}{S_x} = \frac{\pi}{LS_x}\sqrt{JGEI_y} \tag{4.3}$$

The following value for $(I_y)_{eff}$ accounts for the added stability induced by downward deflection (4.3):

$$(I_y)_{eff} = I_y\left[\frac{1}{1 - I_y/I_x}\right] \tag{4.4}$$

For rectangular beams or box sections having a depth of more than twice the width, the resulting increase in I_y is small. In any event (as noted in Ref. A1), Eq. 4.4 should not be applied to cambered members.

Eq. 4.1 can be used for box girders of rectangular cross section. The following close approximation of the torsion constant J, based on Bredt's theory (4.4) for thin-walled box sections (see Fig. 4.4*a*), can be used in Eq. 4.1:

$$J = \frac{4A^2}{\int_s \frac{ds}{t}} = \frac{2b^2d^2}{\frac{b}{t} + \frac{d}{t_w}} \tag{4.5}$$

where A is the total area within the middle planes of the plates making up the box periphery.

The elastic lateral-buckling stress of box girders of usual proportions is far above the yield point, in which case failure will be by inelastic lateral buckling. The inelastic lateral-buckling strength of a beam can be approximated from the basic column strength for any structural metal (see Table 2.3). This approach was validated in the case of the I-beam by Galambos (4.20), as shown in Fig. 4.2, and was suggested by Bleich* for the more general beam-column problem. It is also used by the Aluminum Company of America.† Thus, replacing σ_e and KL/r in Eq. 2.2 by σ_{eb} (elastic buckling stress for the beam) and $(KL/r)_{\text{equiv}}$, respectively,

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \pi\sqrt{\frac{E}{\sigma_{eb}}} \tag{4.6}$$

* See Ref. A1, p. 130.
† See Ref. A19, p. 115.

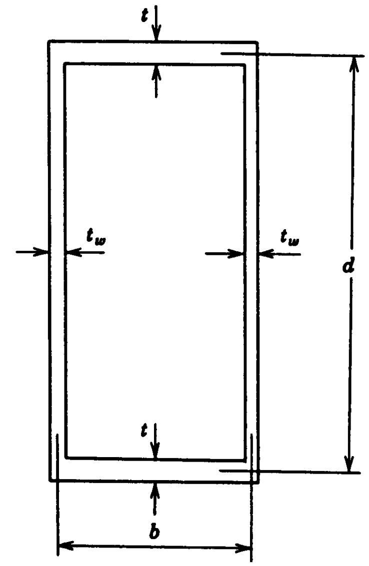


Fig. 4.4*a*. Box-girder cross section.

From Eq. 4.3 σ_{eb} can be determined and substituted in Eq. 4.6; or, alternatively, KL/r can be read directly from Table 2.2, entering the table with σ_{eb} as the argument.

Another alternative would be to replace G in Eq. 4.3 by its equivalent

$$\frac{E}{2(1 + \nu)}$$

If ν (Poisson's ratio) is taken as 0.318 (a compromise between 0.30 for steel and 0.33 for aluminum alloy), the introduction of Eq. 4.3 into Eq. 4.6 yields:

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \sqrt{\frac{5.1LS_x}{\sqrt{JI_y}}} \tag{4.7}$$

This equation is applicable to rectangular or box sections made of either steel or aluminum alloy. The approximation introduced for Poisson's ratio has much less effect than does the normal variation in mechanical properties of the materials. It should be noted that L in this equation refers to the laterally unbraced length and is not necessarily the full span between vertical supports.

Eqs. 4.6 or 4.7 can be used in the design of less conventional sections such as illustrated in Fig. 4.3, in which there are local closed or boxed-in regions which make the torsional resistance of the section relatively large. In such an application it would be necessary to provide intermittent vertical stiffeners to assist the web in maintaining the shape of the complete cross section.

An example will illustrate the application of Eq. 4.7 and demonstrate the great resistance of the box section to lateral-torsional buckling.

Example 4.1

Consider a 50-ft unsupported length of box girder having the cross section shown in Fig. 4.4*b* subjected to pure bending about the strong axis. Neglect the effect of end restraint and determine the critical buckling stress for girders made of the following materials:

(*a*) Structural steel ASTM A36,
(*b*) High-strength steel having 50-ksi yield point, and
(*c*) Aluminum alloy 2014-T6.

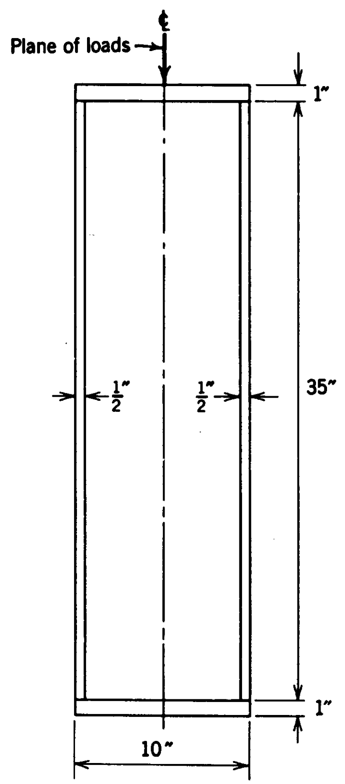


Fig. 4.4*b*.

The following are readily calculated:

$$I_x = 10055 \text{ in.}^4$$
$$I_y = 956 \text{ in.}^4$$
$$S_x = 543.5 \text{ in.}^3$$

By Eq. 4.5,

$$J = \frac{2(9.5)^2(36)^2}{9.5/1 + 36/0.5} = 2870 \text{ in.}^4$$

Then, by Eq. 4.7, the maximum bending stress in the box girder at the lateral-buckling load will be approximated by the compressive strength of a centrally loaded column with a slenderness ratio of

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \sqrt{\frac{(5.1)(600)(543.5)}{\sqrt{(2870)(956)}}} = 31.7$$

Now, from Table 2.3, the maximum (i.e., extreme-fiber) bending stresses at which the girders of the three materials will buckle laterally are found as follows:

Material	*Lateral-Torsional Buckling Stress (ksi)*
Structural steel ASTM A36	34.9
High-strength steel 50-ksi yield point	47.9
Aluminum alloy 2014-T6	49.5

The simplicity and the generality of this procedure are self-evident. The example demonstrates that the strength of a laterally unsupported box beam is not usually governed by lateral buckling, but by either local buckling or the yield stress of the material, the latter of which is closely approached in each of the foregoing cases. Local buckling should be studied as a separate problem.

In the design of box girders, there is little need to tabulate various effective lengths as determined by end restraint and loading conditions, since stress reduction below the full allowable values in bending is hardly ever needed.

4.3 Doubly Symmetric I-Shaped Beams and Girders—Introduction

I-shaped or wide-flange sections are designated as "open" in contrast to the "closed" box-girder section. The torsional rigidity and strength of an open section is but a small fraction of that of a box beam made of the same amount of metal. Thus, the open section is peculiarly susceptible to lateral buckling. Nevertheless, the wide-flange beam is one of the most-used members in bridge and building construction because of its economy and availability, its ease of framing and fabrication, its high bending and shear strength in the plane of the web, and the fact that it is usually not inconvenient to support the compression flange laterally. The plate girder also finds wide use, although built-up box girders are often used for long laterally unsupported members under lateral and/or torsional load conditions, such as crane girders.

In the case of rolled beams of long span, Eq. 4.3 (divided by an appropriate safety factor) could be used as the basis for a design formula. However, as the spans get shorter and the beams deeper, the resulting design would become increasingly conservative since there is an added resistance to twisting provided by the lateral bending rigidity of the flanges. An understanding of this added resistance requires a study of *nonuniform** torsion. In *uniform*† torsion, which occurs when equal and opposite torsional couples act at the unrestrained ends of a prismatic open-section member, there is warping or tilting of all cross-section rectangular elements whose center lines do not pass through the center of twist. In the case of the wide-flange shape under uniform torsion, the center of twist and the centroidal axis are coincident, and the flanges warp freely. In nonuniform torsion (which occurs, for example, when a wide-flange beam buckles laterally), the warping of the flanges is partially restrained and longitudinal stresses arise. Bending moments and shears are thus developed in the flanges, adding appreciably to the torsional resistance.

In the following sections, three different methods of lateral-buckling strength calculation are described. Each of these, within appropriate limits, can be used as a basis for specification design formulas by introduction of the desired factor-of-safety. The three methods are as follows:

Method A, the "basic" design procedure (Sec. 4.4). This requires the use of torsion constants, but gives the most accurate design and greatest economy over a wide range of member proportions. The formulas are more complex than those required in methods B and C.

Method B, the single-formula simplified design procedure (Sec. 4.5). This is less accurate than Method A, but does not require the use of torsion constants.

Method C, the double-formula simplified design procedure (Sec. 4.6). This is less accurate than either Method A or Method B. The lateral-

* See Ref. A9, pages 251–258.

† Commonly termed "St. Venant torsion," St. Venant having been first to develop the general theory.

buckling strength is the maximum value obtained from two different formulas. One of these is written in terms of L/r_y and the other is expressed in terms of Ld/A_f. This double-formula approach has been adopted in the AISC Specification (A13).

4.4 Method A: The Basic Procedure for Doubly Symmetric I-Shaped Beams and Plate Girders

Equation 4.8 closely approximates the elastic buckling stress for I-beams and wide-flange sections, or doubly symmetric plate girders, when such members are loaded by end couples in the plane of the web, or by transverse loads applied at the centroidal axis in the plane of the web. The fundamental relationship can be expressed in terms of the critical moment M_c previously determined by Eq. 4.1 for the rectangle, and the Euler column load, P_{ey}, for buckling in the weak direction, as follows:

$$\sigma_c = \frac{C_1}{S_x}\sqrt{M_c^2 + \frac{h^2}{4}P_{ey}^2} \tag{4.8}$$

where

$$M_c = \frac{\pi}{KL}\sqrt{JGEI_y}$$

$$P_{ey} = \frac{\pi^2 EI_y}{(KL)^2}$$

and C_1 is a coefficient that depends on load distribution and on end conditions, and K is the effective-length factor for column buckling in the *weak* plane of bending. Alternate forms of Eq. 4.8 are given by Eqs. 4.9a, 4.9b, and 4.9c (which give identical numerical results):

$$\sigma_c = \frac{C_1\pi\sqrt{EI_yGJ}}{S_x(KL)}\sqrt{1 + \frac{\pi^2 a^2}{(KL)^2}} \tag{4.9a}$$

$$\sigma_c = \frac{C_1\pi\sqrt{EI_yGJ}}{S_x(KL)}\sqrt{1 + \frac{\pi^2 C_w E}{JG(KL)^2}} \tag{4.9b}$$

$$\sigma_c = \frac{C_1\pi^2 EI_y h}{2S_x(KL)^2}\sqrt{1 + \frac{(KL)^2 JG}{\pi^2 C_w E}} \tag{4.9c}$$

In Eqs. 4.9b and 4.9c, C_w is the torsional warping constant, which can be determined for any shape,* while a in Eq. 4.9a is an alternative version applicable only to the I-shaped section and equal to $(h/2)\sqrt{EI_y/JG}$. C_w for an I-shaped section has the value $h^2I_y/4$. C_w is listed for aluminum

* See p. 120 of Ref. A1 where C_w is listed as Γ for various shapes.

structural shapes in Ref. A19 where it is denoted as C_s; while a is given for rolled wide-flange and I-shapes in Ref. 4.25.

The torsion constant J can be approximated as $\sum(bt^3/3)$ for all shapes made up of rectangular components. Accurate formulas for J for commonly rolled shapes are given in Ref. 4.24. Values of J are listed for rolled steel wide-flange and I-shapes in Ref. 4.25 as K; and are listed for aluminum alloy shapes in Ref. A19.

In Eqs. 4.9a and 4.9b, the value of the radical approaches unity in the case of very shallow and/or stocky beams and girders and the St. Venant torsional resistance is dominant; whereas, for deep and/or thin-walled girders, the radical in Eq. 4.9c approaches unity and the resistance of the compression flange to buckling largely governs.

In Eqs. 4.9 it is assumed that load is applied along the beam centroidal axis. If load is placed on the top flange of the beam there is a tipping effect which reduces the critical load; and conversely, if load is suspended from the bottom flange there is a stabilizing effect which increases the critical load. The applicable equations are:

$$\left.\begin{array}{l}\text{Bottom-flange load } \sigma_c \\ \\ \text{Top-flange load } \sigma_c\end{array}\right\} = \frac{C_1\pi\sqrt{EI_yGJ}}{S_x(KL)}\left[\sqrt{1+\frac{\pi^2a^2}{(KL)^2}(C_2^2+1)} \pm \frac{C_2\pi a}{KL}\right] \quad (4.10a)$$

or

$$\left.\begin{array}{l}\text{Bottom-flange load } \sigma_c \\ \\ \text{Top-flange load } \sigma_c\end{array}\right\} = \frac{C_1\pi^2EI_yh}{2S_x(KL)^2}\left[\sqrt{1+C_2^2+\frac{(KL)^2}{\pi^2a^2}} \pm C_2\right] \quad (4.10b)$$

For more convenient application, Eqs. 4.10 can be written in the following form:

$$\sigma_c = \frac{C_4\sqrt{I_yJ}\sqrt{EG}^*}{S_xL} \quad (4.11)$$

$$^*\sqrt{EG} = \frac{E}{1.414\sqrt{1+\nu}} = \begin{cases}18{,}000 \text{ for } E = 29{,}000 \text{ ksi}\\ 6500 \text{ for } E = 10{,}600 \text{ ksi}\\ 6100 \text{ for } E = 10{,}000 \text{ ksi}\end{cases}$$

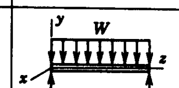
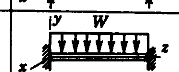
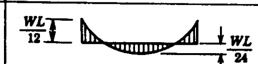
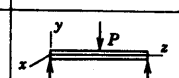
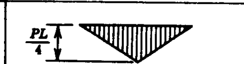
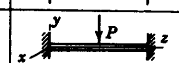
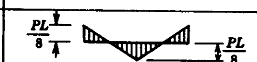
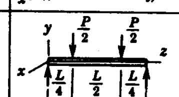
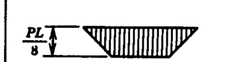
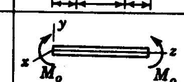
where the constant C_4 is given by

$$C_4 = \frac{C_1}{K}\pi\left[\sqrt{1+\frac{\pi^2a^2}{(KL)^2}(C_2^2+1)} \pm \frac{C_2\pi a}{KL}\right] \quad (4.12)$$

In Eqs. 4.10 and 4.12, the minus sign in the bracketed term is used for load on the top flange, and the plus sign for load on the bottom flange. For load at the centroid, or for end-moment loading, $C_2 = 0$.

Values of coefficients C_1 and C_2 for Eqs. 4.8, 4.9, 4.10, 4.12, and 4.25 for certain loading conditions are given in Fig. 4.5, which is based on information reported by Clark and Hill (4.6). Values of C_4 in Eq. 4.12 for a variety of loading conditions are charted in Fig. 4.6, which is based on Eq. 4.13 and on data taken from Fig. 4.5. C_1 and C_2 depend on load distribution, end restraint, and on the tipping or stabilizing effect due to the position of the load vertically with respect to the centroidal axis. Winter (4.5) developed expressions for the critical load as a more general function of vertical position.

When no transverse loads are carried between lateral braces, as in Fig.

Case	Loading and end restraint* about x axis	Bending-moment diagram	End restraint* about y axis	K	C_1	†C_2
①			None Full	1.0 0.5	1.13 0.97	0.45 0.29
②			None Full	1.0 0.5	1.30 0.86	1.55 0.82
③			None Full	1.0 0.5	1.35 1.07	0.55 0.42
④			None Full	1.0 0.5	1.70 1.04	1.42 0.84
⑤			None	1.0	1.04	0.42
⑥			None	1.0	1.0	0

*All beams are restrained at each end against rotation about the z axis and displacement in the x and y directions.

†$C_2 = 0$ when load is applied at beam centroidal axis. For load applied at top or bottom flange, use appropriate signs in Eqs. 4.10 and 4.12.

Fig. 4.5. Values of coefficients in Eqs. 4.8, 4.9, 4.10, 4.12, and 4.25.

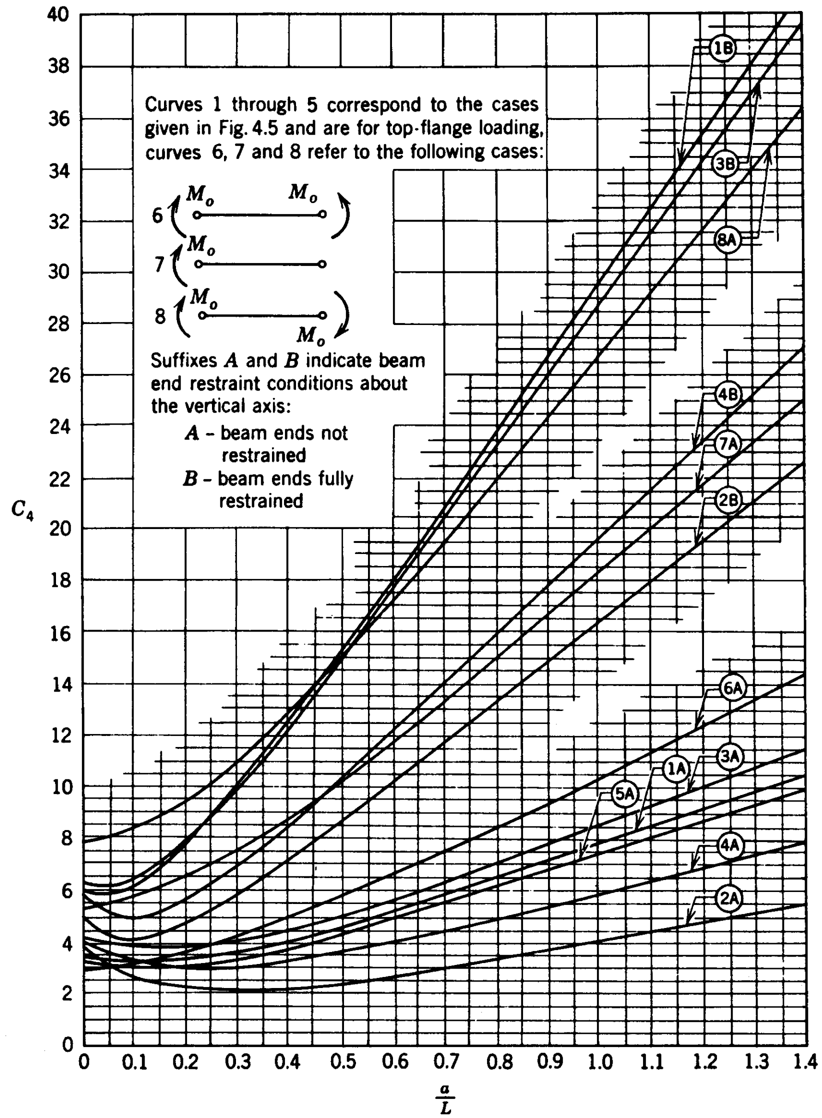


Fig. 4.6. Values of coefficient C_4 (see Eq. 4.12).

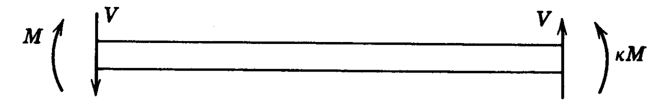


Fig. 4.7. Beam or beam segment with no loads between points of lateral bracing.

4.7, $C_2 = 0$ and C_1 is given conservatively by the following equation, presented by Salvadori (4.27):

$$C_1 = 1.75 - 1.05\kappa + 0.3\kappa^2 \tag{4.13}$$

where κ is the algebraic ratio of the smaller to the larger end moment. C_1 is in no case to be taken as greater than 2.3. ($C_1 = 2.3$ when $\kappa \leqslant -0.46$.) Cases of special interest are as follows:

Moment	κ	C_1
Uniform moment	1	1.0
Moment zero at right end	0	1.75
End moments inducing reverse curvature	−0.46 to −1	2.3

End restraint about the y–y axis is not usually present in practice. It cannot be assumed present in a continuous beam where alternate unbraced spans could buckle in opposite lateral directions. K of 0.5 is a limiting idealization which may be of use as a basis for determination of values of C_1 and C_2 for values of K between 0.5 and 1.0.

In the case of the simple cantilever beam having complete fixity about both axes at the supported end, $K = 1$ and C_1 can be taken conservatively as 1.3 for a concentrated end load, and as 2.05 for a uniform loading.

If the buckling stress, calculated from Eqs. 4.8, 4.9, 4.10, or 4.11, is greater than the proportional limit (which is assumed to be $\sigma_y/2$ in the case of steel), the inelastic buckling stress should be found from Table 2.3 on the basis of the equivalent column KL/r determined by substituting the calculated buckling stress into Eq. 4.6 as σ_{eb} (see Fig. 4.2 for validation of this procedure).

In plate girders, the torsion constant J may be computed as $\sum(bt^3/3)$ for all rectangular component parts. In this summation, the total thickness T of multiple-plate sections may be used between bounding weld or fastener lines, but the thickness t of the individual plates must be used for portions outside such lines. This procedure was originally suggested by de Vries (4.11) and has been validated by Chang and Johnston (4.28). However, if the rivet or bolt pitch exceeds the distance required to provide clamped "integral" action, the component plates will tend to twist individually between fastener lines and the "integral" torsion constant (designated as J_I) must be multiplied by a reduction factor to obtain the "effective" torsion constant J.

The critical fastener pitch p' is approximately

$$p' = A + T \tag{4.14}$$

where A is the diameter of the rivet or bolt head and T the total thickness of the several plates that are riveted or bolted together. If the actual pitch p is greater than p', the effective torsion constant J can be found as follows:

$$J = \left[\frac{1 - C}{1 - Cp'/p}\right] J_I \tag{4.15a}$$

where J_I = integral torsion constant

and

$$C = \left(\frac{p - p'}{0.8b}\right)\left(1 - \frac{1.2}{N^2}\right) \tag{4.15b}$$

where b = width of flange

N = number of component plate thicknesses in flange (including flange angles)

Eqs. 4.15 are simplifications of formulas given in Ref. 4.28, and are not to be considered as precise relationships, but rather as empirical formulas that approximate a highly complex condition. In any event, great accuracy in calculation of the torsion constant is not important, since the buckling of deep plate girders with intermittent lateral support is governed primarily by EI_y rather than JG. (See Example 4.3.)

Two examples will demonstrate the basic procedure for determining the lateral buckling strength for wide-flange rolled shapes and for built-up plate girders.

Example 4.2

A 30W108 simply supported steel beam of ASTM A36 material carries a uniform load on the top flange and is laterally unsupported throughout its span of 22 feet. The end connections are riveted web angles which do not effectively restrict rotation about the beam cross-section axes. The clearance between the column faces and the beam ends is sufficient to permit beam-end warping. Find the permissible bending stress in the beam.

From Ref. 4.25, torsion constants for a 30W108 are found as:

$$J = 5.35 \text{ in.}^4$$
$$a = 117.7 \text{ in.}$$

Hence

$$\frac{a}{L} = \frac{117.7}{264} = 0.445$$

From the AISC Manual,

$$I_y = 135.1 \text{ in.}^4$$
$$S_x = 299.2 \text{ in.}^3$$

From Fig. 4.5, for Case 1,

$$K = 1.0$$
$$C_1 = 1.13$$
$$C_2 = 0.45$$

By Eq. 4.12 (or from Fig. 4.6), C_4 can be found:

$$C_4 = 1.13\pi\left[\sqrt{1 + (0.445\pi)^2(1 + \overline{0.45}^2)} - (0.45)\pi(0.445)\right] = 4.26$$

Then, by Eq. 4.11,

$$\sigma_c = \frac{(4.26)(18000)\sqrt{(135.1)(5.35)}}{(299.2)(22)(12)} = 26.1 \text{ ksi}$$

Since 26.1 ksi is above the assumed maximum elastic buckling stress of $\sigma_y/2$, the equivalent column KL/r for this condition is determined by Eq. 4.6 (or from Table 2.2):

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \pi\sqrt{\frac{29000}{26.1}} = 104.7$$

and, referring to Table 2.3 for $KL/r = 104.7$, the column inelastic buckling stress is found to be 23.6 ksi. This stress should then be divided by the factor-of-safety appropriate to the pertinent specification, giving the allowable bending stress in the beam.

