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Multidirectional Membrane Reinforcement

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A method for analyzing local strength of reinforced concrete membranes reinforced with a multidirectional mesh and subjected to known in-plane internal forces is presented. Such structures occur often in practice, and include shearwalls, floor diaphragms, folded plates, and shells. The effect of concrete cracking on the response of the shell under proportionately increasing loads is examined. The governing system of equations is derived, and a numerical, iterative procedure is developed to calculate local response at any load stage through ultimate. Gupta's conclusion[1] that up to first yield crack direction remains constant and then rotates is confirmed. It is shown that at yield all bars are in tension, and ultimate ductile strength can be calculated from a simple relation; to obtain a time-history response, the method presented here has to be used. The procedure is illustrated with two numerical examples.

Keywords: cracking (fracturing); diaphragms (concrete); **ductility**; folded plates; loads (forces); **load transfer; membranes**; reinforced concrete; **reinforcing steels**; shearwalls; **shells (structural forms); strength**; structural design.

Many reinforced concrete structures transfer forces primarily through in-plane action. Examples are shearwalls, floor diaphragms acting as a part of lateral load-resisting systems, folded plates, and thin shells. In the paper, such structures shall be termed membranes.

Analysis of reinforced concrete membranes of given geometry supported in a defined manner and acted upon by known load systems consists of calculation of deflections and internal forces, followed by an evaluation of strength and serviceability of the structure. In accordance with the current ACI 318 Building Code,[2] evaluation of strength consists of (1) determination of ultimate capacity at any point of the structure, both ductile (yielding of reinforcement) and brittle (crushing of concrete); and (2) demonstrating that appropriately reduced ultimate capacity is at least equal to internal forces due to factored loads.

The calculation of internal forces due to loads usually is performed on the basis of an assumption that the material is linearly elastic, homogeneous, and uncracked. In addition, the effect of reinforcement on the stiffness and internal force distribution is assumed small. These assumptions extend throughout the whole range of magnitudes of applied loads, up to failure. Thus, the distribution of internal forces in a membrane under factored loads (i.e., loads at failure) can be calculated by linear extrapolation from the distribution of forces under working loads. Ultimate strength corresponds to concrete being crushed in compression, or to yielding of reinforcement, the latter being the preferred mode of failure.

A more accurate calculation of concrete shells has been undertaken by a number of investigators[3-6] with the aid of a finite element-based computer analysis. In this approach, the shell is modeled as an assembly of layered elements in which reinforcement is represented as a smeared two or more directional mesh bonded to concrete. The layered elements allow for cracking and incorporation of effects such as tension stiffening and aggregate interlock. This type of analysis, however, is time consuming and expensive and is used as a research asset rather than as a practical design tool. The strength design procedure has been incorporated into codes such as the current ACI 318 Building Code.[2] Since it is somewhat inconsistent (the elastic analysis of the shell as a whole is followed by the plastic analysis of local strength), its use is justified, as in the case of all reinforced concrete structures, by invoking the lower bound theorem and the accumulated experience with completed projects.

Until recently, the effect of possible cracking on local strength was neglected (ACI Committee 334,[7] Flugge,[8] Rosenblueth,[9] and Paduart[10]). Later, in explaining the mechanism of load transfer, a number of researchers proposed to incorporate in the analysis the effect of crack formation. Significant analytical and experimental progress in this direction occurred in the last two decades (Peter,[11] Braestrup;[12] Nielsen;[13] Baumann;[14] Brondum-Nielsen;[15] Conley, White, and Gergely;[16] Vecchio and Collins;[17] and Aoyagi[18,19]), and various theories incorporating cracking of concrete and explaining test results were proposed. (The provisions for reinforcement design in the IASS

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Recommendations[20] were based to an extent on Baumann's[14] research.) In the United States, a principal contributor to this effort has been Gupta, who, in a series of publications,[1,21,22] addressed essentially all aspects of the problem.

The research previously noted was concerned essentially with the case of two-way reinforcement, primarily orthogonal. In the following, analysis of strength of a cracked shell element under membrane loads is extended to the case when the element is reinforced with a multidirectional mesh.