Example 4.3

The riveted plate girder of A36 steel shown in Fig. 4.8 has elastic constants listed below, and spans 75 ft, carrying equal concentrated vertical loads at the one-third points. Lateral and torsional bracing is provided at the one-third points. The middle one-third of the girder is under pure bending moment. Assuming (conservatively) that no bending constraints are introduced at the one-third points, determine the critical stress.

$$I_x = 282{,}600 \text{ in.}^4$$
$$I_y = 2102 \text{ in.}^4$$
$$S_x = 5514 \text{ in.}^3$$

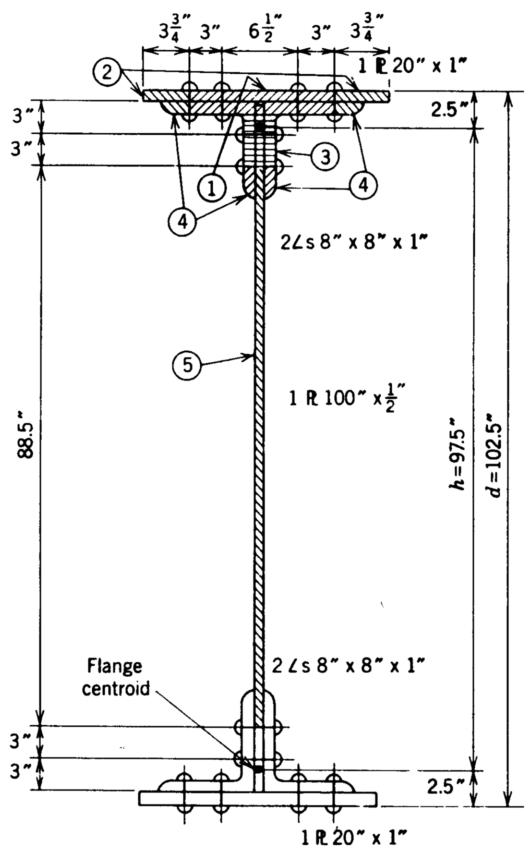


Fig. 4.8. Section of riveted or bolted plate girder.

$$J_I = \sum \frac{bt^3}{3}$$

Sections 1
Horizontal flange portion between outer gage lines $(2)(12.5)(2)^3(\tfrac{1}{3}) = 66.7$

Sections 2
Horizontal flange portion outside of outer gage lines $(4)(3.75)(1)^3(\tfrac{1}{3}) = 5.0$

Sections 3
Vertical portion of flange angles and web between gage lines of angles $(2)(5)(2.5)^3(\tfrac{1}{3}) = 52.1$

Sections 4
Extremities of angles $(8)(2)(1)^3(\tfrac{1}{3}) = 5.3$

Sections 5
Web between outer gage lines of angles $(88.5)(\tfrac{1}{2})^3(\tfrac{1}{3}) = 3.7$

Integral torsional constant $J_I = 132.8\text{ in.}^4$

The rivet pitch in the central one-third of the girder is assumed to be 12 in. The critical pitch for full integral action (Eq. 4.14) is

$$p' = 1 + 2 = 3\text{ in.}$$

and from Eq. 4.15b,

$$C = \frac{9}{16}\left(1 - \frac{1.2}{4}\right) = 0.39$$

Solving Eq. 4.15a,

$$J = \left(\frac{1 - 0.39}{1 - (0.39)(0.25)}\right) J_I = 0.68(132.8) = 90.3\text{ in.}^4$$

The torsion bending constant is

$$a = \frac{h}{2}\sqrt{\frac{EI_y}{GJ}} = \frac{97.5}{2}\sqrt{\frac{2.6 \times 2102}{90.3}} = 378\text{ in.}$$

For the unbraced center girder segment of 25 ft,

$$\frac{a}{L} = \frac{378}{(25)(12)} = 1.26$$

Since the girder in the length under consideration carries uniform moment, $C_1 = 1$ and $C_2 = 0$. Then, from Eq. 4.12,

$$C_4 = \pi[\sqrt{1 + \pi^2(1.26)^2}] = 12.87$$

and by Eq. 4.11, the critical bending stress is found:

$$\sigma_c = \frac{(12.87)(18{,}000)\sqrt{(2102)(90.3)}}{(5514)(300)} = 61.1\text{ ksi*}$$

Since this value exceeds the assumed proportional limit of $\sigma_y/2 = 18.0$ ksi, it is evident that buckling will occur in the inelastic range rather than in the elastic range. Therefore, by Eq. 4.6, the buckling stress for the girder will be equivalent to that of a column having a slenderness ratio of

$$\left(\frac{KL}{r}\right)_{\text{equiv}} = \pi\sqrt{\frac{29000}{61.1}} = 68.4$$

* If the integral torsion constant $J_I = 132.8$ had been used in the computation, a critical stress of 61.8 ksi would have resulted. This illustrates the fact that in the case of intermittently supported deep girders, the contribution of the torsion constaht J to the lateral-buckling strength is negligible.

and from Table 2.3, for ASTM A36 steel, the buckling stress is found as

$$\sigma_c = 30.7 \text{ ksi}$$

By way of comparison, the AREA Specification (A7) allows

$$F = 20{,}000 - 6\frac{L^2}{b^2} = 20{,}000 - 6(15.0)^2 = 18{,}650 \text{ psi}$$

giving, in this particular instance, a factor-of-safety of 1.65. Of course, no general conclusions should be drawn from this single comparison, as the factor-of-safety using the L/b formula will vary considerably as the span, loading condition, and girder cross section vary.

4.5 Method B: The Single-Formula Simplified Procedure for Doubly Symmetric I-Shaped Beams and Plate Girders

Rolled I-shaped sections that are used as laterally unsupported beams usually have a depth of about twice the flange width and a web thickness of about two-thirds the flange thickness. Using these as typical proportions, the following approximate formulas for section properties result:

$$S_x = \frac{2Ar_x^2}{d}$$ A = area of cross section

$$I_y = Ar_y^2$$ L = distance between lateral supports

$$J = 0.27At^2$$ r_y = radius of gyration about web axis

$$h = 0.95d$$ d = beam depth

$$a = \frac{h}{2}\sqrt{\frac{EI_y}{JG}} \approx \frac{1.47dr_y}{t}$$ t = flange thickness

$$C_w = \frac{h^2}{4}I_y = 0.226Ar_y^2d^2$$ h = distance between flange centroids

If the condition of pure bending is assumed with no warping restraint at the ends of the beam ($C_1 = K = 1$ per Eq. 4.13), the following simplified formulas are obtained upon substitution of the above parameters into Eqs. 4.9a and 4.9c:

$$\sigma_c = \frac{3.06E}{Ld/r_yt}\sqrt{1 + 21.48\frac{(d/t)^2}{(L/r_y)^2}} \qquad (4.16a)$$

$$\sigma_c = \frac{14.2E}{(L/r_y)^2}\sqrt{1 + 0.0466\frac{(L/r_y)^2}{(d/t)^2}} \qquad (4.16b)$$

In each of these equations it can be seen that two parameters, L/r_y and d/t, govern the result.

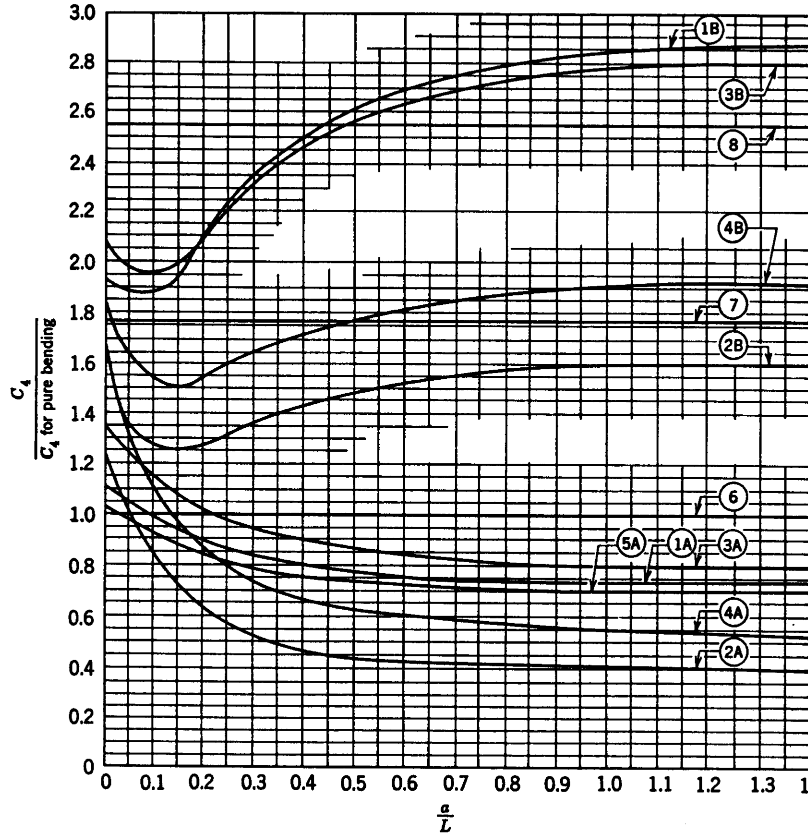


Fig. 4.9. Curves of $C_4/(C_4$ for pure bending) versus a/L. The curves are numbered the same as those in Fig. 4.6.

If either Eq. 4.16a or 4.16b were used in a specification, other load and restraint conditions could be evaluated by means of a multiplier that would be a function of a/L. Such correction factors have been plotted in Fig. 4.9, based on the plotted data of Fig. 4.6.

Eqs. 4.16a and 4.16b give $\sigma_c = 30.7$ ksi for the beam of Example 4.2 under pure bending moment. From Fig. 4.9, with

$$\frac{a}{L} = \frac{1.47 \times 29.82 \times 2.06}{0.760 \times 264} = 0.45$$

a correction factor of 0.78 is obtained for the case of uniform load on the top flange with beam ends not restrained about the vertical axis (Curve 1A), giving an elastic buckling stress of $0.77 \times 30.7 = 23.9$ ksi, as compared with 26.1 ksi from the more exact procedure of Example 4.2. Proceeding as before, the equivalent column KL/r for $\sigma_{eb} = 23.9$ is found from Eq. 4.6 to be 109.4, giving an inelastic buckling strength of 22.5 ksi by Table 2.3. This value corresponds closely to the value of 23.6 ksi determined in Example 4.2.

4.6 Method C: The Double-Formula Simplified Procedure for Doubly Symmetric I-Shaped Beams and Plate Girders

In the case of shallow and/or thick-walled beams, the radical of Eq. 4.16a can conservatively be taken as equal to unity, giving the following approximate formula* for the case of pure bending:

$$\sigma_c = \frac{3.06E}{Ld/r_y t} \tag{4.17a}$$

Similarly, from Eq. 4.16b, in the case of deep and/or thin-walled beams, a satisfactory approximation for the case of pure bending is

$$\sigma_c = \frac{14.2E}{(L/r_y)^2} \tag{4.17b}$$

These two equations give identical results when

$$\frac{L}{r_y} = 4.64\frac{d}{t}$$

Since either equation gives a conservative estimate for the special case of uniform bending moment and simple end supports (see Case 6, Fig. 4.5), the larger value of σ_c should be chosen. Under this premise, it is evident that the following rule will apply:

$$\textit{Use Eq.} \begin{Bmatrix} 4.17a \\ 4.17b \end{Bmatrix} \textit{when } L/r_y \textit{ is} \begin{Bmatrix} \textit{greater} \\ \textit{less} \end{Bmatrix} \textit{than } 4.64(d/t)$$

For example, for the beam of Example 4.2,

$$\frac{L}{r_y} = \frac{(22 \times 12)}{2.06} = 128.2$$

whereas $4.64(d/t) = 4.64 \times 29.82/0.760 = 182.1$. Thus, $128.2 < 182.1$ and by the previous rule, Eq. 4.17b is indicated, giving $\sigma_c = 25.1$ ksi for the case of pure bending. (Eq. 4.17a would give a spurious value $\sigma_c = 17.7$ ksi.) This value can be converted to that for uniform top-flange loading by multiplying by the factor 0.78, obtained from curve 1A of Fig. 4.9 where $a/L = 0.45$, giving a value of 19.6 ksi for the elastic buckling strength. This corresponds to an inelastic buckling strength of 19.5 ksi as

* A similar approach, in terms of b instead of r_y, is the basis for Formula 5 of Ref. A13.

obtained from Table 2.3 for an equivalent column $KL/r = 121$; and 19.5 ksi is conservative in comparison with the "exact" value of 23.6 ksi obtained in Example 4.2.

In the case of application to light-gage steel members, as covered by the AISI Specification (A11), Eq. 4.17a is not needed and the following equation is applicable to sections having strong-axis symmetry:

$$\sigma_c = \frac{2.29Ed^2}{(L/r_y)^2 r_x^2} \tag{4.18}$$

For most light-gage I shapes, $r_x \approx 0.38d$, in which case Eq. 4.18 reduces to the following equation similar to Eq. 4.17b:

$$\sigma_c = \frac{16E}{(L/r_y)^2} \tag{4.19}$$

The allowable-stress formula for I-shaped sections by the AISI Specification (A11) is:

$$F_b = \frac{280{,}000{,}000}{(L/r_y)^2} \text{ (psi)} \tag{4.20}$$

which embodies a factor-of-safety of 1.66 with respect to Eq. 4.19 with $E = 29{,}000{,}000$ psi. These specifications also require use of a transition stress curve in the inelastic range, as follows:

$$F_b = \tfrac{10}{9} F_b^* - \left[\frac{F_b^{*2}}{907{,}000{,}000}\right]\left(\frac{L}{r_y}\right)^2 \tag{4.21}$$

where F_b^* is the basic allowable design stress. When L/r_y is less than $10{,}050/F_b^*$, $F_b = F_b^*$.

The AISC Specification (A13) uses a double-formula procedure for determining the allowable bending stress of an unbraced beam. This specification stipulates† that the allowable compression F_b on extreme fibers of rolled shapes, plate girders, and built-up members having an axis of symmetry in the plane of the web (other than box-type beams) shall be the larger value computed by the following formulas, but not more than $0.60\sigma_y$:

$$F_b = \left[1.0 - \frac{(L/r)^2}{2C_c^2 C_1}\right] 0.60\sigma_y \tag{4.22}$$

$$F_b = \frac{12{,}000{,}000}{Ld/A_f} \tag{4.23}$$

where L is the unbraced length of the compression flange in inches, r is the radius-of-gyration about the web center line of a tee section comprising the compression flange plus the upper one-sixth of the web, A_f is

† See Ref. A13, Formulas (4) and (5).

the area of the compression flange, C_1 is a moment-gradient factor given by Eq. 4.13, and

$$C_c = \pi\sqrt{\frac{2E}{\sigma_y}} \tag{4.24}$$

The following stipulations are also made: (1) C_1 is to be taken as unity when the bending moment at any point within the unbraced length L is larger than that at both ends of this length; and (2) when L/r in Eq. 4.22 is less than 40, this equation need not be applied. This latter exemption introduces a discontinuity in the allowable bending stress ranging from 2.0% to 4.6% (depending on the value of C_1) for steel having a yield point of 36 ksi. The discontinuity is larger for steels of higher strength.

4.7 Summary with Respect to Design Formulas for Doubly Symmetric I-Shaped Beams and Plate Girders

1. Equation 4.11 provides an accurate basis for analysis and can be used conveniently if the constants C_4, J, and a (or C_w) are available.
2. When torsion constants J and a are not readily available, the buckling stress for beams in pure bending can be approximated by using either Eq. 4.16a or Eq. 4.16b. For load conditions other than pure bending, the correction factors plotted in Fig. 4.9 can be used.
3. If equations simpler than Eqs. 4.16a and 4.16b are desired, the greater result obtained from Eqs. 4.17a and 4.17b can be used.
4. For doubly-symmetric girders having shapes or proportions substantially different than those of rolled beams, Eq. 4.8 or Eq. 4.11 is recommended.
5. If the buckling stress is greater than 50% of the specified yield stress in the case of steel, or greater than about 60% of the specified yield stress in the case of aluminum alloy, an equivalent column slenderness ratio should be determined by Eq. 4.6, or by reference to Table 2.2, and the inelastic buckling stress is then found from the appropriate basic column strength values of Table 2.3. Alternatively, the equivalent slenderness ratio can be used in column design formulas to determine the allowable stress directly.

4.8 Girders Symmetrical about the y–y Axis but Unsymmetrical about the x–x Axis

This type of girder is sometimes required in situations where lateral loads are applied to one of the flanges. Such a flange is therefore made stronger with respect to lateral loads than the other flange.

Eqs. 4.8 and 4.10 are special cases of a more general equation which, as Clark (4.6) has shown, gives accurate solutions for either doubly or singly symmetric girders under a wide variety of load and end restraint conditions:

$$\sigma_c = C_1\frac{\pi^2 EI_y}{S_c(KL)^2}\left[C_2 g + C_3 j + \sqrt{(C_2 g + C_3 j)^2 + \frac{C_w}{I_y}\left(1 + \frac{GJ(KL)^2}{\pi^2 EC_w}\right)}\right] \tag{4.25}$$

The following parameters in or related to Eq. 4.25 have not been defined heretofore:

S_c = section modulus for compression,

$$j = e + \frac{1}{2I_x}\int_A y(x^2 + y^2)\,dA, \tag{4.26}$$

g = distance from the shear center of the girder to the point of application of transverse load (positive when the load is below the shear center, otherwise negative),

C_3 = coefficient dependent on girder loading and end restraint, but pertinent only for girders unsymmetrical about the x–x axis since j is zero for those symmetrical about this axis.

e = distance from centroid of girder cross section to the shear center (positive if the shear center lies between the centroid and compression flange, otherwise negative).

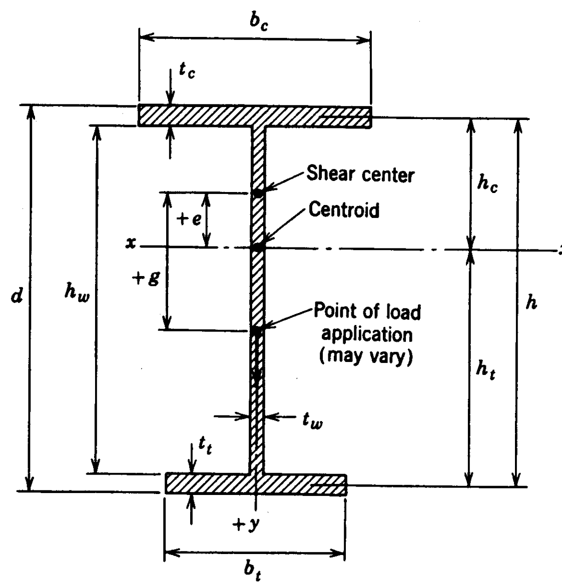


Fig. 4.10 (*see legend on next page*).

Areas

$$A_c = b_c t_c$$
$$A_t = b_t t_t$$
$$A_w = h_w t_w$$
$$A = A_c + A_t + A_w$$

Section Moduli

$$S_c = \frac{I_x}{h_c + t_c/2}$$
$$S_t = \frac{I_x}{h_t + t_t/2}$$

Location of Centroid

$$h_c = \frac{1}{A}\left[A_t h + \frac{A_w}{2}(h_w + t_c)\right]$$

Location of Shear Center

$$e = \frac{I_c h_c - I_t h_t}{I_y}$$

Moment of Inertia about y Axis

$$I_c = \frac{A_c b_c^2}{12} \qquad I_t = \frac{Atb_t^2}{12} \qquad I_y \approx I_c + I_t$$

Moment of Inertia about x Axis

$$I_x = A_c h_c^2 + A_t h_t^2 + \frac{A_w h_w^2}{12} + A_w\left[\frac{h_w + t_c}{2} - h_c\right]^2$$

Constants

$$j = e + \frac{1}{2I_x}\left[h_t\left(I_t + A_t h_t^2 + \frac{h_t^3 t_w}{4}\right) - h_c\left(I_c + A_c h_c^2 + \frac{h_c^3 t_w}{2}\right)\right] \tag{4.27}$$

$$J \approx \tfrac{1}{3}[b_c t_c^3 + b_t t_t^3 + h_w t_w^3] \tag{4.28}$$

(or use tabular values if available from Ref. 4.25)

$$C_w = \frac{h^2 I_c I_t}{I_y}$$

Fig. 4.10. Parameters pertinent to Eq. 4.24 for lateral buckling of singly symmetric I-shaped sections.

C_1 and C_2 are coefficients for various load and end restraint conditions, and have been given in Fig. 4.6. Figure 4.10 shows notation and formulas for the various girder section properties required in connection with Eq. 4.25.

For the case of uniform bending moment, $g = 0$ and C_1 and C_3 are equal to unity (4.6). If j is assumed to be equal to e, as an approximation, Eq. 4.25 becomes the same as a solution by Hill (4.10), as follows:

$$\sigma_c = \frac{\pi^2 EI_y}{S_c(KL)^2}\left[e + \sqrt{e^2 + \frac{C_w}{I_y}\left(1 + \frac{GJ(KL)^2}{\pi^2 EC_w}\right)}\right] \tag{4.29}$$

Eq. 4.29 has been checked against Eq. 4.25, the exact solution for girders having a wide range of proportions, and is found to give conservative

values for the buckling stress when the compression flange is larger than the tension flange, but too-high values when the reverse is true.

For the tee shape the shear center is at the intersection of the middle planes of the flange and web. Thus $C_w \approx 0$, and for the case of pure bending with the flange in compression, with K and C_3 both equal to unity and $C_2 = 0$, Eq. 4.25 yields

$$\sigma_c = \frac{\pi^2 EI_y j}{S_c L^2}\left[1 + \sqrt{1 + \frac{GJL^2}{\pi^2 Ej^2 I_y}}\right] \tag{4.30}$$

For the tee section, the expression for j (Eq. 4.26) is also simplified, since $h_c = e$ and A_t and I_t are zero. The result is

$$j = h_c + \frac{1}{2I_x}\left[\frac{h_t^4 t_w}{4} - h_c\left(I_c + A_c h_c^2 + \frac{h_c^2 t_w}{4}\right)\right] \tag{4.31}$$

Winter (4.5) has also derived a simplified equation for the unsymmetrical I-section in pure bending. His solution gives values of σ_c that are too high when the compression flange is larger than the tension flange, but in good agreement when the tension flange has the greater area. Winter's solution can be obtained from Eq. 4.25 by replacing the quantity j by the expression $h(I_c - I_t)/2I_y$, to give:

$$\sigma_c = \frac{\pi^2 Ed}{2S_c(KL)^2}\left[I_c - I_t + I_y\sqrt{1 + \frac{4GJ(KL)^2}{\pi^2 I_y Ed^2}}\right] \tag{4.32}$$

Since the "exact" solution is bracketed by the approximate solutions of Hill and Winter, a better result than either of these two, quite close to the exact, can be obtained by averaging the two:

$$\sigma_c = \frac{\pi^2 EI_y d}{2S_c(KL)^2}\left[\frac{e}{d} + \frac{I_c - I_t}{2I_y} + \sqrt{1 + \frac{2e^2}{d^2} + \frac{4GJ(KL)^2}{\pi^2 d^2 EI_y}}\right] \tag{4.33}$$

For the special case of the rectangular-shaped flange, with the area of the compression flange greater than that of the tension flange (as is usual in this type of design problem), an equation of the following type has been found to give good (and generally conservative) agreement with the exact solution:

$$\sigma_c = \frac{5.75E}{(L/r_{yc})^2}\left[0.6 + \sqrt{0.75 + \frac{(L/r_y)^2}{12(d/t)^2}}\right] \qquad (A_c > A_t) \tag{4.34}$$

In this equation, r_{yc} and t_c refer to the compression flange. It may be noted that the parameters of this equation are similar to those of Eqs. 4.16 for doubly symmetric sections.

In Eq. 4.34, σ_c is independent of the amount of tension-flange area. Nevertheless, over the extreme range of sections that has been used as a basis for comparison of various formulas by Clark and Hill (4.6), Eq. 4.34 gives results generally as good as, or better than, the more complex approximate formulas.

A paper by Nylander (4.16) may be referred to for additional material on unsymmetrical sections.

4.9 Doubly Symmetric Girders with Variable Flange Area

A single-web plate girder with variable cross section will usually have lateral support that is not continuous, and the lateral-buckling design check must refer to a segment in which the cross section can be assumed constant. However, during construction and prior to erection of lateral bracing, or in some other situation wherein no lateral support is provided, a lateral-buckling design check may be needed for the girder as a whole. This can be obtained by using the following approximate equation:

$$\sigma_{c(v)} = \sigma_c \sqrt{\frac{I_{\min}}{I_{\max}}} \tag{4.35}$$

where $\sigma_{c(v)}$ is the critical stress for a girder of variable cross section and σ_c is that for a girder having a constant I equal to $I_{\max}$, found either by the basic Eq. 4.11 or by simplified Eqs. 4.16. Eq. 4.35 is an empirical simplification of the results of preliminary studies by Austin in which an idealized loading was assumed of a type that would produce, at every cross section, the same maximum fiber stresses in the flanges. The proposed approximate design equation is simpler and more conservative than similar proposals made in Great Britain (4.26).

4.10 Channels and Special Shapes

If a laterally unsupported channel section is both loaded and supported by vertical forces that pass through the centroid of the channel, it will twist as well as bend, except for the special case wherein the loads act normal to the plane of the web, causing bending in the weakest direction. For loadings other than the special case, the channel is subjected to combined bending and torsion.

If, however, such a channel is loaded and supported by forces that pass through the shear center, it will bend without twisting. If, further, the loads are parallel to the plane of the web, bending will be in this plane and lateral-torsional buckling may be a problem if lateral support is absent.

It is usually not practicable to place uniform loading on a channel in such a way that it acts through the shear center, without at the same time

providing continuous lateral as well as rotational support and thus eliminating the buckling problem. However, if an otherwise laterally unsupported channel has concentrated loads brought in by other members than frame into it, such loads can be considered as being applied at the shear center, *provided that the span of the framing member is measured from the channel shear center and the framing connections are designed for the moment and shear at the connection.* The end connections of the channel must provide effective support at the shear center.

Referring to Fig. 4.11, the distance from the middle plane of the channel web to the shear center is

$$e = \frac{ch^2}{4r_x^2} \tag{4.36}$$

The warping and torsion constants are, respectively,

$$C_w = \frac{h^2 I_y}{4}\left[1 - \frac{c(e-c)}{r_y^2}\right] \tag{4.37}$$

$$J \approx \tfrac{1}{3}(2bt_f^3 + ht_w^3) \tag{4.38}$$

Hill (4.9) has shown that the use of critical-stress formulas derived for an I-shaped section will give results for a channel that err by no more than 6%. However, the error is on the *unsafe* side.

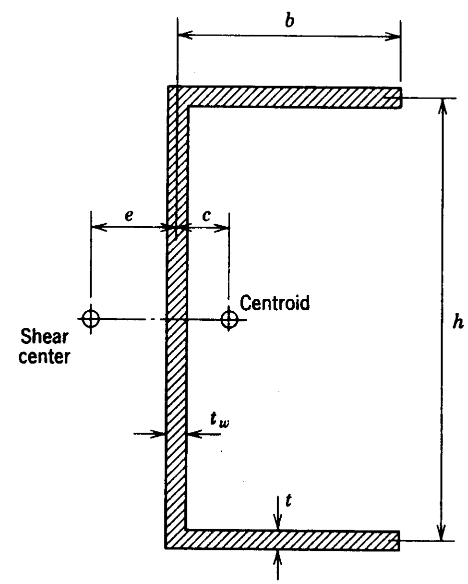


Fig. 4.11.

It is recommended that Eq. 4.3 be used as a simple procedure for determining the allowable bending stress for the unsupported region between loads for a rolled channel section used as a beam loaded through fairly rigid connections that eliminate torsion, as just described. This equation neglects the stabilizing effect of warping rigidity, and is thus not applicable to thin-walled sections where this effect is of primary importance.

Alternatively, if a more precise determination is desired, Eq. 4.8b, which includes the effect of warping constraint, can be used. Modifications for load, end restraint, and inelastic behavior can then be made as outlined for I-shaped beams (see Sec. 4.4).

In the case of the Z section, if concentrated loads are brought in by framed members which prevent lateral deflection, and which are designed to resist the forces resulting from the tendency toward transverse deflection, the permissible stress with respect to lateral buckling can also be computed using Eq. 4.3. In this case, the minimum principal moment-of-inertia should be used in place of I_y, since lateral buckling always tends to occur about the weakest bending axis. The torsion constant J is identical with the value for a channel as given by Eq. 4.35.

Design information for buckling of compression flanges of many light-gage special sections is presented in the AISI Design Manual, *Light Gage Cold-Formed Steel* (see Refs. A11 and A12). This information is based on research by R. T. Douty (4.21).

4.11 Types of Lateral Support

Continuous lateral support is the type supplied by a positively attached continuous floor, roof, or similar system. Embedment of the top flange will also provide such continuous lateral support.

Winter's studies (4.30) provide simple methods for determining minimum requirements for lateral support, continuous or intermittent, both for columns and for beams.

If the compression flange of a beam or girder is supported at intermediate points by special lateral braces or framed beams, the unsupported length should be taken as the distance between such points, and this length should be used in the determination of the permissible bending stress. Both the strength and the rigidity of lateral braces must be adequate. Lateral braces permit higher allowable design stresses in a beam because they induce higher modes of buckling. Thus, a laterally braced beam loaded to failure in buckling will usually develop node points in the buckled configuration at the lateral support points. For a perfectly straight and ideally loaded beam, the design requirement for a lateral brace is rigidity, not strength, as no calculable stress is induced in the brace. If, however, either initial curvature or misalignment of loading is assumed in

a beam, then a calculable force is induced in the lateral brace and a larger rigidity of the brace is required than for the perfectly straight and ideally loaded beam.

On the basis of a study of a number of hypothetical initial-curvature and load-misalignment conditions, Zuk (4.29) confirms the customary practice of designing each lateral support for 2% of the total compressive force that exists concurrently in the compression flange of the laterally braced beam or girder. He also believes that braces designed for such forces will necessarily have sufficient rigidity to prevent the beam or girder from buckling in a lower mode. Recent studies by Lay (4.22) have demonstrated the correctness of this rule as extended to plastic design.

4.12 Effect of Concrete Encasement

If a laterally unsupported, doubly symmetric I-shaped beam is encased within a rectangular section of minimum-strength concrete (as might be done for fireproofing) that is properly held in place by mesh reinforcement, the beam resistance to lateral buckling will usually be increased sufficiently to allow design to the working stress that is permitted for full lateral support.

For example, consider a 24I79.9 beam 500 in. long that is encased in a 26 × 9 in. cross section of concrete. Assume that the concrete has a compressive strength of only 1000 psi, and a modulus of elasticity of one-thirtieth that of steel. If the beam were unencased, the critical stress by Eq. 4.17 would be 10.1 ksi, corresponding to a critical bending moment of 1756 kip-in. The critical moment for a 26 × 9 in. rectangular plain-concrete beam of the same length is 6.0 times this value even for the minimum-strength concrete assumed. Thus, it is safe to state that any concrete-encased steel beam can be expected to reach its yield stress prior to lateral buckling. (The concrete should, as noted, be held in place with mesh reinforcement.)

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Chapter Five

Plate Girders

5.1 Introduction

This chapter deals with the buckling and ultimate strength of plate girders, as influenced by the behavior of the principal component parts, i.e., the web, the flanges, and the stiffeners. The design of these parts for girder loadings both below and above the buckling load of the web will be considered. Post-buckling concepts have already (1966) been introduced into specifications for design of girders for buildings (including light-gage metal structures), but not yet into specifications for design of highway girders, railway girders, or mill-building girders subjected to fatigue loading.

The buckling strength of a plate-girder web that has regularly spaced transverse stiffeners is augmented by its capacity for truss-like behavior. The action of such a girder web loaded beyond its buckling strength can be compared with that of the web members of the early form of bridge truss shown in Fig. 5.1. The double diagonals in each panel of this truss are designed only for tension, and the vertical web members carry only compression, so that under a uniform load those diagonals shown as dashed lines would be in compression and their contribution to the shear resistance of the web would be neglected (although each would, in fact, contribute its small buckling load). Thus, for a plate girder having an adequately stiffened web, the shear resistance between any two transverse stiffeners will be the sum of (1) the shear buckling strength of the web and (2) the vertical component of the web yield strength in direct diagonal tension. This latter is sometimes termed "tension-field" strength.

That the classical buckling theory underestimates the carrying capacity of plate girders has been recognized for a long time, and it is for this reason that many specifications employ a low safety factor with respect to web buckling for plate girders. The evaluation of the ultimate carrying capacity of plate girders has been studied by many investigators, some of whom tried to overcome the shortcomings of the small-deflection buckling theory by employing the large-deflection theory. When lateral deflections

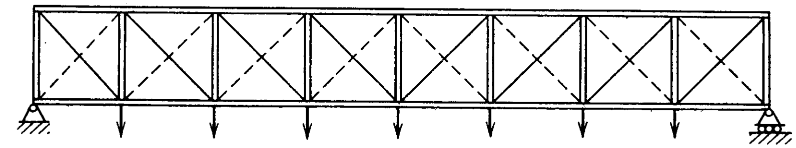

Fig. 5.1. Truss with tension diagonals.

of the web exceed its thickness, membrane stresses (which are neglected in small-deflection theory) develop and produce a stable state of equilibrium at loads above the buckling load. Large-deflection theory considers both bending and membrane stresses in predicting the post-buckling behavior of web plates, and has validity only when the web-panel framing can support the induced reactions.

Because of the complexity of the large-deflection theory, only a few cases have been solved. Solutions for a square plate and for an infinitely long plate subjected to the action of shearing forces along the edges have been obtained by Bergmann (5.26) and by Skaloud (5.27). This theory has not been extended to the inelastic range. Since collapse of a web panel is associated most often with the plastic stretching of the web and the collapse of the framing (flanges and stiffeners), neither of the foregoing theories is able to predict the actual load-carrying capacity of plate girders.

Extensive studies, both analytical and experimental, have been made by Basler and Thürlimann (5.28 through 5.40) to study the post-buckling behavior of the web and the framing. This work led in 1961 to a more realistic basis for the design of plate girders in building construction (A13).

For plate girder web design to be economical, stiffeners must be used. Stiffeners may be transverse and/or longitudinal. Transverse stiffeners increase the resistance of the web to shear buckling, but are not efficient in increasing resistance to buckling in bending unless they are very closely spaced. Longitudinal stiffeners placed in the compression zone of the web effectively increase its resistance to buckling due to bending.

A combination of both transverse and longitudinal web stiffeners is often used. Knowledge of stiffener behavior is not complete and further investigations in this area are needed in order to determine the effect of details, of initial imperfections, of loads in direct bearing on flanges, and end support details for tension-field webs.

Girders with tubular flanges and with closed stiffeners possess a high ultimate strength compared with conventional girders and are discussed briefly in this chapter. AASHO, AREA, and AISC specifications are also discussed, Finally, a brief discussion of possible further developments is given. The long list of references by no means covers all of the available material on plate girders.

5.2 Web Buckling

The theoretical buckling loads of plates under a wide variety of idealized conditions of loading and edge support have been determined. In the small-deflection buckling theory, the following assumptions are made:

1. The thickness of the plate is small compared with its surface dimensions,
2. The plate is perfectly plane prior to loading, and
3. The plate deflections are small in comparison with the plate thickness.

The following notations will be used throughout this chapter:

d = over-all depth of the girder,

d_f = girder depth between flange centroids, and

h = clear depth of the girder web between supporting flange components.

In a simply supported plate girder, the web can be considered as subjected to pure shearing stresses close to the supports, to pure bending at the center, and to combined shear and bending at intermediate locations. Equations for the stress in a rectangular plate at the buckling load in pure shear, in pure bending, and in combined shear and bending can be written in the same form as Eq. 3.1 for the buckling strength of a plate under uniform compression:

$$\begin{Bmatrix} \tau_c \\ \sigma_c \\ \tau_c, \sigma_c \end{Bmatrix} = \begin{Bmatrix} k_s \\ k_b \\ k_c \end{Bmatrix} \frac{\pi^2 E \sqrt{\eta}}{12(1-\nu^2)(h/t)^2} \begin{cases} \text{pure shear} & (5.1a) \\ \text{pure bending} & (5.1b) \\ \text{combined shear and bending} & (5.1c) \end{cases}$$

In these equations k_s, k_b, and k_c are buckling coefficients that depend on the plate boundary support conditions and on the aspect ratio $\alpha = a/h$, where a is the plate length and h its width; and $\eta = E_t/E$, where E_t is the tangent modulus of the material.

Figure 5.2 gives values of the buckling coefficient k_s for plates subjected to pure shear for three conditions of edge support. Source data for the curves are as follows:

1. *Plate simply supported on four edges.* Solutions developed by Timoshenko (5.1), Bergmann and Reissner (5.2), and Seydel (5.3) are approximated by Eqs. 5.2a and 5.2b:

$$k_s = 4.00 + \frac{5.34}{\alpha^2} \quad \text{for } \alpha < 1 \tag{5.2a}$$

$$k_s = 5.34 + \frac{4.00}{\alpha^2} \quad \text{for } \alpha > 1 \tag{5.2b}$$

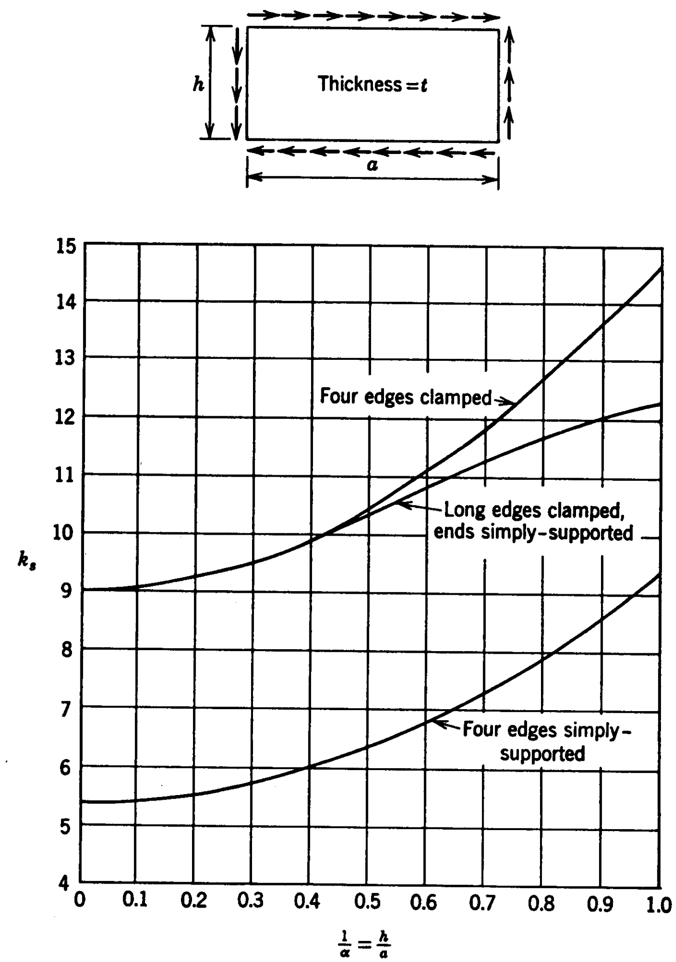

Fig. 5.2. Buckling coefficients for plates in pure shear.

2. *Plate clamped on four edges.* In 1924, Southwell and Skan (5.4) obtained $k_s = 8.98$ for the case of the infinitely long rectangular plate with clamped edges. For the finite-length rectangular plate with clamped edges, Moheit (5.5) obtained

$$k_s = 8.98 + \frac{5.6}{\alpha^2} \tag{5.3}$$

3. *Plate clamped on two opposite edges and simply supported on the other two edges.* A solution for this problem has been given by Iguchi (5.6) for the general case, and by Legget (5.7) for the case of the square plate.

Figure 5.3 gives values of the buckling coefficient k_b for plates subjected to pure bending. The sources of the data are as follows:

1. *Plate simply supported on four edges.* Timoshenko (5.8) computed the values of k_b as plotted in the lower curve of Fig. 5.3. This curve

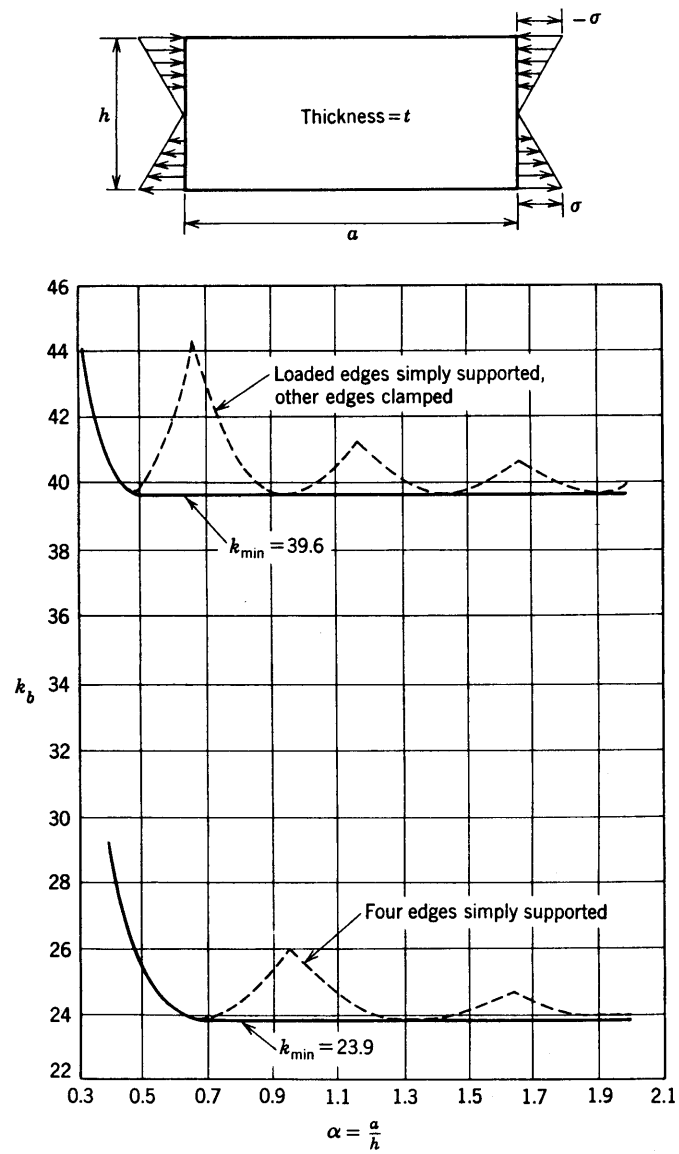

Fig. 5.3. Buckling coefficients for plates in pure bending.

consists of several branches, all of which are tangent to the minimum value $k_b = 23.9$ at integer multiples of $\alpha = \frac{2}{3}$. It is a common practice to use this minimum value of k_b for plates with $\alpha \geqslant 0.67$.

2. *Plate simply supported on loaded edges and clamped on the other edges.* Nolke (5.9) obtained the results plotted in the upper curve of Fig. 5.3, where the branches are tangent to the minimum value $k_b = 39.6$ at integer multiples of $\alpha = 0.47$. For plates having $\alpha \geqslant 0.47$, $k_b = 39.6$ is a satisfactory approximation.

The portion of the web plate close to the intermediate support of a continuous plate girder or at the support of a cantilever section is subjected to combined shearing and bending stresses. For a plate simply supported on four sides, Timoshenko (5.10) obtained a reduced k_c value as a function of τ/τ_c for values of $\alpha = 0.5$, 0.8, and 1.0, where τ is the actual shearing stress and τ_c is given by Eq. 5.1a. This problem was also solved by Stein (5.11) and by Way (5.12), whose results for four values of α are plotted in Fig. 5.4. Chwalla (5.13, 5.14) suggested the following approximate interaction formula, which agrees well with the graphs of Fig. 5.4:

$$\left(\frac{\sigma}{\sigma_c}\right)^2 + \left(\frac{\tau}{\tau_c}\right)^2 = 1 \qquad (5.4)$$

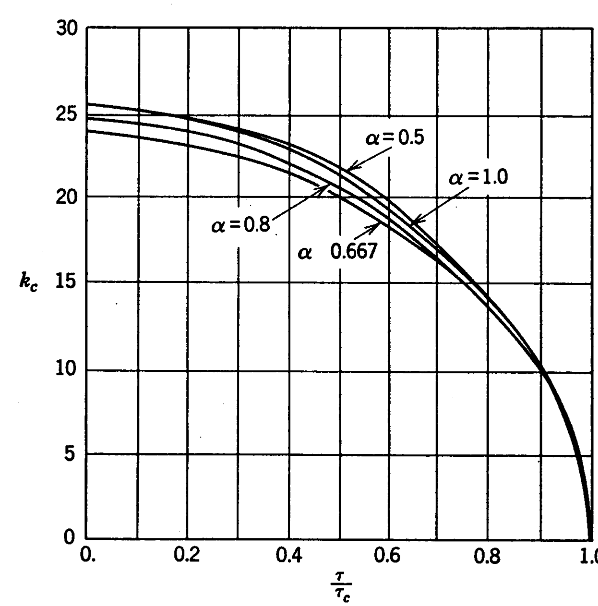

Fig. 5.4. Buckling coefficients for plates in combined bending and shear.

where σ and τ are the actual stresses, and σ_c and τ_c are the buckling stresses, for pure bending and pure shear respectively.

5.3 Plate-Girder Tests Compared with Buckling Theory

Many investigators have tried to check the small-deflection buckling theory by means of tests. In 1907 Lilly (5.18), in the conclusion to his experimental investigations, commented that it is very difficult to determine the buckling load experimentally. In reporting experiments with aluminum alloy girders, Moore (5.19) stated that well-defined buckling loads generally cannot be obtained experimentally and that the theoretical load required to cause the web plate to buckle bore little relation to the ultimate load-carrying capacity of the girder. Sparkes (5.20) showed that plate girders can function as normal load-carrying structures at stresses well beyond the lower critical shear stress obtained theoretically. Wästlund and Bergmann (5.21, 5.22) came to similar conclusions and stated that the ratio of the ultimate load to the theoretical buckling load increases as web slenderness increases. Rockey (5.23) showed by tests that the buckling of the web is not a sudden phenomenon which results in immediate failure of the girder. In addition, Rockey stated that since buckling does not result in any sudden marked change in the load-carrying characteristics of the girder, there is no justification in employing a high factor-of-safety with respect to the theoretical buckling loads when designing plate-girder webs.

Massonnet (5.24) also emphasized that there is no significant change in behavior at critical load, and that it is difficult to determine the critical load experimentally. Moreover, Massonnet stated that the failure by buckling of a web panel surrounded by rigid framing is usually associated with the plastic stretching of the web or with the failure of the framing, and further that these phenomena occur only at a load considerably in excess of the theoretical critical load.

From tests on riveted steel girders, Vasahelyi and others (5.25) concluded that because of unavoidable initial curvature in the web, it starts to deflect laterally as soon as it is loaded; and that the deflection does not become disproportionately large even when the girder is loaded beyond its theoretical elastic buckling load.

This limited discussion covers only part of the experimental research in this area, but it is nonetheless extensive enough to indicate the following important conclusions:

1. It is not possible to obtain precise experimental checks of small-deflection buckling theory in plate-girder tests.
2. Small-deflection buckling theory furnishes a safe but often over-conservative design. If the web is adequately supported, it will carry stresses substantially exceeding theoretical buckling stresses.

Allowable stresses based on girder reserve strength in the post-buckling range may be high enough to require evaluation of the effects of repeated load. Research is underway on this subject, with recent progress reported by Corrado, Mueller, and Yen (5.41). More research is also needed regarding the behavior of thin webs in unsymmetrical girders, such as might be used when a floor system participates as part of the compression flange.

5.4 Ultimate Strength in Bending

It has been shown (5.32, 5.37, 5.38) that when a plate girder is subjected to bending and the web plate has buckled, the web burdens the compression flange with that portion of the bending moment that it cannot resist. Thus, the plate girder does not fail in bending until its compression flange fails. The buckling of the compression flange can be classified into three modes: (1) vertical buckling, (2) lateral buckling, and (3) torsional buckling (see Fig. 5.5).

(1) Vertical Buckling. When a plate girder develops curvature of the compression flange, transverse flange forces develop, which cause a uniform compressive stress σ_n on the upper and the lower edges of the web (see Fig. 5.6). A conservative estimate of the vertical buckling stress can be obtained by Eq. 3.1, taking k conservatively as 1:

$$\sigma_n = \sigma_c = \frac{\pi^2 E}{12(1 - \nu^2)(h/t_w)^2} \tag{5.5}$$

Thus, compression-flange failure in the vertical direction will occur if the web-slenderness ratio h/t_w is high, and it is evident that this kind of failure can be avoided by setting an appropriate upper limit for h/t_w.

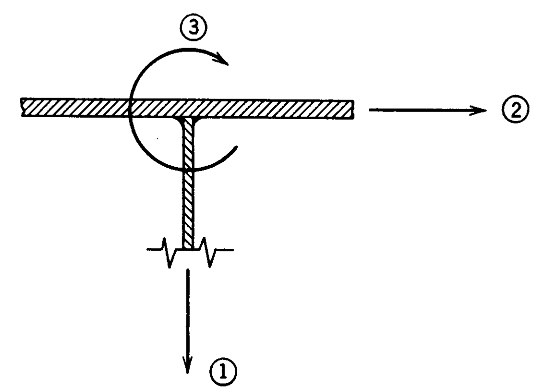

Fig. 5.5. Three modes of compression-flange buckling.

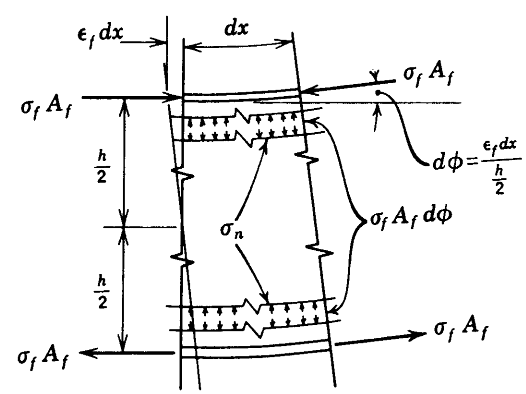

Fig. 5.6. Web stresses induced by girder bending.

To prevent vertical web buckling when the flange has no appreciable bending stiffness in the vertical direction, as is usually the case for welded girders, it is necessary that the applied force be smaller than the resisting force; i.e., from Fig. 5.6, assuming the girder depth as h,

$$\sigma_f A_f \frac{2\epsilon_f}{h} dx < \left(\frac{\pi^2 E}{12(1-\nu^2)(h/t_w)^2}\right) t_w dx$$

from which, approximately, vertical web buckling will not occur if

$$\frac{h}{t_w} < \sqrt{\left(\frac{\pi^2 E}{24(1-\nu^2)}\right)\left(\frac{A_w}{A_f}\right)\left(\frac{1}{\sigma_f \epsilon_f}\right)} \tag{5.6}$$

Since the ratio of web area to flange area is usually more than 0.5 for plate girders, a conservative value of 0.5 for A_w/A_f can be assumed for establishing an upper limit for h/t_w. If every flange fiber reaches yield stress before web failure, then $\sigma_f = \sigma_y$, and $\epsilon_f \approx (\sigma_y + \sigma_{RC})/E$. Making these substitutions in Eq. 5.6, and taking $\nu = 0.3$, the following web-slenderness requirement for prevention of vertical web buckling is obtained:

$$\frac{h}{t_w} < \frac{0.48E}{\sqrt{\sigma_y(\sigma_y + \sigma_{RC})}} \tag{5.7}$$

where σ_y = yield stress

σ_{RC} = maximum residual compressive stress

This critical web-slenderness ratio agrees fairly well with the test observations of structural carbon steel girders by Basler and Thürlimann (5.37, 5.38). However, later studies indicate that the web ratio given by Eq. 5.7 is probably ultraconservative for high-yield-strength, quenched-and-tempered alloy-steel girders. In fact, research still in progress (1966) at Lehigh University suggests that higher ratios may also be safe for structural carbon steel. When the flange itself possesses vertical rigidity, such as in the case of riveted girders with flange angles, Eq. 5.7 is conservative since it does not take account of such rigidity. If camber is introduced, the vertical stress may be neutralized, or even reversed, depending on the amount of deflection, which introduces further conservatism into Eq. 5.7. If external load is applied directly at a point where there are no transverse stiffeners, the web must be further checked for local crippling and overall failure, as will be discussed later.

(2) Lateral Buckling. This subject was discussed in detail in Chapter 4. For deep beams, such as plate girders, the buckling resistance is furnished mostly by the compression flange, and the lateral-buckling stress has been shown by Basler (5.32) to be very near that of a column whose effective cross section is composed of the compression flange and one-sixth of the web:

$$\sigma_c = \frac{\pi^2 E_t}{(L/r)^2} \tag{5.8}$$

where $$r = \sqrt{\frac{I_f}{A_f + \frac{1}{6}A_w}}$$

and the flange moment-of-inertia I_f is taken about the web axis. Eq. 5.8 provides a conservative estimate of the bending stress at which a deep beam will buckle laterally. The column strength curves proposed in Chapter 2 can be used to determine the girder buckling stress in either the elastic or the inelastic range. For deep beams of steel,

$$\frac{\sigma_c}{\sigma_y} = 1 - \frac{\lambda^2}{4} \quad \text{for } 0 < \lambda < \sqrt{2} \qquad \text{(inelastic)} \tag{5.9a}$$

$$\frac{\sigma_c}{\sigma_y} = \frac{1}{\lambda^2} \quad \text{for } \lambda > \sqrt{2} \qquad \text{(elastic)} \tag{5.9b}$$

where

$$\lambda = \frac{1}{\pi}\frac{L}{r}\sqrt{\epsilon_y} = \frac{L}{\pi}\sqrt{\frac{\epsilon_y(A_f + A_w/6)}{I_f}}$$

It should be noted that L is the unbraced length of the compression flange and not necessarily the full length of the girder.

(3) Torsional Buckling. If all rotational flange restraint contributed by the web is neglected, the girder torsional-buckling problem reduces to the problem of buckling of a long flange plate hinged along its centerline and subjected to edge compression at its ends. This problem has been specifically considered in Chapter 3, wherein reference should be made to Eq. 3.1 and

Case 4 of Fig. 3.2. In the inelastic range, if the transition curve for steel proposed by Haaijer and Thürlimann (5.17) is used, the torsional-buckling strength curve for steel beams will be

$$\frac{\sigma_c}{\sigma_y} = 1 - 0.53(\lambda - 0.45)^{1.36} \quad \text{for } 0.45 < \lambda < \sqrt{2} \text{ (inelastic)}$$
$$\frac{\sigma_c}{\sigma_y} = \frac{1}{\lambda^2} \quad \text{for } \lambda > \sqrt{2} \text{ (elastic)} \tag{5.10}$$

where

$$\lambda = \sqrt{\frac{\sigma_y}{\sigma_c}} = \frac{b_f}{2t_f}\sqrt{\frac{12(1-\nu^2)\epsilon_y}{0.425\pi^2}}$$

The girder compression flange should be made as wide as possible to increase its lateral rigidity and consequently its lateral-buckling strength; but if this is done to excess, the flange plate will fail in torsional buckling prior to the onset of lateral buckling. In order to eliminate torsional buckling as a primary cause of failure, the critical stress of the flange plate given by Eq. 5.10 should exceed the critical lateral-buckling stress given by Eq. 5.9. Thus, for steel,

$$\frac{L}{b_f} > \frac{0.73}{\sqrt{1 + \frac{1}{6}(A_w/A_f)}}\frac{b_f}{t_f} \quad \text{for } \frac{b_f}{t_f} > 52$$
$$\frac{L}{b_f} > \frac{40}{\sqrt{1 + \frac{1}{6}(A_w/A_f)}}\left(0.026\frac{b_f}{t_f} - 0.45\right)^{0.68} \quad \text{for } \frac{b_f}{t_f} < 52 \tag{5.11}$$

These correlations between b_f/t_f and L/b_f are plotted in Fig. 5.7. As indicated in the figure, the condition

$$\frac{b_f}{t_f} \leqslant 12 + \frac{L}{b_f} \tag{5.12}$$

would exclude the possibility of primary failure in torsional buckling for girder sections under uniform bending. Alternatively, a tubular or boxed-flange section could be used to eliminate the possibility of torsional buckling.