LOAD TRANSFER MECHANISM

An element of a membrane is shown in Fig. 1(a). It is acted upon by a known system of calculated internal membrane force system N (kip/in. or kN/mm) consisting of axial forces N_x and N_y and shears N_{xy}. Relative magnitudes of these forces are assumed constant. Reinforcement is placed in several directions, each direction being defined by its angle α_i to the x-axis. The reinforcing bars are assumed to be straight. The internal resultant force system N may be such that the element is entirely in compression. In this case, the element fails in a brittle mode, by crushing of concrete, and analysis of ultimate capacity follows a well-established path. Total resistance consists of the sum of the resistance of concrete and reinforcement, as shown in Fig. 1(a), (b), and (c), with reinforcement transferring only a small part of the total force. Of interest here is the case when the state of stress in the element includes tensions so that cracking occurs. Then the classical plane elasticity or membrane shell techniques no longer apply, and the effect of the presence of a crack must be included in the calculation of the local strength of the shell. Failure still can be brittle or ductile, depending on the state of stress.

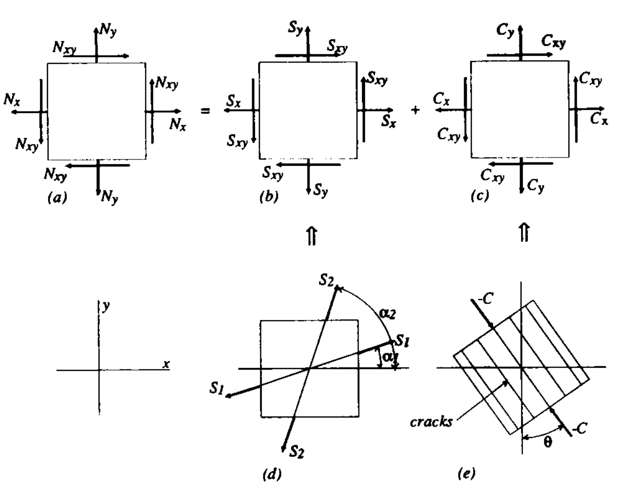

Fig. 1—Force transfer mechanism in a membrane element

As in the case of the uncracked shell, total load transfer mechanism consists of the sum of the resistance of reinforcement and resistance of concrete, as shown in Fig. 1(a), (b), and (c). The following assumptions are made.

The reinforcement is capable of transferring axial forces only. This is shown in Fig. 1(d). Since bars are assumed weak in flexure, their dowel action in the plane along a crack can be neglected (this results in a slight underestimate of the strength of the membrane). The effect of tension stiffening (an increase in the stiffness of bars due to the bond between reinforcement and concrete) is also disregarded (this results in an underestimate of the stiffness of the membrane). Thus, both of these assumptions are somewhat conservative.

Concrete is assumed capable of transferring compression only. Further, it is assumed that no shears can be transferred along a crack plane and aggregate interlock is taken as negligibly small. Therefore, the concrete state of stress C_x, C_y, and C_{xy} must be equivalent to a principal stress system in which there is zero force normal to the crack direction and a compressive force C (kip/in. or kN/mm) acting parallel to the crack, as shown in Fig. 1(e). It is assumed that, at a given point of the membrane, the crack direction may vary depending on the magnitude of internal forces, and that, therefore, any nonlinear effects of crack rotation are negligible.

Note that the assumptions regarding small effect of dowel action, tension stiffening, and aggregate interlock appear justified by the available experimental evidence that shows that the predicted ultimate strength and crack direction are in good agreement with the test values.[4,11,17]

EQUATIONS OF EQUILIBRIUM

In consequence of the assumptions made, a wedge can be isolated from the element, bounded by two element sides, and the plane of the crack inclined at angle θ to the y-axis, as shown in Fig. 2. The force system that maintains the wedge in equilibrium consists of the internal forces N and reinforcing forces S, i.e., S_x, S_y, and S_{xy}, all in the units of force per unit length of the membrane (kip/in. or kN/mm). Then the equations of equilibrium of the wedge are

$$S_x = N_x + (N_{xy} - S_{xy}) \tan\theta \qquad (1a)$$

$$S_y = N_y + (N_{xy} - S_{xy}) \cot\theta \qquad (1b)$$

The forces in the reinforcement S_i are related to the forces S_x, S_y, and S_{xy} as follows (Fig. 3)

$$S_x = \Sigma S_i \cos^2\alpha_i$$

$$S_y = \Sigma S_i \sin^2\alpha_i$$

$$S_{xy} = \Sigma S_i \sin\alpha_i \cos\alpha_i \qquad (2)$$

Eq. (2) follows directly from the equations of transformation of plane stress. The principal compressive force in concrete C (kip/in. or kN/mm) can be expressed in terms of the forces N and S as follows

$$C = (N_x + N_y) - (S_x + S_y) = -\frac{N_{xy} - S_{xy}}{\sin\theta \cos\theta} \quad (3)$$

A negative sign of C, as calculated from Eq. (3), is associated with compression. Eq. (3) is the consequence of the fact that the stress system N is the sum of systems S and C (Fig. 1).

STRESS-STRAIN RELATIONS

Forces in each of the reinforcing bar groups are related to strains ϵ_i through the one-directional Hooke's law as

$$S_i = A_i E_s \epsilon_i \quad (4)$$

where A_i (in.²/in. or mm²/mm) are resultant bar areas and E_s (ksi or MPa) is the modulus of elasticity of steel. Similarly, compression in concrete is related to strain ϵ_{II} through the expression

$$C = t E_c \epsilon_{II} \quad (5)$$

where t (in. or mm) is the thickness of the membrane and E_c (ksi or MPa) is the compressive modulus of elasticity of concrete. Finally, since principal strains in the concrete ϵ_I and ϵ_{II} are oriented, respectively, normal to and parallel to the crack direction, they are related to strains in the reinforcement as follows

$$\epsilon_I = \epsilon_I \cos^2(\theta - \alpha_i) + \epsilon_{II} \sin^2(\theta - \alpha_i) \quad (6)$$

There are as many Eq. (4) and (6) as there are bar directions.

MEMBRANE RESPONSE

Eq. (1) through (6) describe the response of a membrane shell element with known internal forces N. The nature of the response varies with the magnitude of these forces. As internal forces increase proportionately from zero, several phases can be distinguished, as described in the following paragraphs.

Initial phase

Forces N are sufficiently small that none of the reinforcing bars is at yield. Bar forces S_i increase proportionately as forces N increase. In consequence, crack direction θ remains constant throughout the initial phase (see also Gupta and Akbar[1]).

Intermediate phase

At some value of forces N, one of the bars yields. As forces N increase beyond this value, other bars yield in turn so that the force in these bars also reaches yield and

$$S_{ry} = A_r f_y \quad (7)$$

where S_{ry} is the force in bar group r, A_r is the resultant area of bar r, and f_y is the yield stress of steel material. This means that, after the first bar has yielded, the crack angle θ changes direction (rotates) as the intensity of forces N increases. Eq. (2) can be written

$$S_x = \Sigma S_i \cos^2\alpha_i + \Sigma S_{ry} \cos^2\alpha_r$$
$$S_y = \Sigma S_i \sin^2\alpha_i + \Sigma S_{ry} \sin^2\alpha_r$$
$$S_{xy} = \Sigma S_i \sin\alpha_i \cos\alpha_i + \Sigma S_{ry} \sin\alpha_r \cos\alpha_r \quad (2.1)$$