5.5 Ultimate Strength in Shear

The practical importance in design of the post buckling strength of the beam web plate has been recognized for many years and was utilized in aircraft wing design in the 1930s. Before web buckling occurs, the shear stress is equivalent to equal tensile and compressive stresses which act at 90° to each other and at 45° to the girder axis. After web buckling, the compressive stress remains approximately constant because of the inability of the web to carry additional compressive stress, while

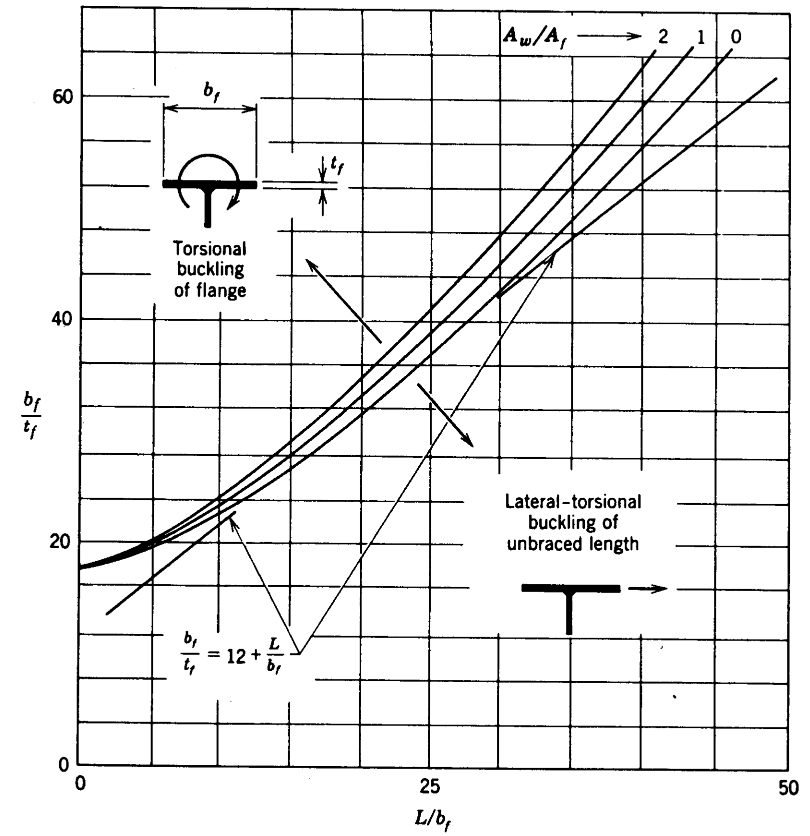

Fig. 5.7. Condition for torsional flange buckling.

the tensile stress increases as additional loads are applied. This results in part of the shear load being carried by the normal shear action, and the remainder by a truss-type action in which the web acts as the diagonal tension members, the flanges act as chords, and the vertical stiffeners act as compression posts.

Van der Neut (5.42) has reviewed the applications of post-buckling theory in aircraft design. Wagner (5.43), in 1931, was the first to develop the diagonal-tension theory. Wagner's work was applicable to some early aircraft aluminum alloy girders with extremely thin webs, where the web buckled at a fraction of its ultimate load. However, for most practical girder designs the web buckling load is far from being negligible, and Wagner's theory is too conservative. Kuhn and others (5.44, 5.45, 5.46), working in aircraft design, developed the so-called "incomplete diagonal-tension theory." In recent work at Lehigh University, Basler (5.33, 5.38) derived the ultimate shear strength of plate girders utilizing the combined beam-action and diagonal-tension field theories. The ultimate web shear strength V_u is the sum of two parts designated as beam action V_τ and tension-field action V_σ:

$$V_u = V_\tau + V_\sigma \tag{5.13}$$

The beam action can be increased up to a limiting value and remains constant thereafter. This limiting value is determined by the critical shear stress according to linear buckling theory:

$$V_\tau = ht\tau_c \tag{5.14}$$

If the flanges are simple rectangles (as in the case of welded girders) they offer negligible resistance against vertical deflection and they will not provide anchorage for the tension field. In such a case, the anchorage must be provided by the vertical stiffeners and effective portions of the web. Thus, from Fig. 5.8,

$$V_\sigma = \sigma_t t_w s \sin\phi \tag{5.15}$$

If we express s as a function of a, h, and ϕ, then for maximum V_σ (i.e., $\partial V_\sigma/\partial\phi = 0$), the tension-field angle ϕ_0 is equal to one half the angle of the panel diagonal with the horizontal. The tension-field angle ϕ_0 will, therefore, vary from a maximum approaching 45° for extremely close stiffeners to a minimum of zero for unstiffened webs (no tension field). By taking a free-body diagram of a portion of the web as indicated in Fig. 5.9, the increase ΔF_f in the flange force can be found from the requirement of equilibrium in the horizontal direction:

$$\Delta F_f = \tfrac{1}{2}\sigma_t a t_w \sin\theta \tag{5.16}$$

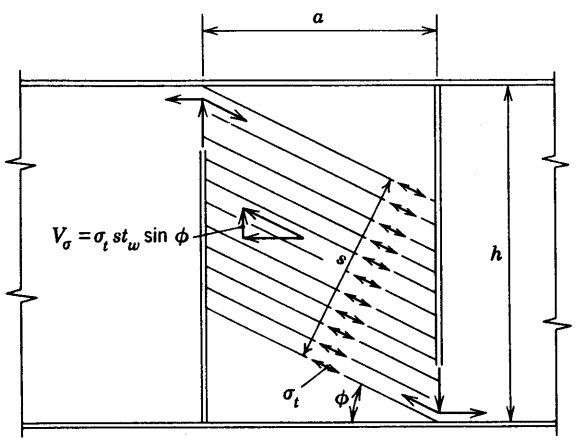

Fig. 5.8. Plate-girder panel. (Aspect Ratio $\alpha = a/h$)

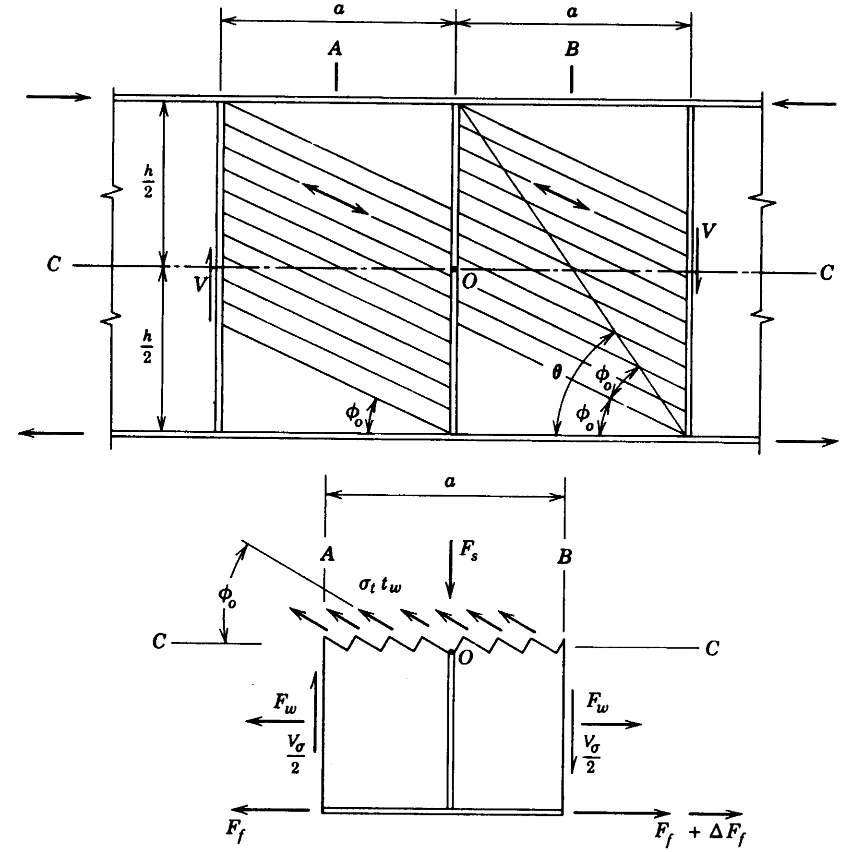

Fig. 5.9. Free-body of portion of plate-girder web.

Taking moments around point O,

$$V_\sigma = \tfrac{1}{2}\sigma_t a t_w \sin\theta \tan\theta \tag{5.17}$$

Finally, the requirement of equilibrium in the vertical direction yields the stiffener force as

$$F_s = \tfrac{1}{2}\sigma_t h^2 \frac{\alpha}{\beta}(1 - \cos\theta) \tag{5.18}$$

where

$$\alpha = a/h$$
$$\beta = h/t_w$$
$$\theta = \tan^{-1}(h/a)$$

The critical shear stress τ_c and the tensile stress σ_t exist simultaneously in the tension band of the web; hence, the magnitude of σ_t is limited by the fact that unrestricted yielding of the tension band will occur when the combination of σ_t and τ_c becomes equivalent to the yield stress in simple tension. Since yielding in this case will occur only for a state of stress between the two extremes of simple tension and pure shear, the yield criterion may be taken approximately as

$$\frac{\sigma_t}{\sigma_y} + \frac{\tau_c}{\tau_y} = 1 \tag{5.19}$$

Substituting Eqs. 5.19 and 5.17 in Eq. 5.13, the ultimate shear strength is given by

$$V_u = ht_w\tau_c + \tfrac{1}{2}\sigma_y\left(1 - \frac{\tau_c}{\tau_y}\right)at_w \sin\theta \tan\theta \tag{5.20}$$

where α and β are as previously defined, and

$$\sin\theta = 1/\sqrt{1+\alpha^2}$$
$$\tan\theta = h/a$$
$$V_p = ht_w\tau_y = \text{full plastic shear strength of the web}$$
$$\tau_y = \sigma_y/\sqrt{3}$$

Dividing Eq. 5.20 by the expression for V_p gives

$$\frac{V_u}{V_p} = \frac{\tau_c}{\tau_y} + \frac{\sqrt{3}}{2}\left(1 - \frac{\tau_c}{\tau_y}\right)\frac{1}{\sqrt{1+\alpha^2}} \tag{5.21}$$

Note that the critical buckling stress τ_c according to the small-deflection theory is given by Eq. 5.1. Eq. 5.21 can be written in the form

$$\tau_u = \frac{\sigma_y}{\sqrt{3}}\left[\frac{\tau_c}{\tau_y} + \frac{1 - \tau_c/\tau_y}{1.15\sqrt{1+\alpha^2}}\right] \tag{5.22}$$

The first term within the brackets represents beam action and the second term is the contribution of tension-field action. As the numerator $(1 - \tau_c/\tau_y)$ of the second term shows, tension-field action starts only after beam action is fully developed. For stocky webs with $\tau_c/\tau_y \approx 1$, beam-action shear can be carried to yielding, and therefore no tension-field action will take place. In such cases, the second term is negligible.

5.6 Ultimate Strength in Combined Bending and Shear

The shear strength as determined in Sec. 5.5 makes no use of the flange as an anchor for the tension field. Therefore, the ultimate shear strength should not be reduced by the presence of a bending moment as long as the flanges alone are able to carry this moment. Hence, it is only that small

part of the bending moment which is assigned to the web that leads to an interaction. The pertinent moments are as follows:

Flange moment: $$M_f = \sigma_y d_f A_f \tag{5.23a}$$

Yield moment: $$M_y = \sigma_y d_f(A_f + \tfrac{1}{6}A_w) \tag{5.23b}$$

Plastic moment: $$M_p = \sigma_y d_f(A_f + \tfrac{1}{4}A_w) \tag{5.23c}$$

V and M can be nondimensionalized by expressing the shear force in terms of the ultimate shear force V_u, and the moment in terms of the yield moment M_y.

Since there will be no reduction in the ultimate shear strength as long as the flanges alone are able to carry the bending moment, then

$$\frac{V}{V_u} = 1, \quad \text{for } 0 < \frac{M}{M_y} \leqslant \frac{M_f}{M_y}$$

If there were no shear present, the maximum moment that could be carried under the most favorable circumstances (disregarding strain hardening) is the plastic moment M_p, i.e., for

$$\frac{V}{V_u} = 0, \quad \frac{M}{M_y} = \frac{M_p}{M_y}$$

Therefore, only those bending moments in the range between M_f and M_p will affect shear-carrying capacity. Basler (5.34) suggested the use of the following interaction curve for this range:

$$\left(\frac{V}{V_u}\right)^2 + \frac{M - M_f}{M_p - M_f} = 1 \tag{5.24}$$

Substituting values of M_f and M_p from Eqs. 5.23a and 5.23c,

$$\sigma = \sigma_y \frac{1 + \frac{1}{4}(A_w/A_f)[1 - (\tau/\tau_u)^2]}{1 + \frac{1}{6}(A_w/A_f)} \tag{5.25}$$

The presence of shear has a beneficial aspect on girder strength because the shear forces always imply a moment gradient and, therefore, only a short girder portion is subjected to the maximum moment. To check the lateral stability of the compression flange, the effect of moment gradient can be evaluated by means of Eq. 4.13. Thus, Eq. 5.9 can be modified to give

$$\frac{\sigma_c}{\sigma_y} = 1 - \frac{\lambda^2}{4C_1} \qquad 0 < \lambda < \sqrt{2C_1}$$
$$\frac{\sigma_c}{\sigma_y} = \frac{C_1}{\lambda^2} \qquad \lambda > \sqrt{2C_1} \tag{5.26}$$

where

$$\lambda = \frac{1}{\pi}\frac{L}{r}\sqrt{\epsilon_y} = \frac{L}{\pi}\sqrt{\frac{\epsilon_y(A_f + A_w/6)}{I_f}}$$

and

$$C_1 = 1.75 - 1.05\kappa + 0.3\kappa^2$$
$$-0.46 < \kappa < +1$$

where κ is the ratio of the smaller to the larger end moment of a longitudinal girder segment laterally braced at each end and free from interspan loads.

5.7 Use of Transverse Stiffeners to Increase Buckling Load

For web plates subjected to pure shear, Moore (5.48), in 1942, attempted to find the minimum stiffener flexural rigidity needed to produce a nodal line in a slightly buckled web. He proposed the following formula for optimum relative stiffness of stiffener and web:

$$\gamma_o = \frac{14}{(a/h)^3} \tag{5.27}$$

where $\gamma_o = \dfrac{EI_o}{Da} = \dfrac{\text{flexural rigidity of stiffener}}{\text{flexural rigidity of corresponding web portion}}$

I_o = optimum moment-of-inertia of stiffener, and

$D = \dfrac{Et_w^3}{12(1-\nu^2)}$ = web-plate flexural rigidity.

To derive this empirical formula, Moore arbitrarily introduced small stiffener deflections. He also showed that when the panel is operating below the buckling stress, there is no evidence to support the practice of designing the vertical stiffeners as columns. For the same problem, Stein and Fralich (5.49) derived theoretical values of the theoretical plate-buckling coefficient k for different values of γ for three stiffener spacings, namely h, $0.5h$, and $0.2h$, assuming the web plate to be simply supported at all four edges.

Theoretically, k continues to increase as γ increases. However, for practical purposes there is an optimum value γ_o beyond which the increase in k is small. Bleich (5.50) derived from three curves presented by Stein and Fralich an approximate equation for the buckling coefficient k as a function of γ and h/a, from which the optimum value γ_o was found as

$$\gamma_o = 4\left\{7\left(\frac{h}{a}\right)^2 - 5\right\} \tag{5.28}$$

Kleemen (5.51) extended Stein and Fralich's work to allow for the effect of torsional rigidity of the stiffeners. However, the torsional rigidity of the

usual open-section stiffener is so small that its effect on web stiffness can be neglected.

Rockey (5.52) obtained from an experimental investigation the following empirical formulas for the optimum value γ_o:

$$\gamma_o = 27.75\left(\frac{h}{a}\right)^2 - 7.5 \quad \text{for double stiffeners} \tag{5.29}$$

$$\gamma_o = 21.5\left(\frac{h}{a}\right)^2 - 7.5 \quad \text{for single stiffeners} \tag{5.30}$$

Rockey recommends the use of these empirical formulas only when the thickness of the attached stiffener leg is equal to or greater than the thickness of the web plate.

In the practical range of stiffener spacing there is considerable divergence among the various recommendations. Hartmann and Clark (5.8) have discussed these differences in relation to design with aluminum alloys (A20, A21, A22).

5.8 Transverse Stiffeners in Tension-Field Design

In a tension-field web each stiffener must act as a compression strut. From Eqs. 5.18 and 5.19, the stiffener force at ultimate load is found as

$$F_s = ht_w\sigma_y\left(1 - \frac{\tau_c}{\tau_y}\right)\frac{\alpha}{2}\left[1 - \frac{\alpha}{\sqrt{1+\alpha^2}}\right] \tag{5.31}$$

Dividing by σ_y, the required stiffener area at ultimate load is obtained as

$$A_s = \frac{1}{2}\left(1 - \frac{\tau_c}{\tau_y}\right)\left[\alpha - \frac{\alpha^2}{\sqrt{1+\alpha^2}}\right]ht_w \tag{5.32}$$

This stiffener-area requirement is needed to supplement the requirement which calls for the necessary rigidity. Massonnet (5.53) found, on the basis of his early experiments in which the load was increased to the point of rupture, that for the stiffeners to remain practically straight almost up to the rupture load of the girder, the optimum relative-stiffness factor γ_o should be multiplied by a factor of 20. Further studies by Massonnet and Skaloud (5.54, 5.55) showed that a multiplying factor of only 3 was adequate.

5.9 Use of Longitudinal Web Stiffeners to Increase Web Buckling Strength

For web plates in girders subjected to pure bending, longitudinal stiffeners increase the web buckling strength. Madsen (5.56) verified this experimentally in his investigation of the buckling characteristics of box girders. Chwalla (5.13, 5.14), in 1963, was the first to investigate the influence of a

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longitudinal stiffener on the buckling strength of a rectangular simply supported plate. He examined the case where the stiffener was placed at a distance of one-quarter the web depth from the compression flange (aspect ratio a/h equal to 0.8). Chwalla did not obtain a general solution in terms of the aspect ratio.

Massonnet (5.57) presented, in 1941, the first general solution for the buckling strength of a rectangular plate having a longitudinal stiffener at one-quarter depth from the compression flange. He derived relationships between k, δ, γ, and α for a longitudinally stiffened plate simply supported at all four edges, where

$$\delta = \frac{\text{area of stiffener}}{\text{area of web}} = \frac{A_s}{ht_w},$$

$$\gamma = \frac{\text{flexural rigidity of stiffener}}{\text{flexural rigidity of web}} = \frac{EI}{Dh}, \text{ and}$$

$$\alpha = \text{aspect ratio of panel} = \frac{a}{h}.$$

The maximum value of k was found to be 101. This value is obtained when the horizontal stiffener has sufficient rigidity to cause a nodal line to form. Bleich (A1) derived from Massonnet's curves the following approximate expression, for $\alpha \leqslant 1.6$:

$$\gamma_o = (12.6 + 50\delta)\alpha^2 - 3.4\alpha^3 \qquad (5.33)$$

Dubas (5.58, 5.59) presented a general solution for plate buckling for the case where all web edges are simply supported and the stiffener is at one-fifth the web depth from the compression flange. The maximum value of k was found to be 129. Rockey and Legget (5.56) studied the case of a longitudinal stiffener at the one-fifth-depth position, at the one-quarter-depth position, and at mid-depth, assuming the unloaded web edges (corresponding to those supported by the girder flange members) to be rigidly clamped against rotation. Figure 5.10 shows the value of γ_o required to develop the maximum k values of 129.4 (Dubas) and 142.6 (Rockey and Legget) when the unloaded web edges are simply supported and clamped, respectively, and the stiffener is at the one-fifth-depth position. The comparison shows the effect of clamped longitudinal edges on the required stiffener flexural rigidity, for web panels with the maximum buckling coefficient k.

Mitchell (5.60), Stüssi, and Charles and Pierre Dubas (5.61, 5.62, 5.63) have shown that the longitudinal stiffener at one-fifth the depth from the compression flange is the most beneficial when all edges are simply supported.

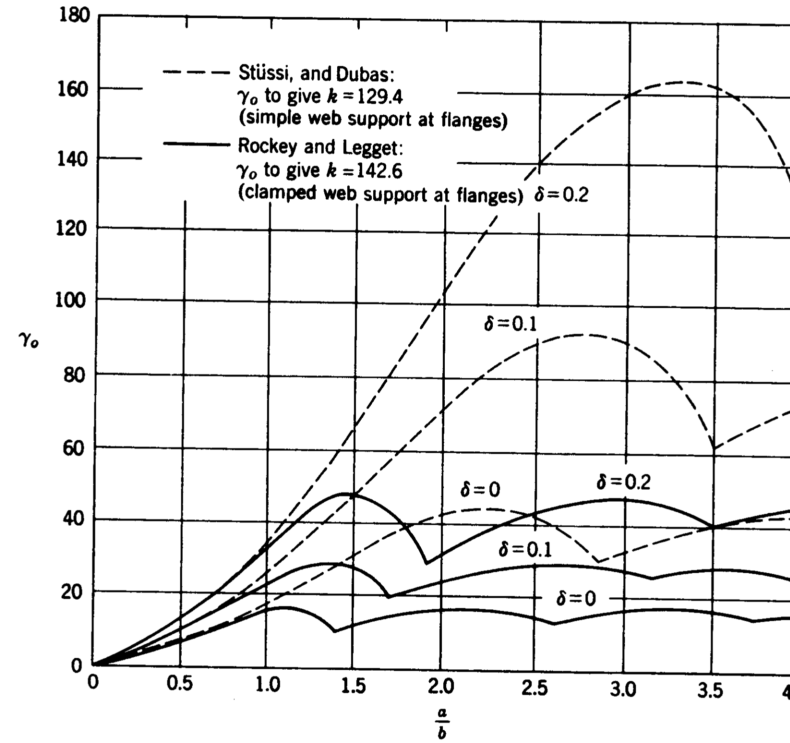


Fig. 5.10. Curves for longitudinally stiffened plates.

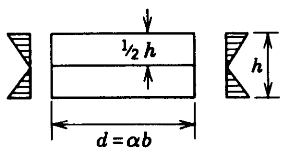
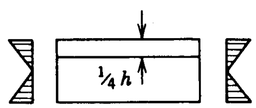
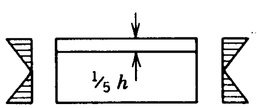
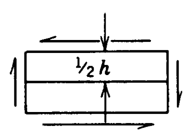
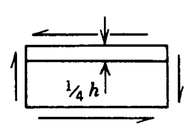
Hampl (5.64) examined the case of a plate having a longitudinal stiffener at mid-depth, and found that this position is not effective in increasing the buckling resistance of the plate under pure bending. Klöppel and Scheer (5.65) provided a solution for the case when there are two longitudinal stiffeners, placed at either one-third or one-quarter of the web depth from both flanges. Only a small increase in the buckling coefficient k is obtained by using such a symmetrical placement of stiffeners. For such cases each stiffener will have an influence on the other, and no accurate rules exist for determining the dimensions of the stiffeners. However, according to Massonnet (5.24), when two or more horizontal stiffeners are used in the same panel, each stiffener can be designed as if it were alone.

If a stiffener with a rigidity less than that corresponding to γ_o is used, the web buckling coefficient k will be less than the maximum. In all of the references given previously, the authors provide curves showing the relation between k and γ for various values of a/h and δ. However, a detailed study carried out by Massonnet (5.24), in which he considered both material and fabrication costs, showed that rigid stiffeners are generally the most economical.

While the optimum location of longitudinal stiffeners for pure bending in a simply supported panel is at the one-fifth-depth position from the

Table 5.1

Optimum Stiffener Relative Rigidity for Various Web Conditions

Web condition	Web aspect ratio	Optimum stiffener relative rigidity
		$\gamma_o = 1.3$
	$\alpha \leq 0.5$ $\alpha > 0.5$	$\gamma_o = 2.4 + 18.4\delta$ $\gamma_o = (12 + 92\delta)(\alpha - 0.3)$ with maximum value $\gamma_o = 16 + 200\delta$
	$0.5 \leq \alpha \leq 1.5$	$\gamma_o = 3.87 + 5.1\alpha + (8.82 + 77.6\delta)\alpha^2$ $\gamma_o = 12.5\,[0.4 + (1 + 6\delta)\alpha^2]$*
	$0.5 \leq \alpha \leq 2$	$\gamma_o = 5.4\alpha^2(2\alpha + 2.5\alpha^2 - \alpha^3 - 1)$
	$0.5 \leq \alpha \leq 2$	$\gamma_o = 7.2\alpha^2(1 - 3.3\alpha + 3.9\alpha^2 - 1.1\alpha^3)$

* A simpler formula used in aluminum alloy specifications (A.21) gives nearly the same result.

compression flange, this is not the case for web panels subjected to combined bending and shear. For pure shear the optimum location of a longitudinal stiffener is mid-height of the web. Massonnet (5.24) gives the optimum location of a rigid stiffener for various ratios of web shear to bending stress (τ/σ). However, it should be kept in mind that the primary function of longitudinal stiffeners is to make the web fully effective in resisting bending stress.

The transverse deflection of a stiffener is proportional to the transverse force acting on it, but the transverse deflections of the web plate are restrained by membrane stresses which act in the middle plane of the plate and which increase progressively with load. Thus, the stiffener is at a disadvantage in comparison with the plate. This phenomenon was observed by Massonnet (5.22), who suggested that for a stiffener to remain practically straight up to the web collapse load, it must have a relative rigidity γ equal to $n\gamma_o$, where values of the optimum relative rigidity γ_o are as given in Table 5.1. Massonnet's tests showed that the value of n depends mainly on the location of the stiffener, and he suggested the following values for design purposes:

Distance between Longitudinal Stiffener and Compression Flange	*Value of* n
$\frac{b}{2}$	3
$\frac{b}{3}$	4
$\frac{b}{4}$	6
$\frac{b}{5}$	7

5.10 Combination of Transverse and Longitudinal Stiffeners

Rockey (5.67) investigated the case of a web subjected to pure shear and reinforced by both transverse stiffeners and a central longitudinal stiffener. He found that if the transverse stiffeners have a relative rigidity equal to the optimum value of γ_{ot}, then the optimum relative rigidity for the longitudinal stiffener is

$$\gamma_{ol} = 11.25\left(\frac{h}{a}\right)^2$$

Rockey found also that the weight of longitudinal and transverse stiffeners required to achieve a given web buckling stress can be as little as 50% of the stiffener weight required when only transverse stiffeners are used.

Combinations of transverse and longitudinal stiffeners other than those investigated by Rockey have not been studied. If the web is reinforced by one or more longitudinal stiffeners, its ultimate strength is considerably improved. In such cases the transverse stiffener rigidity must be greater than that specified heretofore, so that it will remain straight until this increased ultimate strength is obtained. Massonnet and Skaloud (5.54, 5.55) proposed the concept of an "equivalent web," having a thickness t_e which is determined from the condition that the buckling stress of the equivalent web *without* longitudinal stiffeners must equal the buckling stress of the given web of thickness t_w *with* longitudinal stiffeners. The equivalent thickness t_e is to be used for the design of the transverse stiffeners, and this results in greater rigidity.

5.11 Stiffened Plates under Combined Bending and Shear

Little has been published on the design of stiffeners for webs subjected to combined stresses. Milosavljevitch (5.68) dealt with the case of transverse stiffeners at the one-third points of a panel and a longitudinal stiffener at one-fourth the depth from the compression flange. Massonnet (5.24) analyzed the case of a panel reinforced by a median vertical stiffener, and found that the optimum relative rigidity of a stiffener assumed to be rigid is never greater than the optimum value for either bending alone or for shear alone. Young and Landau (5.69) proposed that in a girder subjected to combined shear and bending, the necessary flexural rigidities of the horizontal and vertical stiffeners for the separate stress conditions should be calculated, and then combined to find the requisite section.

Wittrick (5.70) has provided charts to calculate the buckling stress of a plate subjected to any combination of shear and longitudinal compression. These charts can be used to find the buckling stress of a web panel bounded by both vertical and horizontal stiffeners. He found that only small errors will be involved if the nonuniform longitudinal compression loading is replaced by an equivalent uniform compression.

5.12 Stiffener Details

The moment-of-inertia of double stiffeners is taken about the centerline of the web plate, and that of single stiffeners is usually taken about the surface of the web plate in contact with the stiffener. According to Massonnet (5.24) the latter assumption favors the use of single stiffeners, but this preference should be limited to design levels below the buckling load, for which levels bending stiffness rather than column action is the criterion. Massonnet recommends taking the moment-of-inertia of the single stiffener about the neutral axis of the cross section composed of

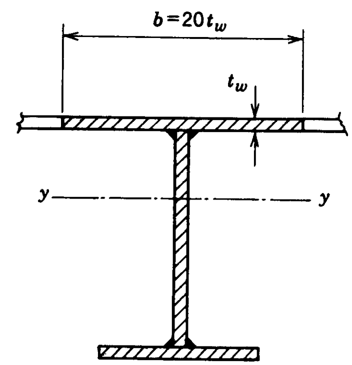


Fig. 5.11. Web stiffener.

the stiffener and an effective web width of $20t_w$, as shown in Fig. 5.11. For this purpose, Bleich (A1) recommends the use of an effective web width of $30t_w$.

In post-buckling design, the stiffener axial force resulting from the tension field is applied in the plane of the web, thus loading the single stiffener eccentrically. For this reason, a single stiffener will not be as efficient as a double stiffener in carrying the compression load, and the single stiffener would need to be larger in cross-sectional area than the double. Basler (5.31) recommends as a minimum that the area be 2.4 times larger in the case of a single plate, and 1.8 times larger in the case of a single equal-legged angle with one leg flat against the web.

Basler (5.33) also recommends that the axial force F_s in the stiffener be developed over a distance of one-third of the web depth, in order to provide for possible variations in shear flow.

Welding to the tension flange will reduce fatigue strength and may also be objectionable with regard to brittle fracture. In addition, it is expensive to fit a stiffener against both flanges. Both disadvantages can be overcome by fitting stiffeners to the compression flange only, and by leaving clearance at the tension flange. In test girders used by Basler (5.29, 5.38) and others, a clearance of 1 in. was left between the stiffeners and the tension flange. During the tests, no movement of the tension flange with respect to the stiffeners was observed until after the ultimate load was reached. It was concluded that stiffeners can be cut short of the tension flange without impairing girder strength. Basler suggested that this clearance should not be more than four times the web thickness. However, Basler (5.38) prescribed that stiffeners should always be fitted to the compression flange. In the case of single stiffeners, a substantial weld should be provided to the compression flange as a safeguard against flange torsion; and in the case of double stiffeners, a nominal weld is needed to guard against flange rotation due to accidental transverse loads.

5.13 Other Considerations

The ultimate strength of a plate girder in bending is determined primarily by the flanges rather than by the web. When shear governs, the ultimate strength depends primarily on the web, and it is obvious that a tension field will develop in the web regardless of whether or not it is initially perfectly plane. These considerations show that initial web out-of-flatness has little effect on girder strength. Basler (5.30) recommends that this imperfection be ignored. Attempts to correct out-of-flatness by means of spot heating will introduce residual stresses and embrittlement.

As mentioned by Basler (5.31), loads in direct bearing on the flanges can introduce two hazards in cases where proper web stiffeners are absent. The resulting bearing pressure on the web can cause local web yielding followed by web crippling. Alternatively, the web may collapse as a result of overall buckling.

The vertical buckling problem due to transverse loading resulting from girder curvature was discussed in connection with Eq. 5.5. The transverse (vertical) normal stress induced by direct loads on the flange are less conducive to buckling, since the stress decreases almost linearly from the top to the bottom boundary. Thus k in Eq. 5.5 can be taken as 2 if there is no flange restraint to the web plate, and as 5.5 when the flange fully restrains the web (both values are approximate).

Considering now the case where web stiffeners are used, and assuming that they do not carry any load in direct bearing, then

$$\sigma_n = k \frac{\pi^2 E}{12(1 - \nu^2)(a/t_w)^2} \tag{5.34}$$

A conservative estimate of $k = 4$ is obtained if σ_n is assumed to be constant over a depth equal to the distance between stiffeners.

For an approximately square web panel, the stresses just given can be combined to give approximate buckling stresses as follows:

(a) Loaded edge clamped, others simply supported:

$$\sigma_n = \left(5.5 + \frac{4h^2}{a^2}\right)\left[\frac{\pi^2 E}{12(1 - \nu^2)(h/t_w)^2}\right] \tag{5.35a}$$

(b) All edges simply supported:

$$\sigma_n = \left(2 + \frac{4h^2}{a^2}\right)\left[\frac{\pi^2 E}{12(1 - \nu^2)(h/t_w)^2}\right] \tag{5.35b}$$

Note that when $h \gg a$, Eq. 5.35a approaches Eq. 5.34. If the applied load is acting as a concentrated load, it may be converted into an equivalent

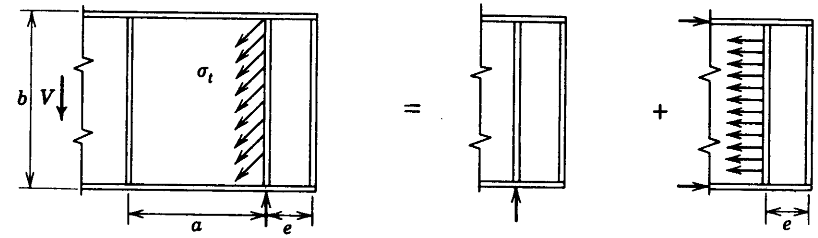


Fig. 5.12. Girder end-post analysis.

transverse uniform stress σ_n by dividing it by the area at_w if $a < h$, and ht_w if $h < a$. Some conservatism is desirable in applying the foregoing rules, since no account is taken of the added effect of the longitudinal compressive normal stress, which is also conducive to buckling.

In the end panel of the web, there is no continuing web plate to serve as an "anchor" for a tension field. One way to handle this problem is to use a shortened stiffener spacing for the end panel, such that the computed average working shear stress does not exceed τ_c/N, where N is the factor-of-safety. This will eliminate the possibility of the development of a tension field in the end panel. An alternative choice is to supply an end post strong enough to resist the tension field. This can be done by carrying the top flange around the end of the girder, or by welding a separate plate to the end (see Fig. 5.12).

If for simplification the angle of inclination ϕ of the tension field is assumed to be 45°, then the vertical force in the end post from the tension field is equal to $V - V_\tau$ (where V_τ is given by Eq. 5.14) and the horizontal force is of the same magnitude. Then the maximum bending moment to which the end post is subjected is equal to $V_\sigma h/8$. This moment will produce compressive and tensile forces equal to $V_\sigma h/8e$ in the end plate and in the bearing stiffener, respectively. The bearing stiffener will be adequate since the added force is tension, while the end plate must be made strong enough to resist the compressive force.

5.14 Plate Girders with Tubular Flanges and Stiffeners

Since the collapse of a girder is usually associated with the plastic stretching of the web or with the failure of the web frame (flanges and stiffeners), the ultimate strength of a plate girder can be increased by using tubular flanges and stiffeners. This was proposed in Germany by Dornen (5.75), who did not emphasize the increase in the stability of the flange but considered the main advantage to be reduction of the web depth. Bornscheuer (5.76) showed that tubular stiffeners are greatly superior to

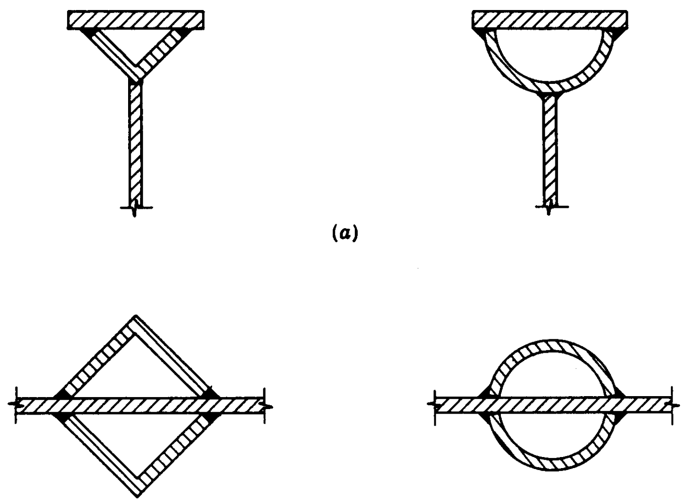


Fig. 5.13. (*a*) Tubular flanges. (*b*) Tubular stiffeners.

the usual type because of their very high torsional rigidity. They also provide the flange with greater bending stiffness vertically and thus induce a more favorable orientation of the tension field.

Tubular sections can be formed using large angles or semicircular elements (Fig. 5.13). Tests carried out by Massonnet (5.77) on the tubular type of plate girder showed their excellent behavior, even with web slenderness ratios h/t_w as high as 500. Evidently, the only deterrent to the use of such girders is the cost of fabrication.

5.15 Design Trends and Research Needs

Shanley (5.87) has traced the evolution of plate girders in the aircraft industry, starting with buckling-resistant design and going full circle through utilization of both post-buckling strength in direct compression and tension-field web strength in the design of wing girders; and returning finally, in the case of high-speed aircraft, to buckling-resistant design for functional reasons and, in certain cases, in order to obtain the minimum weight-strength ratio.

There seems to be a place for the application of both buckling-resistant and post-buckling strength analysis and design. In both railway (A7) and highway (A6) bridge design practice, buckling-resistant girder design is the rule. Localized stress concentrations and web wrinkling may occur as a result of tension-field design, and when these are superimposed on the more readily calculable stresses due to bending and shear, acceleration of fatigue failure is a possibility. This has retarded acceptance of post-buckling strength concepts in bridge specifications. For the same reason, tension-field design is presently not accepted in the design of crane runway girders in mill buildings.

The use of longitudinal stiffeners to permit large web depth-thickness ratios was pioneered in the design studies of Moisseiff and Lienhard (5.15), sponsored by the aluminum industry. Longitudinal stiffeners were introduced into specifications for traveling-crane design (5.47) in 1942, and subsequently into highway bridge design specifications (A6) and into specifications for aluminum alloy structures (A20, A21, A22).

The utilization of post-buckling-strength concepts in the aircraft industry and in industrial applications of light-gage metals has resulted primarily from use of thin material as a partition, deck, skin, or roof element. When structural strength is also needed, as is usually the case in such applications, the functional use is dual and the material provides both strength and effective space separation. In the AISI Specification (A11), post-buckling strength and tension-field action have not been utilized for thin beam webs because of the unsuitability of the necessary transverse stiffeners in this type of light-gage construction.

As recounted herein, steel building specifications in both Canada (A15) and the United States (A13) were quick to adopt buckling strength combined with partial tension-field action as the basis for plate-girder design, as proposed by Basler and Thürlimann. Both of these specifications have been supplemented by commentaries which explain the manner in which the experimental and analytical research has been interpreted and applied.

Research is underway, and more is needed, to study the fatigue strength of all-welded girders designed on the basis of tension-field action. Studies are needed to determine the degree to which closed-tube or box-flange design can improve the effectiveness of the tension field.

The development of metal and combination metal-and-concrete composite bridge deck and girder units, usually involving what has been termed "orthotropic construction," opens the need for a wider knowledge of web behavior when the member cross section is unsymmetrical, and tensile stresses resulting from bending are predominant over most of the section depth.

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Chapter Six

Beam-Columns

6.1 Introduction

The term "beam-column" denotes a member which is subject simultaneously to axial force and bending moment. Bending moment may result from transverse forces and/or from a known eccentricity of the axial force at one or both ends. As the bending moment approaches zero, the member tends to become a centrally loaded column, a problem that has been treated in Chapter 2. As the axial force approaches zero, the problem becomes that of a beam, which, for the laterally unsupported case, involves the buckling and design problems that have been treated in Chapter 4.

The determination of the ultimate strength of a beam-column is a problem in which inelastic action must be considered. If an isolated beam-column is either laterally supported or bent in its weakest plane, it fails without twisting; otherwise the lateral-torsional mode of failure must be considered.

A number of studies and reviews of the beam-column problem have been sponsored by Column Research Council (6.1, 6.2, 6.3, 6.4). Early developments, including the work of von Kármán, Ros and Brunner, Chwalla, Westergaard and Osgood, Jezek, and others, have been reviewed by Bleich (A1) in a monograph initiated through Column Research Council. Sidebottom and Clark (6.6) developed a convenient semigraphical scheme for determining the deflections and failure loads of eccentrically loaded columns.

Ketter, Kaminsky, and Beedle (6.7) have presented, and corroborated by tests, a method for determining the plastic behavior of laterally supported wide-flange shapes under combined moment and axial force, including the effect of residual stresses. Galambos and Ketter (6.8, 6.41) have presented numerically determined interaction curves for the maximum strength of beam-columns under various end conditions and including the effect of residual stress, correlating the analysis with earlier tests by Johnston and Cheney (6.19), Campus and Massonnet (6.20), and Mason. Fisher, and Winter (6.10).

Most analyses and tests of beam-columns have been concerned with the isolated member, whereas practical beam-columns are usually a part of a frame. Recent research has given increasing attention to the behavior of complete frames and to the effects of end restraint on beam-column behavior. Studies of framed beam-columns have been stimulated by the development of procedures for plastic design of structures (A7, A25). The effect of end restraint on the eccentrically loaded column has been explored by Bijlaard, Winter, Fisher, and Mason (6.9, 6.10). Ojalvo and others (6.42, 6.43, 6.44, 6.45) have developed nomograms which permit matching of compatible moments transmitted between the ends of the beam-column and the remaining portions of the structure to which the column ends are attached, in terms of the angular rotation at the joint. Tests on structural subassemblages by Lay and Galambos (6.40) have confirmed the general validity of the analytical techniques used in the aforesaid work. Neal and Mansell (6.24) have applied the same approach to truss members.

Beam-columns unsupported in the weak direction and subjected to forces or moments acting in the strong direction are found frequently in practice. If such a column is short and torsionally strong, as in the case of a box member, and is subjected to large bending moments, it will bend essentially in the plane of the applied forces and will develop as much strength as if it were restrained from deflecting in the weak direction. A long slender box-column with small bending moments in the strong plane will buckle in a direction normal to the plane of bending and will behave essentially the same as if loaded concentrically. For intermediate-length box-columns, some combination of the two failure modes can be expected.

A torsionally weak beam-column of open cross section (such as a wide-flange member, a tee, or an angle) is apt to twist as well as bend during buckling failure. An exception is the case wherein bending moment is introduced only in the weaker principal plane of the section.

The development of the theory of torsional buckling, with and without bending, is reviewed in detail by Timoshenko (6.11). Wagner introduced the initial theory of pure torsional buckling, a special case of the more general theory of combined torsional and bending failure under central load, which was later developed by Kappus and simplified by Goodier (6.12). The general problem of the column of open cross section under eccentric end load, with the bending moment caused by the eccentric load assumed to be constant over the length, has also been treated by Goodier (6.13). Hill and Clark (6.14, 6.15) have found that neglect of moments resulting from deflections may make a significant difference.

Salvadori (6.16, 6.17) has found the elastic critical loads of obliquely loaded I-shaped columns (unequal eccentricities at the ends) whose ends

are free to rotate in the plane of the web and are either elastically restrained or fixed in the plane parallel with the flanges. Each end is prevented from twisting around the column axis, and the tendency to twist under central load is neglected. The solutions are presented in the form of interaction curves.

Nylander (6.18) has found the critical loads for lateral-torsional buckling of monosymmetrical I-shaped columns (unequal top and bottom flanges) and tee columns, subjected to eccentric loading in the strong plane and having simply supported ends.

Instead of determining the buckling load, a lower bound to the strength of steel beam-columns can be found by calculating the maximum stress due to (1) bending moment and transverse loading acting in the plane of the web, and (2) accidental initial imperfections normal to the plane of the web. It is postulated that failure occurs when the maximum fiber stress reaches a specified limiting value. Using this approach, Zickel (6.22) and Thürlimann (6.21) have studied the stresses which arise in initially twisted beams, columns, and beam-columns. Nylander (6.18) has derived a practical design procedure based on a simple solution for the maximum stresses in a beam with initial deflections. His procedure can be applied to monosymmetrical and bisymmetrical I-shaped and tee-section columns with either oblique loading or eccentric loading. Finally, Horne (6.37) has presented a complete design procedure based upon the limiting-stress concept.

When a column of open cross section is loaded eccentrically with respect to both principal axes, both torsional and bending moments are produced. The direct stresses due to restraint of warping that accompanies non-uniform torsion add to the direct stresses caused by bending. Thürlimann (6.21), in an extension of the work of Zickel (6.22), has shown that, for certain loading conditions, the additional stresses due to torsion may be greater than those caused by bending. He also showed that a large increase in stress may result from initial twist in a centrally loaded column—comparable to the effect of the initial eccentricity assumed in the application of the secant formula to the design of centrally loaded steel columns.

Two simple approximate methods have been developed for estimating the strength of beam-columns. The first method is based upon the concept that the load which produces initiation of yielding in the fibers subjected to maximum stress provides a lower bound to the failure load. This method is described in the following section. The second approach is based upon the interaction formula described in Sec. 6.3.

6.2 Beam-Column Design Based on Load at Initial Yield

The allowable-stress formulas developed for this procedure apply only to members that fail by bending in the plane of the applied loads, and a separate design check has to be made with regard to possible lateral-torsional buckling. The procedure applies best for material having a linear elastic stress-strain relationship.

If the initial-yield theory is to be used to calculate the strength of columns with small lateral loads or small known eccentricities, it is necessary to assume that some *unintentional* "equivalent" eccentricity exists, in order that the "initial yield-strength" curve will approach a proper limit for concentrically loaded* columns. The assumed equivalent eccentricity includes the effects of all imperfections, such as initial crookedness, nonhomogeneity, and residual stresses. Thus, prior to the recognition of the importance of residual stress, the secant formula for the eccentrically loaded column provided a plausible and valuable explanation for the behavior of centrally loaded columns, and became the basis for the current AREA (A17) and AASHO (A16) column formulas for both central (concentric) and eccentric loads. In each of these specifications a simple parabolic formula provides a satisfactory approximation of the secant formula for centrally loaded columns having L/r less than 120.

The first step in developing the initial-yield criterion is the analysis of maximum combined stress caused by axial load, applied bending moment, and bending moment due to deflection. The load that causes the maximum combined stress to reach the yield point is divided by the desired factor-of-safety to give the working load. The special case of this procedure where the end loads have equal eccentricities gives the secant formula:

$$\sigma_{\max} = \frac{P}{A}\left[1 + \left(\frac{ec}{r^2} + \frac{e_o c}{r^2}\right)\sec\frac{L}{2r}\sqrt{\frac{P}{AE}}\right] \tag{6.1}$$

where $\sigma_{\max}$ denotes the maximum stress (at midlength of the column), e is the eccentricity of the applied end loads, and e_o is the assumed equivalent eccentricity representing defects and so forth.

The limiting average stress F_a for design use can be obtained from Eq. 6.1 by replacing P with nF_aA (where n is the factor-of-safety) and $\sigma_{\max}$ with σ_y, and then solving for F_a:

$$F_a = \frac{\sigma_y/n}{[1 + (ec/r^2 + e_o c/r^2)\sec L/2r\sqrt{nF_a/E}]} \tag{6.2}$$

A more comprehensive incipient-yield procedure for the design of eccentrically loaded columns can be based upon the solution for unequal end eccentricities. This procedure was presented by Young (6.23) and was discussed fully by Timoshenko (A9). Ketter (6.41) has shown that it can be expressed in an interaction equation. An elaborate extension

* In this case the "concentrically loaded" column is one without *intentional* end eccentricity.

of this approach, including consideration of both load eccentricity and column curvature, with a variety of end loading and restraint conditions, has been developed by Stephenson (6.25, 6.26) and includes detailed tables and charts which minimize the inherent difficulties in direct application of the complex formulas that are involved.

Julian (6.27) outlines a procedure, based on a convenient nomogram, for finding the maximum fiber stress at any section of a compressed member with equal or unequal end moments.

A good approximation of the maximum moment in a beam-column can be found from the following equation:

$$M_{\max} = M_o + \frac{P\delta_o}{1 - P/P_e} \tag{6.3}$$

where M_o and δ_o are the moment and deflection, respectively, *without regard to the added moment caused by deflection.* Eq. 6.3 can conveniently be written:

$$M_{\max} = M_o\left(\frac{1 + \psi P/P_e}{1 - P/P_e}\right) \tag{6.4}$$

in which, for a simply supported constant-section member,

$$\psi = \frac{\pi^2\delta_o EI}{M_o L^2} - 1$$

If the approximate maximum moment given by Eq. 6.4 is used to determine the maximum combined stress in a beam-column owing to direct load and bending, the following formula is obtained:

$$\sigma_{\max} = \frac{P}{A} + \left(\frac{1 + \psi\alpha}{1 - \alpha}\right)\frac{M_o c}{I} \tag{6.5}$$

where $\alpha = P/P_e$.

Since formulas for lateral deflection δ_o under many different loading conditions are readily available in handbooks, Eq. 6.5 greatly simplifies calculation of maximum stresses in beam-columns. Values of ψ for various conditions can be determined readily and the error, in comparison with exact analysis, is less than 2% in all common cases. For example, in the case of a member subjected to concentric axial loads and uniform lateral loading,

$$\psi = \frac{\pi^2[(5/384)(wL^4/EI)]EI}{(wL^2/8)L^2} - 1 = \frac{40\pi^2}{384} - 1 = 0.028$$

Several common cases, including the foregoing, are presented in Table 6.1.

In a beam-column, the relationship between the axial load and the maximum combined stress due to axial load and bending moment is, of

Table 6.1

Loading Condition	Parameter ψ in Eq. 6.4 and 6.5
Constant moment	+0.233
Concentrated lateral load at midlength	−0.178
Uniform lateral load	+0.028

course, nonlinear. Thus, in designing to a desired factor-of-safety with respect to a load causing maximum stress equal to yield, it is essential to calculate this (yield-producing) *load* and divide it by the factor-of-safety. This same procedure was used in developing the "exact" design formula, Eq. 6.2, for the eccentrically loaded column. If the approximate formula, Eq. 6.5, is used for the same design case (equal end eccentricities e, $M_o = Pe$, and $\psi = +0.23$), the allowable average stress becomes:

$$F_a = \frac{\sigma_y/n}{1 + \left(\dfrac{1 + 0.23n\alpha}{1 - n\alpha}\right)\dfrac{ec}{r^2}} \tag{6.6}$$

This equation gives results in close agreement with those obtained from Eq. 6.2, within 1% for $n\alpha$ less than 0.80, and within 2.5% for $0.80 < n\alpha < 0.95$.

For the case of the beam-column with uniform total lateral load $W = kP$, indicating proportional increase of lateral load W with axial load P, the same approach as just given yields the following equation, taking $\psi = 0.028$:

$$F_a = \frac{\sigma_y/n}{1 + \left[\dfrac{1 + 0.028(nF_a/\sigma_e)}{1 - (nF_a/\sigma_e)}\right]\left(\dfrac{k}{8}\right)\left(\dfrac{L}{r}\right)\left(\dfrac{c}{r}\right)} \tag{6.7}$$

Where $c = r$, El Darwish and Johnston (6.39) have prepared tables based on Eq. 6.7 for commercially available steels with yield points ranging from 33 to 100 ksi, load ratios k from 0.01 to 0.30, and L/r values from 10 to 200. These tables, with the introduction of suitable modifications of k that are also given, can be used to obtain approximate solutions for beam-columns having a variety of loading conditions (including eccentric end loads) and a variety of cross sections. As in the use of the secant formula, a separate design check is required for lateral-torsional buckling.

The secant formula, adapted for design use by means of charts or tables, is an adequate and economical approach to the design of built-up box-

type compression chords. Such members are not susceptible to torsional buckling, have unavoidable end eccentricities because of load nonaxiality or because of secondary stresses, and have relatively thin material that is likely to buckle at or near (but not significantly below) the yield point. In this type of application the main defect of the secant formula as used in bridge specifications lies in the irrational nature of the arbitrary equivalent-length factors that are introduced. However, the procedure can be applied more rationally to framed columns if the equivalent end eccentricities are determined by means of an analysis that includes the effect of axial load on bending stiffness.