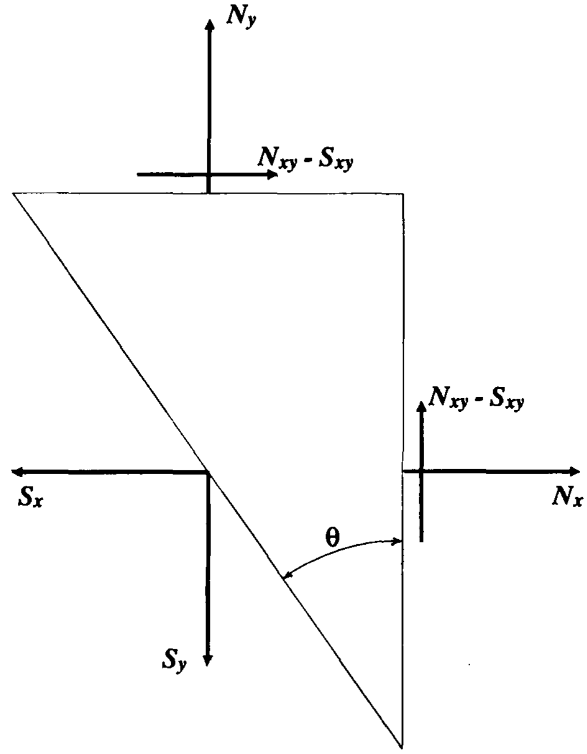

Fig. 2—Equilibrium of a wedge

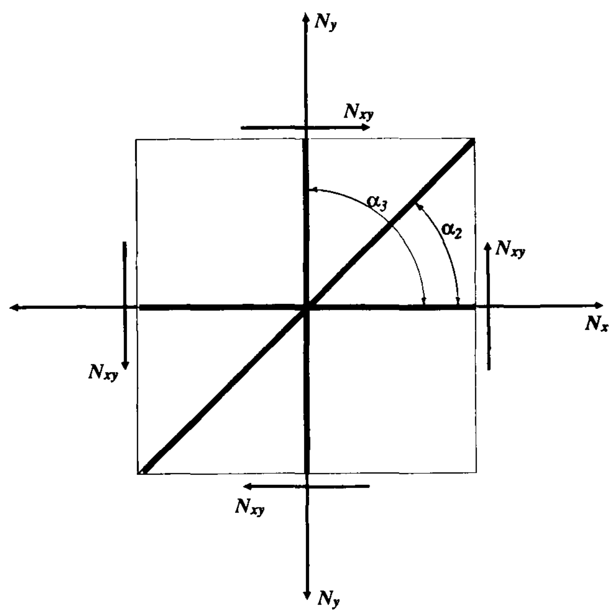

Fig. 3—Example 1—Three-way mesh at 0, 45, and 90 deg

In Eq. (2.1), summations are performed over bar group i (not yielded) or r (yielded).

Ultimate phase

All bars are assumed to have yielded, and the element is said to have reached its ultimate ductile strength. This condition is, perhaps, of primary interest to the shell designer.

At any stage, it is possible also to calculate the principal compression in concrete C from Eq. (3). In principle, the crushing strength of concrete might have been reached before the reinforcing bars yielded. In practice, this condition is to be avoided, as leading to an undesirable mode of failure.

GENERAL SOLUTION

In general, as seen from Eq. (1), the problem is internally statically indeterminate, and it is necessary to solve the complete system of Eq. (1) through (7). A direct solution is not practical since one of the unknowns, the crack angle θ, appears in the equations not directly, but as an argument of several trigonometric functions that multiply some of the other unknowns. Only for the particularly simple case of a two-way orthogonal mesh was the solution obtained in the form of a fourth-degree polynomial in $\tan\theta$ independently by Perdikaris, White, and Gergely,[23] and by Gupta and Akbar.[1]

In the following, the system of Eq. (1) through (7) is reduced to a set of three equations in ϵ_I, ϵ_{II}, and θ, with forces N considered known. First, bar forces S_i are expressed in terms of strains ϵ. Next, using Eq. (2.1), these are substituted into Eq. (1) and (3). After some algebraic manipulation, the result is

$$\epsilon_I E_s \Sigma A_i \cos^2(\theta - \alpha_i)\cos\alpha_i K_i + \epsilon_{II} E_s \Sigma A_i \sin^2(\theta - \alpha_i)\cos\alpha_i K_i = N_x + N_{xy}\tan\theta - \Sigma S_{ry}\cos^2\alpha_r \tag{8a}$$

$$\epsilon_I E_s \Sigma A_i \cos^2(\theta - \alpha_i)\sin\alpha_i K_i + \epsilon_{II} E_s \Sigma A_i \sin^2(\theta - \alpha_i)\sin\alpha_i K_i = N_y\tan\theta + N_{xy} - \Sigma S_{ry}\sin^2\alpha_r \tag{8b}$$

$$\epsilon_I E_s \Sigma A_i \cos^2(\theta - \alpha_i) + \epsilon_{II}[E_s \Sigma A_i \sin^2(\theta - \alpha_i) + E_c t] = N_x + N_y - \Sigma S_{ry} \tag{8c}$$

where

$$K_i = \cos\alpha_i - \sin\alpha_i \tan\theta \tag{8d}$$

For any given value of internal forces N, the iterative solution proceeds as follows. First, a value of the angle θ is assumed, and a solution for strains ϵ_I and ϵ_{II} is obtained from any of the possible pairs of Eq. 8(a), (b), (c) (in choosing the pair of equations to use, possible singularities should be considered). The calculated values of strains are substituted into the remaining equation, and a new value of θ is obtained. The procedure is repeated until the calculated value of θ at the end of any step of iteration differs by not more than some acceptably small amount from the value of θ used at the start of the same step.