For the limiting case of the centrally loaded column, the secant formula can provide a reasonable basis for design by the assumption that an accidental eccentricity exists in all columns. The assumed eccentricity can be adjusted empirically so that the results obtained from the secant formula are almost the same as those found by taking residual stress as a basis for the column strength curve.

Nevertheless, in summary, four objections to the initial-yield criterion for beam-column design can be made:

1. It cannot be applied rationally to beam-columns made of a material with a nonlinear stress-strain curve.
2. The effect of residual stress cannot rationally be taken into account.
3. Design on the basis of initial yield may be over-conservative in certain cases, for example, for an I-shaped member having large end eccentricities and subjected to bending about the minor (y–y) axis.
4. For the I-shaped column that is bent about the major (x–x) axis and is laterally unsupported in the weak direction, a separate lateral-buckling check must be made.

The interaction formulas now to be discussed provide procedures for beam-column design that overcome all of these objections to the initial-yield procedure. In the range where the initial-yield approach is applicable, they will give about the same results.

6.3 Beam-Column Strength in Bending Without Lateral Buckling

This section deals with interaction formulas for beam-columns that are subjected to bending in the weak direction, and to beam-columns subjected to strong-direction bending provided that they are adequately braced against lateral buckling. Shanley (6.28) has discussed interaction formulas in considerable detail, and was an early proponent (6.29) of this approach to design. In Ref. 6.29, Shanley reviews early developments of the design of columns in the aircraft field.

The strength of members subjected to flexure combined with compressive axial load can be expressed conveniently by interaction formulas in terms of the ratios P/P_u and M/M_u, where

P = thrust at actual failure,

P_u = ultimate load for the centrally loaded column for buckling in the plane of the applied moment,

M = maximum bending moment at actual failure, and

M_u = ultimate bending moment in the absence of axial load.

The following equation is the basis for several such interaction formulas:

$$\frac{P}{P_u} + \frac{M}{M_u} \leqslant 1 \tag{6.8}$$

In the elastic range, an approximation of the maximum bending moment for beam-columns subjected to bending forces producing maximum moment at or near the center of the member is obtained from Eq. 6.4 by setting $\psi = 0$:

$$M_{\max} = \frac{M_o}{1 - (P/P_e)} \tag{6.9}$$

where P = applied axial load,

P_e = elastic critical load for buckling in the plane of applied moment, and

M_o = maximum applied moment, not including contribution of axial load interacting with deflections.

Substituting Eq. 6.9 into Eq. 6.8 gives

$$\frac{P}{P_u} + \frac{M_o}{M_u[1 - (P/P_e)]} \leqslant 1 \tag{6.10}$$

For eccentrically loaded columns having equal end eccentricity e at both ends, Eq. 6.10 takes the following form:

$$\frac{P}{P_u} + \frac{Pe}{M_u[1 - (P/P_e)]} \leqslant 1 \tag{6.11}$$

The simpler straight-line interaction formula (Eq. 6.8 with $M = M_o$) has been used in some design specifications. When L/r is large and the applied moment is small, with maximum moment at or near midlength of the member, Eq. 6.8 generally overestimates the carrying capacity. This fact has been demonstrated by tests reported by Hill, Hartmann, and Clark (6.30, 6.31), and by Mason, Fisher, and Winter (6.10). Therefore, the trend in specifications is toward the inclusion of the amplification factor $1/[1 - (P/P_e)]$, as shown in Eqs. 6.10 and 6.11.

Galambos and Ketter (6.8) present dimensionless interaction curves for the ultimate strength of typical wide-flange beam-columns bent in the strong direction and having (*a*) equal axial-load eccentricities at both ends, and (*b*) eccentricity at one end only. Interaction curves are given which define the initial-yield condition, the maximum load capacity neglecting residual stresses, and the maximum load capacity considering residual stresses. Calculations for maximum capacity (in the inelastic range of bending) involve application of the Newmark numerical procedure (6.32). The parameters used for the maximum-strength formulas are P/P_y and M/M_u, and the interaction formulas take the following form:

$$A\frac{M}{M_u} + B\frac{P}{P_y} + C\left(\frac{P}{P_y}\right)^2 \leqslant 1 \tag{6.12}$$

where A, B, and C are empirical coefficients that are functions of L/r and of the loading condition, and P_y is the column axial load at the full-yield condition ($P_y = A\sigma_y$). The Galambos-Ketter curves are in good agreement with prior tests made at Wisconsin (6.33), Lehigh (6.19, 6.34), and Liège (6.20).

The AISC Manual *Plastic Design in Steel* (6.35) and the AISC Specification (A13) use a complete tabulation of the Galambos-Ketter coefficients of Eq. 6.12 for beam-columns loaded as described in the preceding paragraph.*

In a third case of strong-direction bending, where a beam-column is bent in double curvature by moments producing plastic hinges at both ends, the AISC (A13) recommends a strength formula that can be written

$$0.85\frac{M_o}{M_p} + \frac{P}{P_y} \leqslant 1 \tag{6.13}$$

This has the same form as Eq. 6.12, but is independent of L/r. M_p is the plastic bending moment ($M_p = Z\sigma_y$, where Z is the plastic modulus).

Galambos and Prasad (6.46), using the curves that formed the basis for Eq. 6.12, as well as additional information supplied by Ketter (6.41), have furnished more complete tables for the ultimate strength of eccentrically loaded beam-columns for all ratios of M_o/M_p and for L/r values between 0 and 120. Although presented in nondimensional form, the tables are for steel with a yield stress of 33 ksi. If it is assumed that the residual stress has the same distribution pattern over the section and the same proportion of the yield stress for all steels, the tables can be applied to steels of other yield points by substituting a modified L/r:

$$\left(\frac{L}{r}\right)_{\text{mod}} = \frac{L}{r}\sqrt{\frac{33}{\sigma_y}}$$

* See Ref. 6.35, pages A2 and A3; and Ref. A13, Tables 5.33 and 5.36.

6.4 Strength of Laterally Unsupported Beam-Columns

The preceding section pertains to beam-columns that are either bent in their plane of weakness or bent in their strong plane and laterally supported. In the case of I-shaped laterally *unsupported* beam-columns bent in the strong plane, Hill and Clark (6.15) have found that the following modified interaction formula is quite satisfactory for both the elastic and inelastic range:

$$\frac{P}{P_u} + \frac{M_o}{M_u[1 - (P/P_e)]} \leqslant 1 \tag{6.14}$$

where P = applied axial load,

P_u = axial load producing failure in the absence of bending moment,

P_e = elastic critical load for buckling in the strong plane,

M_o = maximum applied moment, *not* including contribution of axial load interacting with deflections, and

M_u = bending moment producing failure in the absence of axial load.

It should be noted that the terms P_u and M_u of Eq. 6.14 have essentially the same meaning respectively as P_u and M_u of Eq. 6.10, except that in the case of Eq. 6.14 the possibilities of buckling in the weak plane and of lateral-torsional buckling are incorporated.

Eq. 6.14 is conservative and shows good agreement with the test results for eccentrically loaded aluminum alloy columns that fail by lateral-torsional buckling. The corresponding simpler straight-line interaction formula obtained by dropping the magnification factor is

$$\frac{P}{P_u} + \frac{M_o}{M_u} \leqslant 1 \tag{6.15}$$

This equation gives results with greater scatter and overestimates the strength of slender members for which the applied moment is maximum at or near midlength, and where P/P_u is large in comparison with M_o/M_u. Campus and Massonnet (6.20) also found that Eq. 6.14 agrees with test data better than Eq. 6.15 does, and that Eq. 6.15 gives too high an estimate of strength in some cases.

6.5 Evaluation of Interaction Design Formulas

Interaction formulas have a simple form, are convenient to use, and have a wide scope of application. Allowable stresses determined from interaction formulas vary continuously and in a smooth transition from stresses for concentrically loaded columns at one limit to stresses for beams at the other.

In building construction, Eq. 6.10, modified to an allowable-stress basis, is used by both the AISI (A11) and the AISC (A13), but both permit

omission of the amplification factor when the ratio of average axial stress to allowable axial stress in a member is less than 0.15. In plastic design, Eqs. 6.12 and 6.13 are used by AISC (A13) in conjunction with appropriate tables. Suggested specifications (A21, A22) for design of aluminum alloy members also call for use of a formula similar to Eq. 6.14; however, the aluminum specifications recommend a straight-line formula when the bending moment at the center of the span is not more than one-half the maximum moment in the span.

In specifications for both railway (A17) and highway (A16) bridges, secant-type formulas have been retained for design on an allowable-stress basis, as discussed in Sec. 6.2.

Allowable-stress interaction formulas for design are essentially empirical and are devised to give a desired factor-of-safety against some limiting condition, which may be either excessive deflection or the initiation of yielding. The nature of an interaction formula obviously depends upon the expressions chosen to define the allowable stresses F_a and F_b due to axial load and bending moment respectively. The allowable compressive stress F_a may be chosen with a suitable factor-of-safety against buckling (taking into account residual stress) or initial yielding (taking into account assumed accidental end eccentricities).

6.6 Beam-Columns having Unequal End Moments

When designing a beam-column subjected to unequal end moments, it may be overconservative to use the maximum of these in an interaction formula for design of the member, especially where the end moments are of opposite sign. This is because the interaction formula assumes the maximum moment to be at or near the center of the span. For cases where the moment diagram is a straight line (no loads in the span), Massonnet (6.20) has developed an "equivalent uniform moment" as follows:

$$M_{eq} = \sqrt{0.3(M_a^2 + M_b^2) + 0.4M_aM_b} \tag{6.16}$$

where M_a and M_b are the end moments. On the basis of a simpler analysis, Horne (6.37) recommends the ratios M_{eq}/M_a listed in Table 6.2, where M_a is the numerically larger end moment. The AISC (A13) and AISI (A11) Specifications concur in the adoption of a still simpler criterion, which can be expressed as follows:

$$\frac{M_{eq}}{M_a} = 0.6 + 0.4\frac{M_b}{M_a} \geqslant 0.4 \tag{6.17}$$

The Massonnet and Horne recommendations and those of the AISC and AISI Specifications are compared in Table 6.2.

Table 6.2. Equivalent Uniform Moment M_{eq}

M_b/M_a*	+1.0	+0.5	0	−0.5	−1.0
M_{eq}/M_a by Horne	1.000	0.762	0.565	0.429	0.391
M_{eq}/M_a by Massonnet	1.000	0.759	0.548	0.416	0.447
M_{eq}/M_a by AISC and AISI	1.000	0.800	0.600	0.400	0.400

* M_a = numerically larger end moment.

If the moment diagram is not a straight line, and the maximum moment is not at the center of the span, a conservative procedure for using Eq. 6.16 or Eq. 6.17 is to choose a substitute straight-line moment diagram that is external to the actual moment diagram at every point (neglecting the moment contribution of the axial loading interacting with deflections).

The maximum stress at the end of a beam-column should be checked, irrespective of what other interaction formulas are used, by use of the following equation:

$$\frac{f_a}{F_{ao}} + \frac{f_b}{F_b} \leqslant 1 \tag{6.18}$$

where F_{ao} and F_b are the unreduced allowable stresses for a column and a beam, respectively.

6.7 Beam-Columns in Biaxial Bending

Adaptations of both the secant formula and the interaction-type formula have been applied to beam-columns in biaxial bending in bridge and building specifications. For example, Eq. 6.10 can be expanded to cover biaxial bending, as follows:

$$\frac{P}{P_u} + \frac{M_{o(x-x)}}{M_{u(x-x)}[1 - (P/P_{e(x-x)})]} + \frac{M_{o(y-y)}}{M_{u(y-y)}[1 - (P/P_{e(y-y)})]} \leqslant 1 \tag{6.19}$$

Equation 6.19 neglects torsional moments that are generally present in a biaxially loaded beam-column and which may require consideration in the case of open sections. Birnstiel and Michalos (6.47) have analyzed the effect of biaxial moments on wide-flange sections under axial loading only, in the inelastic range, including the torsional behavior.

El Darwish and Johnston (6.39) have considered the design problem of a closed-section column consisting of four point-areas located at the corners of a square, bent biaxially by transverse loads in the two principal planes of the cross section. The general problem of analysis of biaxially bent beam-columns is exceedingly complex and only a beginning has currently (1966) been made.

6.8 Beam-Columns in Frames

Most beam-columns do not occur as isolated, simply supported members, but as parts of frameworks, which often have rigid connections. In such cases a rational design procedure should take account of the end restraints afforded and end loads transmitted by the adjacent framing members, which may in turn be loaded axially and/or by lateral loads. At the connection between the beam-column ends and the restraining members, there is a common value of rotation (assuming continuity), and, in addition, the member end moments must be in equilibrium. The continuous-frame problem, in either the elastic or inelastic range, can be programmed for incremental solution by means of a digital computer.

For structural-grade steel with a yield point of 33 ksi, Ojalvo and others (6.42, 6.43, 6.44) made use of a concept originated by Chwalla in which the deflected axis of any beam-column is represented by a portion of a basic column deflection curve. Charts were developed for finding both the end moments in the restraining members and the slenderness ratio of the beam-column, for the fully continuous condition. Companion sets of curves were prepared for load ratios P/P_y of 0.12, 0.2, 0.3, 0.4, and 0.6. The charts apply specifically to cases where moments are introduced to the beam-column at its ends. Ojalvo (6.42) shows how the method can be applied to a member with any set of end moments and end restraints. Levi (A26) has applied similar procedures to restrained columns with sidesway. Lay (6.5) has shown how the charts can be generalized and extended to steels of different yield points.

Many of the important advances in the study of restrained beam-columns, as reviewed in the preceding paragraphs, have occurred during the five-year period between the first and second editions of this *Guide*. However, general procedures are not yet available for the design of restrained beam-columns for any condition of load and end restraint, and for any cross section, yield point, and so forth. In the absence of such general procedures, the design procedure for restrained beam-columns developed by Winter and associates (6.38) can be used. The Winter approach is simple and conservative, but at the same time considers the strength afforded by the end restraints of adjacent members.

Eccentrically loaded, end-restrained columns can be designed safely by replacing the restrained beam-column by an equivalent, hinged-end beam-column having a length equal to the effective length (between points of contraflexure) of the real, restrained beam-column, and analyzing this equivalent beam-column for axial compression plus those portions of the total joint moment which are resisted by the real beam-column alone. It is sufficiently accurate to use moment distribution to determine those portions of the total joint moments which are resisted by the real beam-column, neglecting the reduction in effective column stiffness attributable to the axial force. (This approximation is conservative, since it will result in an overestimate of bending moment resisted by the column.)

Using this simplified procedure for the design of a rigid frame, the designer would determine the end moments of each vertical member by conventional moment analysis, without regard to change of effective stiffness caused by axial load. He would then dimension the member for the normal force and the end moments so obtained, using the effective length KL instead of the real length L in determining the slenderness ratio. This method is applicable regardless of the particular code provision for calculating stresses due to combined axial force and bending (secant formula, interaction formula, and so forth).

For members having real eccentricities (such as truss members with joint eccentricities and multi-story columns with offset axes) which result in end moments M_e, the designer can calculate the effective eccentricity e_n of each such member framed to a given joint from the results of the moment distribution analysis, i.e., from the following equation:

$$e_n = \frac{M_e}{P_n} \frac{(I/L)_n}{\sum (I/L)} \tag{6.20}$$

where P_n is the axial compression force and $(I/L)_n$ the end rotational rigidity of the nth compression member; and $\sum(I/L)$ is the sum of the rigidities of all members connected to the given joint. Each member can then be designed for simultaneous axial load P_n and bending moment $M = P_n e_n$, using the effective length KL for determining the slenderness ratio.

The following outline summarizes the Winter simplified procedure for design of a restrained beam-column:

1. Select a trial section (using an appropriate approximate method) and compute the bending stiffness parameter (I/L) for the member;
2. Estimate the effective length of the member, using the charts given in Fig. 2.21, or any other convenient procedure;
3. Calculate the bending-moment distribution in the frame;
4. Check the trial beam-column selection using the member effective length;
5. If necessary, repeat the procedure, using other trial sections; and finally,
6. Check the stresses at the ends of the member to make certain that they do not exceed the basic allowable values (i.e., with no reduction for buckling).

The most rational approach to beam-column design is to use the ultimate strength of the complete frame as the basis for the determination of permissible working loads, regardless of whether the allowable-stress or

the plastic-design procedure is being used. Research is underway (1966), particularly at Lehigh University, to apply the principles of plastic design to tier-building frames (6.48). New developments are to be expected in both the allowable-stress and the plastic-design techniques for restrained beam-columns.

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Chapter Seven

Pony Trusses

7.1 Introduction

A pony-truss bridge is one in which top-chord lateral bracing members cannot be used (because of vertical clearance or other requirements), so that the truss compression chords must be braced laterally by the vertical and diagonal web members.

It may be noted that procedures for the lateral-buckling analysis of the top chord of a pony truss can also be applied to the lateral buckling of the compression flange of a plate girder, if the girder tension flange is braced both laterally and torsionally at vertical-stiffener locations (see Sec. 7.6).

The compression chord of a pony truss receives elastic lateral support at the panel points. The design of the compression chord may be based on the computed buckling load; or, because of initial crookedness and because of moments introduced by bending of the floorbeams, it may be based on combined-stress calculations that include the effect of deflection. The latter approach is a rational one but has not as yet been simplified sufficiently to make it a practical design procedure. On the other hand, the buckling-load analysis gives only an upper bound to the actual strength of the member. Current design rules are based on a semiempirical procedure in which adequate stiffness of the compression-chord lateral supports is obtained by designing them for fictitious horizontal loads introduced at the top-chord panel points normal to the plane of the truss. The buckling load of the chord with elastic lateral supports at the panel points is then determined, and the design load is found by dividing this buckling load by a suitable factor-of-safety.

The development of the buckling-load analysis and to a much lesser extent the combined-stress procedure will be reviewed in this chapter. The design of pony truss transverse frames (floorbeams, truss verticals, and connecting knee braces) has a direct bearing on both procedures. If the transverse frames are not designed adequately, failure of the compression chord, and subsequently of the bridge, may result.

Towards the end of the nineteenth century the failure of several pony-truss bridges focused attention on the top-chord buckling problem. Engesser, between 1884 and 1893 (7.1, 7.2), was the first to present a simple, rational, and approximate formula for the required stiffness, C_{req}, of elastic supports equally spaced between the ends of a hinged-end column of constant section. An equivalent uniform elastic support was assumed in the Engesser analysis.

Chwalla (7.3) presented a more general solution for the problem of a column under constant compression and supported on an elastic foundation. He assumed that the ends are supported by rotational elastic restraints of arbitrary stiffness.

References 7.4 to 7.18 are a succession of reports on later investigations that provide more general solutions directly related to the pony-truss problem. These developments are reviewed by Bleich in Chapter 8 of Ref. A1. The results of these studies cannot be applied readily in practice because of their complexity.

Hu (7.19), using the energy method, has studied the problem of elastically supported chords. He considered nonuniform axial forces, chord cross sections, and variable-stiffness spring supports for both simple and continuous pony-truss bridges.

Holt (7.20, 7.21, 7.22, 7.23), in work sponsored by Column Research Council, has presented a method of analysis for determination of the buckling load of a pony-truss top chord which is essentially "exact" in that it includes most of the secondary effects which influence the behavior of the pony truss. In a similar manner, Lee (7.24, 7.25) studied the stability of pony-truss bridges.

The effect of floor-system deflections on the top-chord stresses was studied in another CRC-sponsored project by Barnoff and Mooney (7.26). Experimental work on the primary problem of the instability of elastically supported columns has been done by Engesser (7.27), Schibler (7.28), and Lazard (7.29). Tests on models of pony-truss bridges have been conducted by Holt (7.23) and by Lee (7.24).

7.2 Buckling of the Compression Chord

The buckling problem of the compression chord of a pony truss can be reduced to that of a column braced at intervals by elastic springs whose spring constants correspond to the rigidity of the truss transverse frames. The top-chord axial compression and the top-chord stiffness vary from panel to panel, and the rigidity of the transverse frames varies from panel point to panel point, thus complicating the theoretical problem. In addition, there are secondary factors such as the following:

1. The stiffening effect of the truss diagonals.
2. The torsional stiffness of the chord and web members.
3. The initial crookedness of the chord and the eccentricity of the axial load.
4. For nonparallel-chord trusses, the effect of chord curvature.

Engesser's solution (7.1, 7.2) is based on the following simplifying assumptions:

1. The top chord, including the end posts, is straight and of uniform cross section.
2. Its ends are taken as pin-connected and rigidly supported.
3. The equally spaced elastic supports have the same stiffness and can be replaced by a continuous elastic medium.
4. The axial compressive force is constant throughout the chord length.

Engesser's analysis can be applied with reasonable accuracy to the case where the lateral support is supplied by equally spaced springs, provided that the half wavelength of the buckled shape of the continuously supported bar is at least 1.8 times the spring spacing; and this will be true if the bar is stable as a two-hinged column carrying the same axial load and having a length no less than 1.3 times the spring spacing. Since the flexibility of the end supports is neglected, the Engesser solution can best be used as a preliminary design tool with a more accurate subsequent evaluation by Table 7.1 as hereinafter discussed.

Engesser's solution for the required stiffness of a pony-truss transverse frame is

$$C_{\text{req}} = \frac{P_c^2 l}{4EI} \tag{7.1}$$

where C_{req} is the elastic transverse frame stiffness at a panel point that is required to ensure that the overall chord having panel lengths l and flexural rigidity EI will attain buckling load P_c. If the proportional limit of the column material is exceeded, E should be replaced by the tangent modulus E_t.

Eq. 2.3a for basic column strength can be written as follows for a column of length l:

$$E_t I = \frac{P_c (Kl)^2}{\pi^2} \tag{7.2}$$

Taking l in this equation as the panel length of the pony-truss compression chord, we can substitute Eq. 7.2 into Eq. 7.1, obtaining the required spring constant as

$$C_{\text{req}} = \frac{\pi^2 P_c}{4K^2 l} \tag{7.3}$$

This equation has been shown* to be adequate when the half wavelength of the buckled chord is no less than $1.8l$; and this limiting value corresponds to a K-factor of 1.3. It is not applicable to short bridges with a small number of panels.

It should be noted that P_c is both of the following:

1. The buckling load of the entire compression chord, laterally supported by the transverse frames (and assumed to be pin ended).
2. The buckling load of the portion of the compression chord *between* the transverse frames (with ends restrained consistent with the factor K).

According to the Engesser theory, the maximum compression-chord buckling load and the corresponding required spring constant of each support can be determined as follows for a member of given cross section having area A:

1. Determine the critical load P_c for the member between spring supports, using the expression

$$P_c = A\sigma_c \tag{7.4}$$

Obtain σ_c from an appropriate column strength curve, taking the equivalent column slenderness ratio as Kl/r, with $K = 1.3$ and r estimated on the basis of probable shape and size of member.

2. Determine the spring-constant C_{req} such that the buckling load of the chord member as a whole is equal to P_c:

$$C_{\text{req}} = 1.46 \frac{P_c}{l} \tag{7.5}$$

It may be noted that Eq. 7.5 follows from Eq. 7.3, taking

$$\frac{\pi^2}{4K^2} = \frac{\pi^2}{4(1.3)^2} = 1.46$$

The Engesser simplifying assumption of taking the chord ends as pin connected may result in significantly unsafe errors in C_{req} in the case of short pony trusses. Holt (7.31, 7.32) provides an alternate design procedure which does not require this simplifying assumption. Holt's solution for the buckling load of the compression chord of a pony truss is based on the following assumptions (see Fig. 7.1):

1. The transverse frames at all panel points have identical stiffness.
2. The radii-of-gyration of all top-chord members and end posts are identical.

* See Ref. 7.19, p. 275.

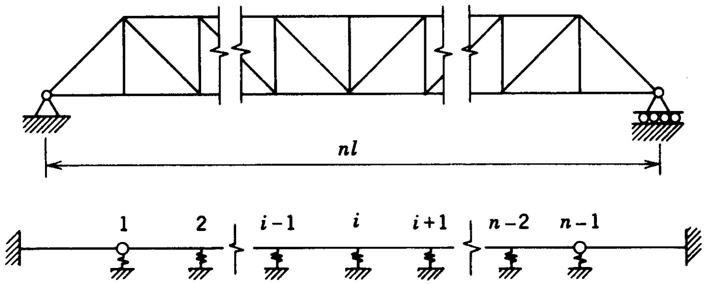

Fig. 7.1. Pony truss and analogous top chord.

3. The top-chord members are all designed for the same allowable unit stress, hence their areas, and (from 2) their moments of inertia are proportional to the compression forces.
4. The connections between the top chord and the end posts are assumed pinned.
5. The end posts act as cantilever springs supporting the ends of the top chords.
6. The bridge carries a uniformly distributed load.

The results of Holt's studies are presented in Table 7.1, which gives the reciprocal of the effective-length factor K as a function of n (the number of panels) and of Cl/P_c (where C is the furnished stiffness at the top of the least-stiff transverse frame).

Table 7.1, where applicable, provides a rapid design aid in checking the stability of a pony-truss compression chord. The procedure is as follows:

1. Design the floorbeams and web members for their specified loads.
2. Calculate the spring-constant C furnished at the upper end of the cross frame having the least transverse stiffness.
3. Calculate the value of parameter Cl/P_c, where P_c is the maximum design chord stress multiplied by the desired factor-of-safety.
4. Enter the table with n and Cl/P_c, and find the corresponding value of $1/K$ for a compression-chord panel, interpolating as necessary.
5. Determine the value of Kl/r for the compression-chord panel (note that this value of Kl/r is to be applied to all panels).
6. Determine the allowable top-chord compressive unit stress corresponding to this value of Kl/r, using the appropriate column curve or table.

Values of $1/K$ less than 0.5 (i.e., $K > 2$) are only of academic interest, since usual bridge proportions and transverse-frame stiffnesses lead to

Table 7.1. $1/K$ for Various Values of Cl/P_c and n

$1/K$	n						
	4	6	8	10	12	14	16
1.000	3.686	3.616	3.660	3.714	3.754	3.785	3.809
0.980		3.284	2.944	2.806	2.787	2.771	2.774
0.960		3.000	2.665	2.542	2.456	2.454	2.479
0.950			2.595				
0.940		2.754		2.303	2.252	2.254	2.282
0.920		2.643		2.146	2.094	2.101	2.121
0.900	3.352	2.593	2.263	2.045	1.951	1.968	1.981
0.850		2.460	2.013	1.794	1.709	1.681	1.694
0.800	2.961	2.313	1.889	1.629	1.480	1.456	1.465
0.750		2.147	1.750	1.501	1.344	1.273	1.262
0.700	2.448	1.955	1.595	1.359	1.200	1.111	1.088
0.650		1.739	1.442	1.236	1.087	0.988	0.940
0.600	2.035	1.639	1.338	1.133	0.985	0.878	0.808
0.550		1.517	1.211	1.007	0.860	0.768	0.708
0.500	1.750	1.362	1.047	0.847	0.750	0.668	0.600
0.450		1.158	0.829	0.714	0.624	0.537	0.500
0.400	1.232	0.886	0.627	0.555	0.454	0.428	0.383
0.350		0.530	0.434	0.352	0.323	0.292	0.280
0.300	0.121	0.187	0.249	0.170	0.203	0.183	0.187
0.293	0						
0.259		0					
0.250			0.135	0.107	0.103	0.121	0.112
0.200			0.045	0.068	0.055	0.053	0.070
0.180			0				
0.150				0.017	0.031	0.029	0.025
0.139				0			
0.114					0		
0.100						0.003	0.010
0.097						0	
0.085							0

values of $1/K$ reasonably near 1.0, and this results in economical use of material.

Hu (7.19) developed the curves shown in Fig. 7.2. These curves give the stiffness of the compression-chord transverse supports that is required to make each panel of chord buckle as one half wave. Hu assumed that both the flexural rigidity (EI) and the axial force (P) of the compression chord vary as a symmetrical second-degree parabola along the chord length.

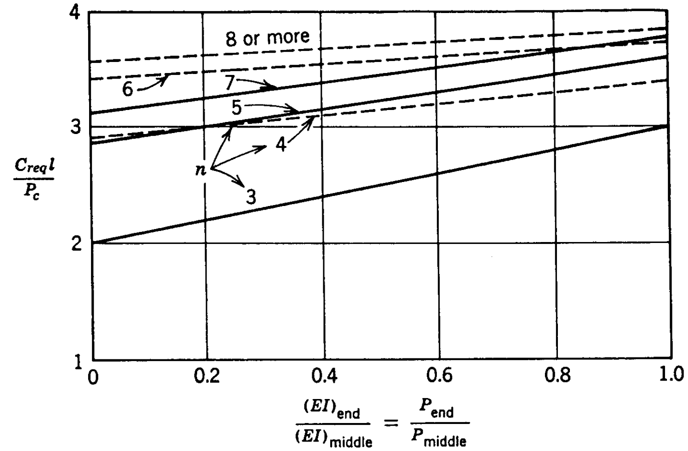

Fig. 7.2. Effect of variation in compression chord on transverse stiffness requirements.

Hu's results for a chord of constant section ($(EI)_{end}/(EI)_{middle} = 1.0$ in Fig. 7.2) can be compared with Holt's work for $1/K = 1$ (first line of Table 7.1) for the cases $n = 4$, 6, and 8, which were considered by both investigators. Hu's results give stiffness requirements approximately 7% less than those of Holt for $n = 4$, and 5% greater for $n = 6$ or 8. Thus, the results are in reasonable agreement, even though the procedures are somewhat different.

Hu (7.19) also studied the effect of the variation of C_{req} caused by parabolic variation of the length of the pony-truss verticals, and the effect of parabolic variation of C_{req}. In both cases the value of C_{req} will be less than that for the case where C_{req} has the same value at each transverse frame.

Because of the uncertainties involved in the analysis of pony-truss top chords, it is reasonable to require a factor-of-safety for overall top-chord buckling somewhat greater than that used for designing hinged-end columns.

The transverse-frame spring-constant C that is actually furnished can be determined for the frame loaded as shown in Fig. 7.3 by means of the following equation:

$$C = \frac{E}{h^2[(h/3I_c) + (b/2I_b)]} \qquad (7.6a)$$

The first term within the denominator brackets represents the contribution of the truss vertical, and the second term represents the contribution of the floorbeam. Thus, the contributions of the top-chord torsional strength and the web-diagonal bending strength to the frame stiffness are neglected

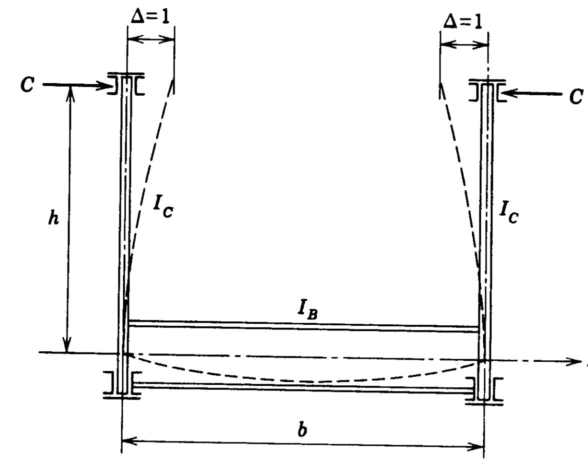

Fig. 7.3.

in this equation. It is evident that if the floorbeam is very stiff in comparison with the truss vertical, the frame stiffness is approximately

$$C = \frac{3EI_c}{h^3} \qquad (7.6b)$$

When the two chords tend to move in the same direction, the stiffness C will be greater than that given by Eq. 7.6a; therefore, C as found from Eq. 7.6a is always the lower bound.

7.3 Effect of Secondary Factors on Buckling Load

The consideration of secondary factors involves procedures which require a large amount of computation. Most of these procedures use the usual methods of indeterminate structural analysis to set up a system of simultaneous, linear, homogeneous equations. The stability criterion is that the determinant of the coefficients of this system of equations must vanish.

Holt (7.21) considered the following secondary factors in his analysis:

1. Torsional stiffness of the chord and the web members.
2. Lateral support given to the chord by the diagonals.
3. Effect of web-member axial stresses on the restraint provided by them.
4. Effect of nonparallel-chord trusses.
5. Error introduced by considering the chord and end posts to be a single straight member.

Holt's analyses show that the error introduced by neglecting all of these factors is quite small, and that satisfactory results in calculating the compression-chord buckling load can be obtained by assuming that the chord is a straight elastically braced column whose length is the total length of the chord and end posts. These conclusions are in agreement with those reached by Schibler (7.30), who finds that the torsional stiffness of the top chord and the support furnished by the web diagonals increase the chord buckling strength only slightly.

7.4 Top-Chord Stresses due to Bending of Floorbeams and to Initial Chord Eccentricities

The compression chord of a pony truss will be displaced laterally at some panel points as a result of live load on the bridge, and because of initial crookedness and unintentional eccentricities of the chord. Such lateral deflections will, of course, reduce the maximum load capacity of the chord (and of the bridge), just as end eccentricity and initial curvature will reduce the compression strength of any column.

Design procedures that take account of such imperfections are not presently available. It is difficult to take them into account, because of both the complexity of the necessary calculations and the lack of knowledge with regard to probable initial imperfections. The calculations involve the top-chord stiffness in both bending and torsion.

Holt (7.23) has developed an empirical procedure for estimating bending moments in the top chord and end posts that is in agreement with his test results. He recommends that the end post be designed as a simple cantilever beam to carry the axial load combined with a transverse load of 0.5% of the axial load, applied at the upper end. Tentatively, a value of 1% should probably be used, in line with German specifications.*

7.5 Design Procedures

In the design of half-through truss spans, AASHO Specifications† require that "The top chord shall be considered as a column with elastic lateral supports at the panel points. The critical buckling force of the column, so determined, shall exceed the maximum force from dead load, live load and impact in any panel of the top chord by not less than 50 percent." Thus, a load factor of 1.5 is considered adequate, and this is less than that required by the same specification in the determination of allowable compressive stresses in hinged-end columns.‡ This is presumably justified on the basis that all pony-truss top-chord compression members cannot be stressed simultaneously up to the same proportion of critical buckling load. However, it seems evident that in cases where maximum compressive

* See Ref. A18, Vol. I, Sec. 12.1.
† Ref. A16, Art. 1.6.70.
‡ See Ref. A16, Int. Spec. 7(62), Art. 1.4.2.

stress may occur simultaneously in the entire length of the compression chord, the safety factor should be higher than that used for general column design, rather than lower.

German Buckling Specifications* base the design of pony-truss compression chords on the Engesser solution for the buckling load, with the recommendation that K be kept the same for all panels and between limits of 1.2 and 3.0. The formula for required transverse-frame stiffness is given as:

$$C_{\text{req}} = \frac{2.50}{K_m^2}\frac{P_c}{l} \tag{7.7}$$

For $K_m = 1.3$, Eq. 7.7 gives very nearly the same result as Eq. 7.5, which is to be expected since both equations are based on Engesser's solution.

For design of the web verticals, the AASHO Specification reads:† "The vertical truss members and the floorbeams and their connections in half-through truss spans shall be proportioned to resist a lateral force of not less than 300 pounds per linear foot, applied at the top-chord panel points of each truss." The German Buckling Specifications‡ specify a transverse force in either direction of $1/100K_m$ times the compressive force in the adjacent chords, increased by the impact coefficient, for intermediate frames. K_m is the average of the K values for all panel-length compression chords. For end frames the same applies, except that K_m is omitted.

The problem of the lateral stability of a pony-truss compression chord will now be illustrated by means of a design example, using, in part, AASHO Specifications (A16).

Design Example 7.1 (see Fig. 7.4)

Consider a pony truss that has 12 panels of 13 ft 4 in. for a span of 160 ft. The transverse frame is shown in the sketched cross section. The 27W84 floorbeams are required by bridge deck loads. The top chord is a 10-in.-square box section, with wall thickness to be determined by design requirements for a maximum compressive stress of 360 kips (dead load, live load, and impact). The verticals are 10W rolled sections.

The approximate properties of the compression-chord cross section are as follows:

Area: $A = 4td = 40t \text{ in.}^2$

Moment-of-inertia: $I = \dfrac{Ad^2}{6} = 670t \text{ in.}^4$

Radius-of-gyration: $r = \dfrac{d}{6} = 4.08 \text{ in.}$

* See Ref. A18, Vol. II, Sect. 12.12.
† Ref. A16, Art. 1.6.70.
‡ See Ref. A18, Vol. I, Sec. 12.1.

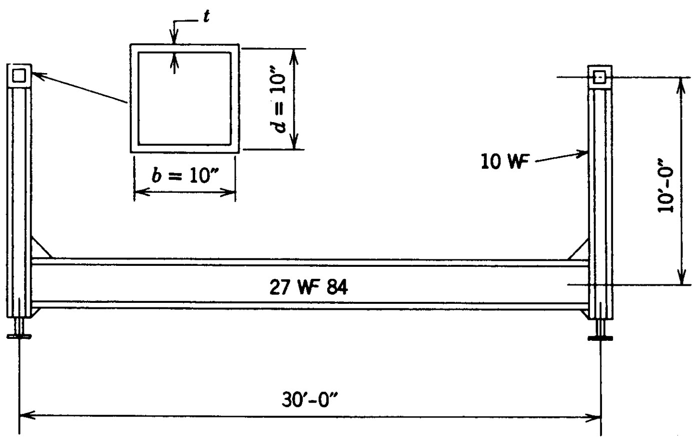

Fig. 7.4.

By AASHO requirements, the lateral force to be resisted at the upper panel points is

$$0.3 \text{ kips/ft} \times 13.33 \text{ ft} = 4 \text{ kips*}$$

The maximum moment in the transverse frame will be at the joint of the web vertical and the floorbeam, and is

$$M = 4 \text{ kips} \times 120 \text{ in.} = 480 \text{ kip-in.}$$

Assume that the maximum tension in the web vertical due to the bridge load is 24 kips in the region where the compression chord is most highly stressed. Taking r as approximately $0.42d$, and the allowable stress as 20 ksi due to combined bending and direct stress, the required cross-sectional area of the vertical is

$$A = \frac{1}{f_a}\left(P + \frac{Mc}{r^2}\right) = \frac{24 + (480 \times 5)/(4.2)^2}{20} = 8.0 \text{ in.}^2$$

Try a 10WF33, for which $A = 9.71$ in.^{2} and $I = 170.9$ in.^{4} The I of the floorbeam is 2825 in.^{4}, and from Eq. 7.6a the transverse-frame spring constant is:

$$C = \frac{29000}{120^2[120/(3 \times 170.9) + 360/(2 \times 2824.8)]} = 6.76 \text{ kips/in.}$$

Assuming the factor-of-safety against buckling to be 2.0 (only 1.50 is required by AASHO), the compression chord must be designed for a

* By the German specifications (Ref. A18), assuming $K_m = 1.3$, the maximum lateral force would be 360 kips/130 = 2.77 kips.

buckling strength of $P_c = 2.0 \times 360$ kips = 720 kips. By Eq. 7.5, the buckling load of the chord as a whole will be 720 kips provided that C is no less than $(1.46 \times 720)/160 = 6.57$ kips/in. Since the actual C has been found to be 6.76, this requirement is met.

Taking the effective length to be $1.3l$, as assumed in Eq. 7.5, the effective slenderness ratio of each panel length of the compression chord is

$$\frac{Kl}{r} = \frac{1.3 \times 160}{4.08} = 51$$

Using the AASHO formula for pin-ended columns of A36 steel, the allowable compression is

$$f_a = 16000 - (0.39)(51)^2 = 15.0 \text{ ksi}$$

The required chord area for a 360-kip load is, therefore,

$$A = \frac{360}{15.0} = 24.0 \text{ in.}^2$$

Since $A = 40t$, the required wall thickness of the compression chord is readily found as

$$t = \frac{24.0}{40} = 0.60 \text{ in.} \quad \text{Use } \tfrac{5}{8}\text{-in. wall thickness}$$

As a more accurate alternative and preferred procedure to the use of Eq. 7.5, Kl/r can be determined from Table 7.1. Using the actual supplied C, and the value of $P_c = 720$ kips,

$$\frac{Cl}{P_c} = \frac{6.76 \times 160}{720} = 1.50$$

Entering Table 7.1 in the column for $n = 12$ panels and $Cl/P_c = 1.50$, $1/K$ falls between 0.800 and 0.850. The interpolated value is $1/K = 0.81$, or $K = 1.23$. The slenderness ratio of the chord between panel points is then

$$\frac{Kl}{r} = \frac{1.23 \times 160}{4.08} = 48.2$$

which is very close to the value of 51 found previously. The allowable compressive stress and the required wall thickness are not appreciably different from the previous values; however, it should be kept in mind that for bridges having less than 10 panels, C_{req} as found by Holt's procedure (Table 7.1) may be appreciably greater than the value found by the Engesser method.

7.6 Plate Girder with Elastically Braced Compression Flange

Although most of the research and references presented herein concern the pony-truss bridge, the design recommendations that have been given are also applicable to the design of plate girders with elastically braced compression flanges. Such girders will customarily have full-depth vertical stiffeners serving the dual function of web stiffener and top-flange transverse support. The lack of girder-web diagonal members does not invalidate the comparison, since no advantage was taken of web diagonals in the pony-truss design derivations. In applying the design procedures to girders, the girder top flange, including one-sixth of the web area, should be introduced in place of the top chord of the pony truss.

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Appendix B

Technical Memoranda of Column Research Council

Technical Memorandum No. 1: The Basic Column Formula*

The Column Research Council has brought out that it would be desirable to reach agreement among engineers as to the best method for predicting the ultimate load-carrying capacity in compression of straight, prismatic, axially loaded, compact members of structural metals. It was proposed that Research Committee A of the Council be assigned the problem of reporting on the correctness and desirability of the tangent-modulus column formula. This formula involves simply the substitution of the tangent modulus, E_t, for E in the Euler formula. This formula may be written

$$\frac{P}{A} = \frac{\pi^2 E_t}{(KL/r)^2}$$

where P = the ultimate load (lb),

A = the cross-sectional area (sq in.),

E_t = the compressive tangent modulus (slope of the compressive stress-strain curve) of the material in the column at the stress P/A (lb per sq in.),

r = least radius-of-gyration of cross section (in.),

L = the length of the column (in.), and

K = a constant depending on end conditions:

- $K = 2$ for one end fixed and the other end free,
- $K = 1$ for both ends simply supported,
- $K = 0.7$ for one end fixed and the other end simply supported,
- $K = 0.5$ for both ends fixed.

For materials which exhibit upper and lower yield points in compression, the lower yield point is to be considered as the limiting value of P/A.

Information and reference to literature supporting the foregoing statement will be made available on request to the Secretary of the Column Research Council.

It is the considered opinion of the Column Research Council that the tangent-modulus formula for the buckling strength affords a proper basis for the establishment of working-load formulas.

* Issued May 19, 1952.

The column formula presented here differs in form from the familiar Euler formula only in that the tangent modulus-of-elasticity is substituted for the ordinary modulus-of-elasticity. There is, however, a great practical difference between the two formulas, for, whereas the Euler formula can be solved directly for the average stress corresponding to any given slenderness ratio, the tangent-modulus formula cannot. It is not the intention to advocate the use of the tangent-modulus formula in design, but rather to propose it as the basis for relating the compressive stress-strain properties of the material to the column strength of the material. The formula furnishes the information for approximating to the average stress in terms of the ratio of slenderness, for any type of centrally loaded column under consideration, by making suitable assumptions with respect to such items as accidental eccentricity, initial curvature of member, residual stresses, and variation in properties of the material.

Technical Memorandum No. 2: Notes on Compression Testing of Metals*

It is desirable to have compressive stress-strain curves of material from as many different sources as possible, together with the usual identifying tensile properties obtained from specimens adjacent to specimens used for determining the compressive stress-strain curves. In order that the compression specimens and the tension specimens be as nearly as possible equally representative of the material, the cross sections of compressive specimens in general should be the same as those of the adjacent tension specimens.

Specimens taken from a flange with parallel faces or from the web of a rolled shape or from a plate should be rectangular in cross section. They should be machined only on the two cut sides and on the ends. In a thin specimen, however, which requires lateral support to prevent premature buckling, it is permissible to remove just enough material from the supported faces of the specimen, if necessary, to make them plane.

The ends of compression specimens should be plane and normal to the longitudinal axis of the specimen. The ends should be parallel within close limits. In most cases this requirement necessitates the turning or grinding the ends.

In general, compression specimens should be no longer than necessary to accommodate a compressometer or strain gages and leave between each end of the specimen and the adjacent end of the gage length a length of specimen equal to the greatest cross-sectional dimension. The compressometer should meet specifications for ASTM class A extensometers,† which limit the error in indicated strain to ± 0.00001 in. per in. In order to obtain a representative stress-strain curve, the gage length should be not less than the greatest cross-sectional dimension; and in order to keep the specimen short, the gage length should not exceed twice this dimension. If the length of a rectangular specimen is more than about 4.5 times the length of the shorter side of the rectangle or of

* Reprinted from the ASTM Bulletin, July 1956, pp. 61, 62.

† 1965 Book of ASTM Standards, Part 30.

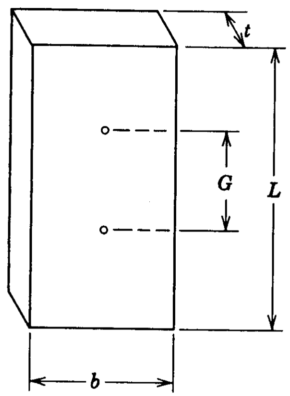


Fig. B1. Test specimen.

a circular specimen more than 4 times the diameter, difficulty may be expected in avoiding premature bending (column action), and special precautions must be taken to prevent excessive bending.* Referring to Fig. B1, the following relationships summarize the above requirements for a member of rectangular cross section:

$$G \stackrel{=}{>} b$$
$$G \geqq t$$
$$4.5t \stackrel{=}{>} L \stackrel{=}{>} G + 2b$$

where G = gage length,
b = width of specimen,
L = length of specimen, and
t = thickness of specimen.

The dimensions of each test specimen should be given. The specimens should be measured with a micrometer reading to 0.001 in. Nominal dimensions should not be used.

Both ends of a compression specimen should bear on smoothly finished plane surfaces. The bearing blocks should be made of or faced with suitably hard material such that the faces of the blocks will not suffer permanent deformation during the test. The blocks should be at least as thick as the smallest cross-sectional dimension of the specimen and should project beyond the area of contact a distance at least half as great as the smallest cross-sectional dimension.

Precautions should be taken to insure uniform distribution of strain over the cross section and to prevent relative rotation of upper and lower surfaces throughout the test. The following are suggested methods:

(1) Capping with a thin layer of plaster of Paris between the upper bearing

* For rectangular specimens see ASTM Methods of Compression Testing of Metallic Materials: (E 9–61).

block and the head of the testing machine. While the plaster is setting, a load should be maintained that will bring those machine surfaces into contact that will normally be in contact at higher loads.

(2) Use of a subpress loaded through a push rod acting at the lower end of a hollow plunger.

(3) Use of bearing blocks which will permit adjustment for parallelism of bearing surfaces.*

The deviation in strain of one gage from the average of all gages at 50% of the estimated yield strength of the material should be no more than ±5%.

Adjustable bearing blocks cannot be depended upon to compensate for tilting of the heads of the testing machine during loading of the specimen and should be used only if appreciable relative tilting of the heads does not occur. If a spherical bearing block is used, it should be at the upper end of the specimen (for specimens tested with the axis vertical). It is desirable that the center of the spherical surface lie in the flat face which bears on the specimen. It is important that the center of the spherical surface be placed close to the axis of the specimen, so that the eccentricity of loading may not be great enough to overcome the friction necessary to rotate the block.

If the length of the specimen does not exceed the maximum recommended length (4.5 or 4 times a cross-sectional dimension for a rectangular or a circular specimen, respectively), strains should be measured with an averaging compressometer or with two strain gages mounted opposite each other. In the case of longer specimens tested without lateral support, reasonable certainty of uniform distribution of strain can be obtained only with the use of not fewer than two strain gages on the wide sides of a rectangular specimen, four gages on a square specimen, or three on a circular specimen. Strains measured on only one gage length are usually unsatisfactory. The stress-strain curve should extend from zero stress and strain to values for which the ratio of stress to strain is at least as low as $0.7E$ or to a strain of at least 0.01 in. per in., whichever results in the larger strain.

Since the properties of the material are a function of the rate of loading, if the loading is continuous, the rate of loading should be recorded. Compressive stress-strain curves should be plotted with stress as ordinate and strain as abscissa to as large a scale as the accuracy of the data justifies. The individual values of stress and strain should also be reported. When applying the above procedures to material which is suspected of showing a considerable variation in properties, the specimens should be taken from a sufficient number of locations to define the extent of the variation in properties.

Determination of Typical Stress-Strain Curve from a Number of Individual Stress-Strain Curves. It is assumed that the compressive stress-strain relationships of enough specimens will be determined so that all variations of the material likely to be submitted under a given specification will be represented. The yield-strength values determined from the individual tests should be

* See Fig. 1 in ASTM Methods of Compression Testing of Metallic Materials: (E 9–61).

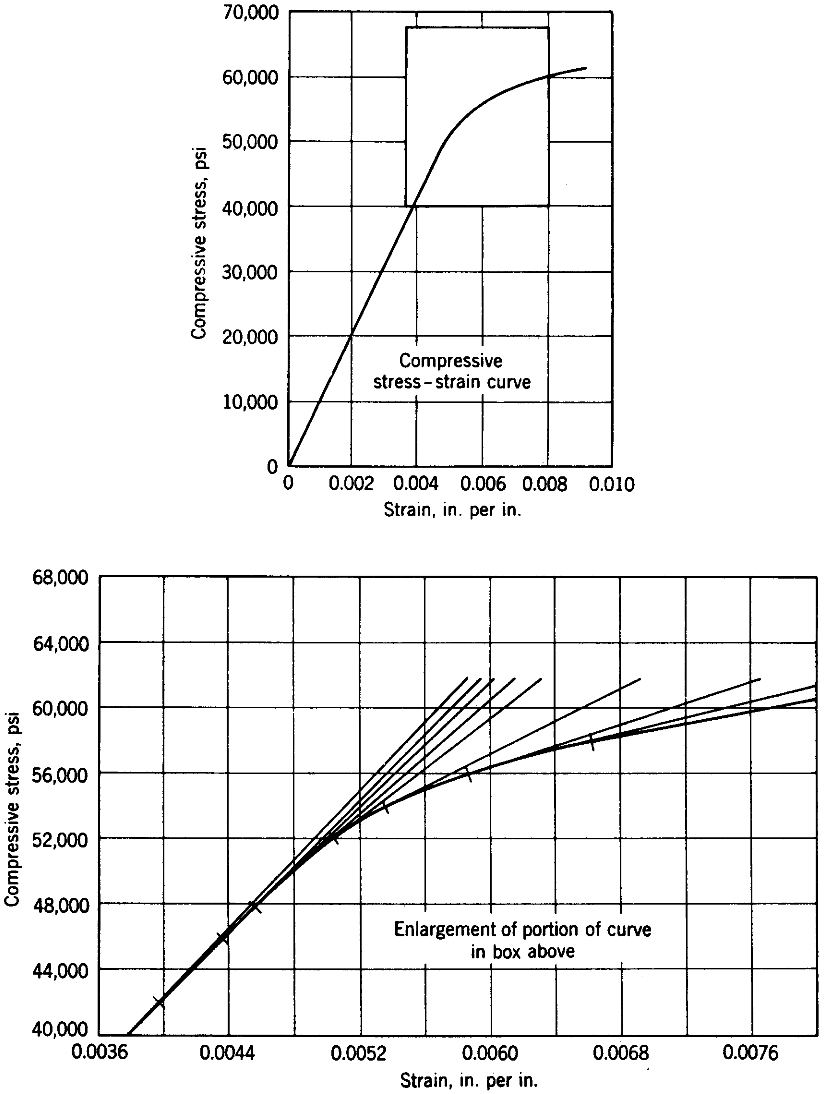


Fig. B2. Typical compression stress-strain curve of a high-strength aluminum alloy.

presented in the form of a distribution plot in which the percent of the total number of tests for which the yield strength is within a certain range is plotted against the average value of the range.

Several different methods can be used for obtaining a typical stress-strain curve from a number of individual stress-strain curves. One method which is fairly simple and has had considerable use is described generally as follows:

1. Record the strain departures from the modulus line for various fixed percentages of the particular individual yield-strength value. These percentages should cover stresses from the proportional limit to above the yield strength.
2. Average all offset values for each of the fixed percentages. (For steel beams it is recommended that the offsets be weighted in proportion to relative flange and web areas.) A curve may then be plotted in which the ordinate is the percent of yield strength and the abscissa is the strain offset from the initial modulus line.
3. For any appropriate yield-strength value, a typical stress-strain curve can then be plotted by adding the offset values to the strain consistent with the elastic-modulus value.

Figure B2 shows a typical compression stress-strain curve of a high-strength aluminum alloy. Lines have been drawn tangent to this curve at different values of stress P/A. The slopes of these lines define the corresponding tangent, modulus, E_t,* essential to the determination of the basic column strength.†

Technical Memorandum No. 3: Stub-Column Test Procedure‡

THE STUB-COLUMN TEST

Definition

A stub column is a column whose length is sufficiently small to prevent failure as a column, but long enough to contain the same residual stress pattern that exists in the column itself.