Once a solution at any given yield phase has been obtained, the corresponding sign of the principal concrete force C (which must be compressive, or negative) and the signs of bar forces S_i (which are not necessarily all tensions) can be verified.

ULTIMATE DUCTILE STRENGTH

By hypothesis, all bars have reached yield, so that $S_i = S_{ry}$. Then, elimination of θ between Eq. (1) yields the relation

$$\tan\theta = \frac{N_x - S_x}{S_{xy} - N_{xy}} = \frac{S_{xy} - N_{xy}}{N_y - S_y} \tag{9}$$

Since the relative values of forces N are known and since, in view of Eq. (7), forces S_x, S_y, and S_{xy} are known from Eq. (2), the values of forces N corresponding to the ultimate local strength of the membrane can be obtained directly from Eq. (9).

Thus, an important simplification is possible: since all the reinforcement is in tension, the problem of calculation of the ultimate strength becomes statically determinate and is independent of the direction of the crack angle θ. All bar forces at yield are known a priori, and values of forces N at yield can be calculated directly from Eq. (9). After forces N have been found, the direction of the crack at failure can be calculated, again from Eq. (9).

With forces N known, the corresponding value of concrete compression C can be found from Eq. (3). In any specific case, it must be less than the crushing strength of the membrane, which depends on the cylinder strength of the concrete, on the thickness of the membrane, and on the local force pattern N. If necessary, the thickness of concrete, or concrete cylinder strength, or both, can be increased to insure the ductile mode of failure.

REINFORCEMENT DESIGN

In the design of membranes, the directions and the amount of reinforcement are usually assumed on the basis of shell geometry and in the light of past experience. After the analyses have been completed, the adequacy of the assumed reinforcement is verified using the procedure just described.

For a two-way orthogonal mesh, a procedure for estimating areas of reinforcement was proposed by Gupta.[21] Important work on design of reinforcement in membranes was undertaken also by Fialkow.[24,25] By choice, Fialkow omits from his analysis any consideration of crack direction. Consequently, while his calculation of ultimate ductile strength is valid, Fialkow's analysis of concrete compression[26] is based on the implied assumption that crack direction is normal to principal membrane tension; the validity of this assumption is limited to an isotropic two-way mesh, as demonstrated by Nielsen's[13] interpretation of Peter's[11] experiments, and as shown in this paper.

ESTIMATE OF CRACK WIDTH

For the sake of completeness, it is noted that an estimate of the width of cracks can be obtained by making assumptions regarding the spacing of cracks.

Assuming that the strain ϵ_I in the direction perpendicular to the crack is known and that the spacing of cracks is s, the crack width is

$$\text{crack width} = \epsilon_I s \quad (10)$$

Taking s as equal to the spacing of reinforcement, say not more than three times the thickness of the shell, the range of values of s is between 5 and 10 in., and the crack width (in in.) can be estimated at between 5 to 10 times the magnitude of the strain ϵ_I.

NUMERICAL EXAMPLES

Example 1

Three-way mesh at 0, 45, and 90 deg. The element is shown in Fig. 3. Calculated internal forces are N = (0.5, −0.5, 1) kip/in. (0.088, −0.088, 0.175 kN/mm). Material properties are E_s = 30,000 ksi (206,850 MPa), E_c = 3500 ksi (24,732 MPa), and f_y = 40 ksi (276 MPa), with shell thickness t = 3 in. (76.2 mm). Bar areas are A_i = t(0.01, 0.02, 0.01) in.²/in. (mm²/mm), corresponding to the three bar directions (0, 45, 90 deg). The solution is obtained in accordance with the procedure described in this paper.