Application

Column strength may be expressed as a function of the tangent modulus determined from the stress-strain relationship of the stub column test.¶ Hence, a stub column test is an important tool in the investigation of the strength of columns.

* Convenient and accurate techniques are available for determining the tangent modulus; one such technique is described in NACA TN 2640, "Interaction of Column and Local Buckling in Compression Members," by P. P. Bijlaard and G. P. Fisher.

† See "The Basic Column Formula," Technical Memorandum No. 1, Column Research Council, May 19, 1952. (Presented previously in this Appendix. Ed.)

‡ This Document was No. X-282-61 approved by the International Institute of Welding at the Annual Conference, Oslo, June 1962, as a Class C Document. It was prepared originally by Lambert Tall under the technical guidance of Task Group 1 of the Column Research Council as Fritz Laboratory Report No. 220A.36, February, 1961, and was revised by an IIW Working Group consisting of H. Louis, Chairman (Belgium), M. Marincek (Yugoslavia), and L. Tall (U.S.A.).

¶ Column strength is not a direct function of the tangent modulus. For example, for an H-shape bent about the strong axis the function is approximately direct, whereas about the weak axis, the strength is a function, approximately, of the cube of the tangent modulus. Further, the tangent modulus theory will give a conservative estimate of column strength. For other shapes, such as box shapes, there is no direct or simple relationship. (Refer to Chapter 2, Section 2.3. Ed.)

The difference between the Young's modulus and the tangent modulus at any load level, determined from a compression test on the complete cross section, essentially reflects the effect of residual stresses. This may be realized when one considers that the cross section, hitherto completely elastic under load, becomes elastic-plastic at the proportional limit. The presence of residual stresses in the cross section implies that some fibers are in a state of residual tension while others are in a state of residual compression. The fibers in a state of residual compression are the first to reach the yield point under load.

The difference between the behavior of a column free of residual stresses and one containing residual stresses lies in the fact that both the tangent modulus and the effective moment-of-inertia are greater for the column free of residual stresses. On the other hand, the behavior of a stub column reflects only the effect of residual stresses on the tangent modulus;* the reduction of the moment-of-inertia due to plastification has no effect on the behavior. Under load, some parts of the cross section will yield before others, leading to a decrease of the effective moment-of-inertia and hence in the strength of the column, since those portions of the cross section which have yielded play no further role in strength consideration, provided the effect of strain hardening is neglected.

Therefore it may be seen that the residual-stress distribution in the cross section, through its influence on the effective moment-of-inertia, supplies the connecting link between the strength of a column and the tangent modulus of the stress-strain relationship of the stub column.

References

Extensive literature exists to show that residual stresses are, indeed, the major factor contributing to the strength of axially loaded, initially straight, columns, and that a conservative value for this strength may be specified in terms of the tangent modulus determined from the results of a stub-column test. (See pp. 53–54 of this *CRC Guide* for most of these references. Ed.)

STUB-COLUMN TEST PROCEDURE

1. *Object*

To determine the average stress-strain relationship of the complete cross section by means of the stub column.

2. *Preparation*

(a) The stub-column length should be cut a distance at least equal to the section depth away from flame-cut sections. The presence of any cold-bending yield lines would modify the resulting stress-strain relationship.

* When the cross section is free of residual stresses, the tangent modulus coincides with Young's modulus.

(b) The length of the stub column should be

$2d + 10''$, or, $2d + 25$ cm, or, $3d$ minimum

$20r_y$, or, $5d$ maximum

where d = depth of section

r_y = radius-of-gyration about weak axis

(c) The ends of the stub column are to be milled flat and perpendicular to the longitudinal axis of the column.*

(d) The thickness of flanges and webs, and the length and area of the stub column should be measured.

3. *Gaging*

Mechanical dial gages or electrical resistance gages should be used, although the use of dial gages over a comparatively large gage length is to be preferred since they provide a better average for the cross section.

The dial gages should read to 1/10,000 inch when read over a 10″ gage length, or to 1/1000 inch when used between base plates over the complete length of the stub column.† Where it can be demonstrated that electrical resistance gages give the same results, they may be used instead of dial gages.

The gage length should be placed symmetrically about the mid-height of the stub column. At least two gages in opposite positions should be used and the average of the readings taken. Corner gages over the complete column length are used for alignment; mid-height gages are used for the measurement of stress-strain relationship. When four mid-height gages instead of two are used, the corner gages may be dispensed with. (This is possible with the flange tips of an H-shape.)

Figure B3 gives typical gage arrangements for a structural shape.

For uniformity, the following gaging procedure for H-shapes is advised:

(a) Four 1/1000-inch dial gages over the complete length of stub column, at the four corners; to be used for alignment.

(b) Two 1/10,000-inch dial gages on opposite sides over a 10″ gage length at the mid-height; to be used in the determination of the stress-strain relationship. The points of attachment for the gage length are to be at the junction of the flange and web, to afford freedom from local flange crippling. When early local flange crippling will not occur, four 1/10,000-inch dial gages over 10″ gage length at mid-height may be clamped to each flange tip; alignment corner gages are not then needed.

(c) The specimens are whitewashed before testing. Flaking of the mill scale during testing gives a general idea of the progress of yielding.

* The alignment is greatly simplified when the tolerance across the milled surface is ±0.001 inch (±0.02 mm).

† A length of 25 cm. would correspond to the 10″ gage length. Dial gages used on a 25 cm. base should read to a precision between limits of 1/10,000 cm. and 1/4,000 cm.

4. *Set-Up*

The specimen should be set up in the testing machine so that it rests between flat bearing plates. These plates should be thick enough to ensure a uniform distribution of load through the specimen.

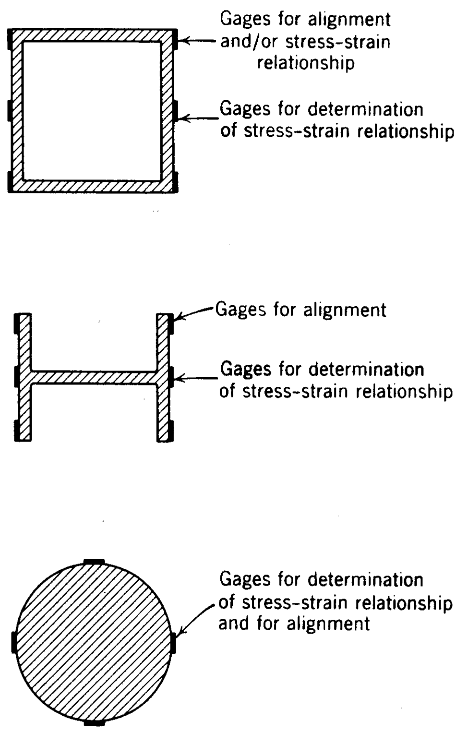
Alignment is achieved preferably by the use of special beveled bearing plates, or else by the use of spherical bearing blocks which are fixed by wedges after alignment to prevent rotation.

The test set-up is shown in Fig. B4.

5. *Alignment*

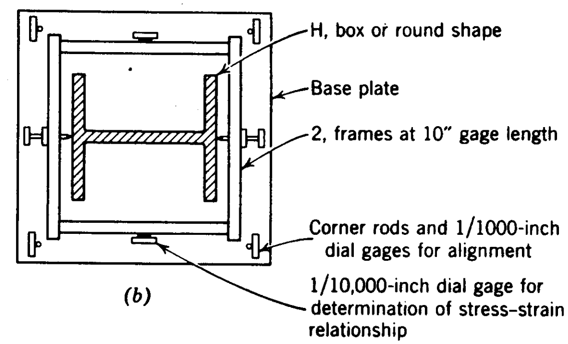
The alignment should be carried out at a range of loads less than the proportional limit. For rolled H-shapes of mild structural steel, this limit is about 1/2 of the predicted yield load; for welded shapes, the limit may be as low as 1/4 of the yield load.

The alignment is carried out by noting the variation of strain at the four corners of the specimen. The variation of individual strains at the four corners should be less than 5% from their average at the maximum alignment load.

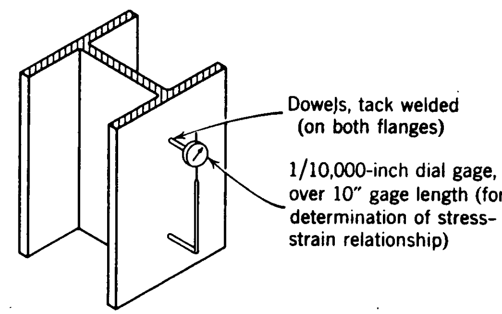


(*a*)

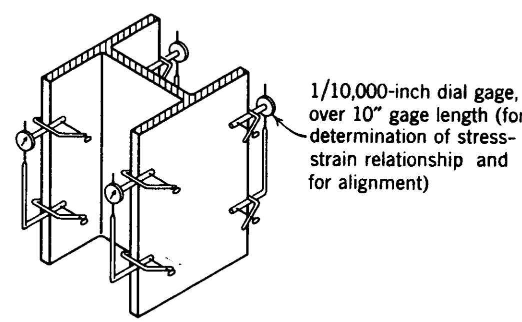
Fig. B3. Position of gages for alignment and testing.



(*b*)



(*c*)



(*d*)

Fig. B3 (concluded).

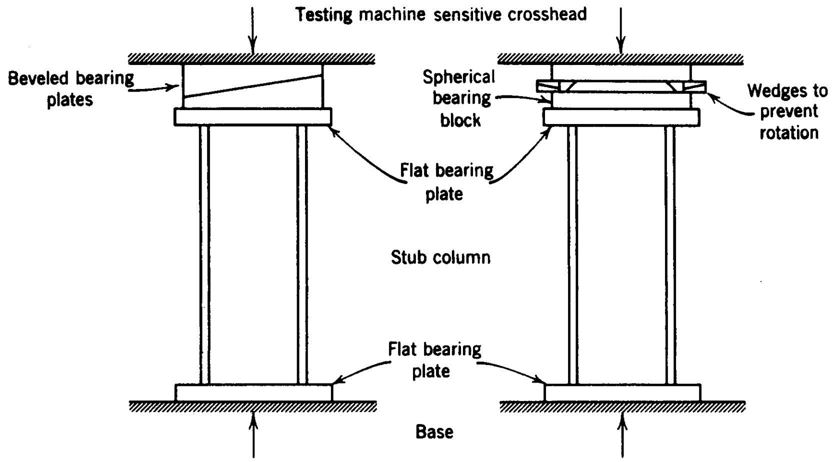


Fig. B4. Stub column set up for testing.

Alignment at very low loads is unsatisfactory. The alignment should consist of about ten increments up to the maximum alignment load.

To check that the load is below the proportional limit, the stress-strain relationship may be plotted during the test and its linearity observed. It is inadvisable to exercise this control by observing the whitewash for yielding of the mill scale, indeed this method is unsatisfactory since it indicates yielding at a load value in excess of the actual proportional limit as indicated by the plotted stress-strain relationship.

6. *Testing*

The stress-strain curve should be constructed from as many experimental data points as possible. To this end, the load increment in the elastic region should be at about 1/30 of the expected yield load. After the proportional limit the load increments should be reduced so that there are sufficient data points to delineate the "knee" of the stress-strain curve.

The proportion limit* will be marked by the beginning of the deviation of the stress-strain relationship from the linear behavior, and the development of yield lines (made clearly visible by whitewash) will indicate the progress of yielding. This is further covered in Item 10.

After the onset of yielding, readings should be recorded when both load and strain have stabilized. The criteria used to specify when data may be recorded depend on the type of machine used for testing. This is explained further below in Item 7.

* It is assumed that the residual stresses are symmetrical and constant in the longitudinal direction, so that the proportional limit does not indicate localized yielding.

To ensure correct evaluation of the yield level and other compressive properties, the test should be continued until one of the following conditions is satisfied:

(a) For an immediate drop in load due to buckling, the test should be continued until the load has dropped to about 50% of the load at the predicted yield level.

(b) For a specimen that exhibits a plastic region of considerable extent, the test should be continued until the load has dropped to about 80% of the load at the predicted yield level.

(c) For a specimen that strain hardens without apparent buckling, or which strain hardens without a plastic range, the test should be continued until the load is about 25% above that at the predicted yield level, for mild steel.

The load and strain at all critical levels should be recorded. This is further outlined in Item 9.

It may become necessary to remove some of the mechanical gages before the completion of the test to prevent damage due to local buckling.

7. *Criteria for Stabilization of Load*

Standard criteria should be followed for the recording of test data when the load is greater than the proportional limit. The choice of the criterion will depend upon the type of testing machine, either hydraulic or mechanical. That is,

(a) With a mechanical testing machine, the criterion is for no further decrease of load, and

(b) With a hydraulic testing machine, the criterion is for no further movement of the sensitive cross head, with the loading valve closed, provided the machine does not leak. (When leakage is suspected, the criterion is a simulation of that used for a mechanical testing machine: that is, for no movement of the crosshead controlled by the loading valve, the load is allowed to stabilize until there is no further decrease of load.)

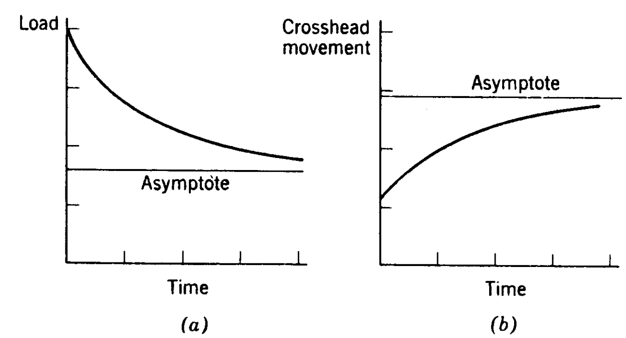


Fig. B5. Yield criteria.

These criteria are best used by plotting the load change, or cross-head movement, on graph paper, and noting the value corresponding to the asymptote. See Fig. B5.

The test data are recorded when

(a) The asymptotic load is approached, with the load criterion, or

(b) The asymptotic cross-head movement is approached, with the cross-head movement criterion.

Readings should not be recorded until the asymptote is definite. Experience will indicate the time intervals required, but three-minute intervals are usually satisfactory. The cross-head movement should be measured by a 1/10,000-inch mechanical dial gage.

8. *Evaluation of Data*

The following methods may be used for the evaluation of the test data:

1. Plot the test data during the test to detect any inconsistencies.
2. Translate the test data to those of stress versus strain (from a knowledge of the exact cross-sectional area), and plot to an enlarged scale for the stress-strain relationship.
3. Determine the tangent-modulus curve from the stress-strain relationship. This is best determined by use of a strip of mirror; the mirror is held normal to the curve and a line drawn along the mirror. The normal is determined from the continuity of the stress-strain relationship and its mirror image at the tangent point considered.

9. *Data to Report*

The following information should be obtained from the stress-strain relationship given by a stub-column compression test:

(a) Young's modulus-of-elasticity
(b) Proportional limit
(c) Yield strength
(d) Yield stress level
(e) Elastic range
(f) Elastic-plastic range
(g) Plastic range
(h) Onset of strain hardening
(i) Strain-hardening range
(j) Strain-hardening modulus

The occurrence of local buckling, and any other phenomena during the test, should be recorded.

The stress-strain diagram is shown in Fig. B6.

10. *Definition of Terms*

The above terms should be defined and measured as follows:

(a) *Young's Modulus*, E, is the ratio of stress to strain in the elastic range.

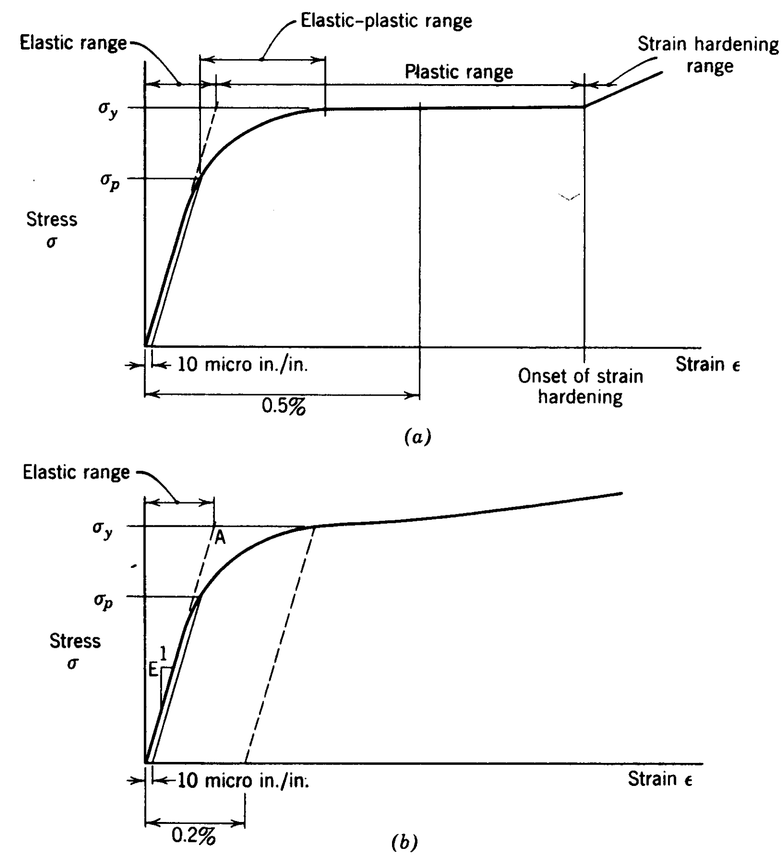


Fig. B6. The stress-strain diagram.

(The method of measuring is defined by ASTM Standard E 111-61 (1961), "Determination of Young's Modulus at Room Temperature.")

(b) *Proportional Limit*, σ_p, is the load corresponding to the strain above which the stress is no longer proportional to strain. It is best measured by the use of an offset of 10 micro in./in.

(c) *Yield Strength* is "the stress, corresponding to the load which produces in a material, under the specified conditions of the test, a specified limiting plastic strain." This is the definition of ASTM Standard A370-64 (1964), and an offset of 0.2% is suggested. (The yield-strength criterion is normally used when there is yielding without constant stress.) (For stub-column stress-strain curves, the *yield stress level* is mainly used, as it is a more representative value; it is an average value in the plastic range.)

(d) *Yield Stress Level*, σ_y, is the stress corresponding to a strain of 0.5%. This stress will usually correspond to the constant stress under yield when the stress-strain relationship is such as shown in Fig. B6a.

(e) *Elastic Range* may be defined as the increment of strain between zero strain and the strain at the point A in Fig. B6a.

(f) *Elastic-Plastic Range* is the increment of strain corresponding to the increment of stress between the proportional limit and the first value of stress equal to the yield-stress level.

(g) *Plastic Range* may be defined as the increment of strain between the elastic range and the onset of strain hardening.

(h) *Onset of Strain Hardening* may be defined as the strain corresponding to the intersection on the stress-strain curve of the yield stress level in the plastic range with the tangent to the curve in the strain-hardening range. This tangent is drawn as the average value in an increment of 0.002 in./in. after the apparent onset of strain hardening.

(i) *Strain-Hardening Range* is the range of strain after the plastic range where the cross section no longer strains at a constant or near-constant stress.

(j) *Strain-Hardening Modulus* is the ratio of stress to strain in the strain-hardening range. It is measured as the average value in an increment of 0.005 in./in. strain after the onset of strain hardening.

Appendix C

Computer Analysis and Design

Since 1960, when the first edition of the *Guide* appeared, electronic digital computers have come into common use in the field of structures, and they are being used increasingly in the field of column analysis and design.

Computers can be used to great advantage in two types of structural engineering problems: those presenting analytical difficulties which result in excessively lengthy and burdensome calculations, and those which must be solved a great number of times with changes in some parameters. The problem of column analysis and design falls usually, but not always, into the second of these categories. Computer programs have been written which allow the designer to analyze columns under a variety of boundary and loading conditions, including those which involve some of the fairly complicated formulas of the preceding chapters. Computers have also been used to solve column design problems in accordance with a number of different specifications.

In computer-aided design, the computer can be made to obtain the minimum-weight column. A common procedure is to arrange all available sections in order of increasing cross-sectional area (i.e., weight) and to begin the design analysis with the lightest. The computer can then automatically analyze and check successive sections until one is found which meets all criteria. The advantage of this method is that the first section satisfying the design criteria is the lightest. (The remainder of the list need not be investigated because none of the remaining sections is lighter.)

Elsewhere in the *Guide*, solutions to certain buckling problems have been presented. There are, of course, many additional problems of this type where the exact solution is either unknown or is very complicated. Such problems, as has been stated, can be attacked using numerical procedures (such as energy methods, finite-difference methods, numerical integration, and methods of successive approximations) in combination with the digital computer. Examples of buckling problems for which solutions have not been presented in the *Guide*, and which are amenable to computer analysis, are as follows:

1. Buckling loads of columns of nonuniform cross section having various end restraint conditions.
2. Buckling loads of columns carrying axial loads applied at the ends and also at several points along their length.

3. Buckling loads of columns carrying distributed axial loads along their length.
4. Buckling loads of columns carrying dynamic axial loadings.
5. Lateral-torsional buckling loads of beams and compression members having variable cross section.
6. Problems involving beam-columns having uniform or nonuniform cross section and various end restraint conditions.
7. Problems involving beam-columns supported at the ends and elastically supported at intermediate points, carrying distributed axial loads.
8. Critical loads of complicated frames.
9. Buckling coefficients of plates and shells.

The following references, and also general references A1 and A9, may be of use in connection with computer analysis of buckling problems:

C1 Newmark, N. M., "Numerical Procedure for Computing Deflections, Moments and Buckling Loads," *Trans. ASCE*, Vol. 108 (1943), p. 1161.

C2 Salvadori, Mario G., "Numerical Computation of Buckling Loads by Finite Differences," *Trans. ASCE*, Vol. 116 (1951), p. 590.

C3 Salvadori, Mario G., "Lateral Buckling of Eccentrically Loaded I-Columns," *Trans. ASCE*, Vol. 121 (1956), p. 1163.

C4 "Digital Computer Solutions of the Dynamic Column Buckling Equations," by E. Sevin, *Conference Papers*, First Conference on Electronic Computation, ASCE, Kansas City, Missouri (1958), p. 237.

C5 Zar, M., and Beck, C. F., "Computer Design of Structural Steel for Buildings," *Conference Papers*, Second Conference on Electronic Computation, ASCE, Pittsburgh, Pennsylvania (1960), p. 35.

C6 Anaston, George P., "Optimum Design of Transmission Towers," *Conference Papers*, Second Conference on Electronic Computation, ASCE, Pittsburgh, Pennsylvania (1960), p. 69.

C7 Sylvester, R. J., and Foll, R. R., "Computer Solutions to Linear Buckling Problems," *Conference Papers*, Second Conference on Electronic Computation, ASCE, Pittsburgh, Pennsylvania (1960), p. 429.

C8 Galambos, Theodore V., and Ketter, Robert L., "Columns Under Combined Bending and Thrust," *Trans. ASCE*, Vol. 126, Part I (1961), p. 1.

C9 Brandt, G. Donald, "Area Properties from Coordinates," *ASCE J. Struct. Div.*, Vol. 88, No. ST3 (Jun., 1962), p. 197.

C10 Renton, John D., "Stability of Space Frames by Computer Analysis," *ASCE J. Struct. Div.*, Vol. 88, No. ST4 (Aug., 1962), p. 81.

C11 Welch, E., "Computer Solutions for a Beam-Column of Non-Uniform Cross-Section," Symposium of the Use of Computers in Civil Engineering, Lisbon, Portugal (1962), p. 3.1.

C12 Ojalvo, M., and Levi, V., "Columns in Planar Continuous Structures," *ASCE J. Struct. Div.*, Vol. 89, No. ST1 (Feb., 1963), p. 1.

C13 Schmit, Lucien A., and Morrow, William M., "Structural Synthesis with Buckling Constraints," *ASCE J. Struct. Div.*, Vol. 89, No. ST2 (Apr., 1963), p. 107.

C14 Birnstiel, Charles, and Michalos, James, "Ultimate Load of H-Columns under Biaxial Bending," *ASCE J. Struct. Div.*, Vol. 89, No. ST2 (Apr., 1963), p. 161.

C15 Yang, Cheng Y., Hansen, Robert J., and Reinschmidt, Kenneth F., "Dynamic Response of Elastic-Viscous-Plastic Columns," *ASCE J. Eng. Mech. Div.*, Vol. 89, No. ST3 (Jun., 1963), p. 43.

C16 Bailey, Herbert R., "Dynamic Bending of Elastic Columns," *ASCE J. Struct. Div.*, Vol. 89, No. ST4 (Aug., 1963), p. 95.

C17 Sherbourne, A. N., "Numerical Methods in Bending and Buckling of Plates," *ASCE J. Struct. Div.*, Vol. 89, No. ST4 (Aug., 1963), p. 137.

C18 Beck, C. F., and Zar, M., "Steel Column Design for Multistory Rigid Frames," *ASCE J. Struct. Div.*, Vol. 89, No. ST4 (Aug., 1963), p. 537.

C19 Stevens, Leonard K., and Schmidt, Lewis C., "Determination of Elastic Critical Loads," *ASCE J. Struct. Div.*, Vol. 89, No. ST5 (Dec., 1963), p. 137.

C20 Hauck, George F., and Lee, Seng-Lip, "Stability of Elasto-Plastic Wide-Flange Columns," *ASCE J. Struct. Div.*, Vol. 89, No. ST6 (Dec., 1963), p. 297.

C21 *"Bibliography on the Use of Digital Computers in Structural Engineering," Progress Report by Task Group, Subcommittee on Publications, Committee on Electronic Computation, Structural Division, ASCE; *ASCE J. Struct. Div.*, Vol. 89, No. ST6 (Dec., 1963), p. 461.

C22 Johnston, Bruce G., "Buckling Behavior Above the Tangent Modulus Load," *Trans. ASCE*, Vol. 128, Part I (1963), p. 819.

C23 Gere, James M., and Carter, Winfred O., "Critical Buckling Loads for Tapered Columns," *Trans. ASCE*, Vol. 128, Part II (1963), p. 736.

C24 Prawel, Sherwood P., Jr., and Lee, George C., "Biaxial Flexure of Columns by Analog Computers," *ASCE J. Eng. Mech. Div.*, Vol. 90, No. EM1 (Feb., 1964), p. 83.

C25 Lee, S. L., and Hauck, G. F., "Buckling of Steel Columns under Arbitrary End Loads," *ASCE J. Struct. Div.*, Vol. 90, No. ST2 (Apr., 1964), p. 179.

* Contains a comprehensive list of texts on numerical methods.

Author Index

Subject Index