Initial phase — At the outset of the initial phase, none of the bars have yielded, and the quantities S_{ry} (Eq. 7) are identically zero. Solution of Eq. (8) is

$$\theta = 29.103 \text{ deg}$$

$$\epsilon_I = 5.526E\text{-}4,\ \epsilon_{II} = -1.229E\text{-}4$$

It follows from Eq. (6) that

$$\epsilon_i = (3.928E\text{-}4,\ 5.019E\text{-}4,\ 3.692E\text{-}5)$$

Then, from Eq. (4)

$$S_i = (0.354,\ 0.903,\ 0.033) \text{ kip/in.}$$
$$(0.062,\ 0.158,\ 0.006 \text{ kN/mm})$$

Bar forces at yield reach the values

$$S_{iy} = A_i f_y = (1.2,\ 2.4,\ 1.2) \text{ kip/in.}$$
$$(0.210,\ 0.420,\ 0.210 \text{ kN/mm})$$

Comparing the values of S_i to S_{iy}, it is seen that the second bar (at 45 deg) will be the first to yield. All quantities of interest calculated in the initial phase can be divided by the ratio S_i to S_{iy} of this bar, so that, at first yield, the results are

N = (1.328, −1.328, 2.657) kip/in.
(0.233, −0.233, 0.465 kN/mm)
θ = 29.103 deg
ϵ_I = 1.468E-3, ϵ_{II} = −3.264E-4
ϵ_i = (1.043E-3, 1.333E-3, 9.808E − 4)
S_i = (0.939, 2.4, 0.088) kip/in.
(0.164, 0.420, 0.015 kN/mm)
C = −3.427 kip/in. (−0.600 kN/mm)

where principal compression in concrete C can be obtained either from Eq. (3) or (5); this may be used as a check on the accuracy of numerical calculations.

Intermediate phase — The second bar (at 45 deg) has already yielded so that, at the outset of the intermediate phase, bar forces are

S_i = (0.939, 0.088) kip/in. (0.164, 0.015 kN/mm) (Bars 1 and 3)

S_{ry} = 2.4 kip/in. (0.420 kN/mm) (Bar 2)

The objective is to determine the value of internal forces N and all other quantities of interest at the point when the next bar (either Bar 1 or 3) reaches yield. To do this, an incremental value $N = N + \delta N$ is used (together with the constant value of S_{ry}) in solving Eq. (8). It is found that the next bar to yield is Bar 1 (at 0 deg), and the results are

N = (1.429, −1.429, 2.857) kip/in.
(0.250, −0.250, 0.500 kN/mm)
θ = 30.377 deg
ϵ_I = 1.916 − 3, ϵ_{II} = −3.618E-4
ϵ_i = (1.333E-3, 2.207E-4) (Bars 1 and 3)
S_i = (1.2, 0.199) kip/in. (0.210, 0.035 kN/mm)
(Bars 1 and 3)
C = 3.799 kip/in. (−0.665 kN/mm)

It is seen that only Bar 3 (at 90 deg) has not yet reached yield.

Ultimate strength — All bars are assumed to have reached yield, and it is seen that, in this particular example, all are in tension. Accordingly, bar forces at ultimate strength are

S_i = S_{iy} = (1.2, 2.4, 1.2) kip/in.
(0.210, 0.420, 0.210 kN/mm)

Using these values in Eq. (2) and (9) yields the following

N = (1.526, −1.526, 3.052) kip/in.
(0.267, −0.267, 0.534 kN/mm)

θ = 25.257 deg

It follows that

ϵ_I = 1.254E-3, ϵ_{II} = −4.571E-4

ϵ_i = 1.333E − 3 for all bars

C = −4.8 kip/in. (−0.841 kN/mm)

The value of C represents principal compression in concrete, which occurs at the time when all bars have just yielded. The results may be summarized as follows:

Bars yield in the sequence, Bar 2 (at 45 deg), Bar 1 (at 0 deg), and Bar 3 (at 90 deg). All bars yield in tension. Therefore, if only the response at ultimate yield were of interest, the problem could have been solved directly from Eq. (9), as previously noted.

Ultimate ductile strength is N = (1.526, −1.526, 3.052) kip/in. (0.267, −0.267, 0.534 kN/mm), and the associated compression in concrete is C = −4.8 kip/in. (−0.841 kN/mm).

The direction of crack angle θ up to initial yield is 29.10 deg; following initial yield the crack rotates, increasing to 30.38 deg at second yield, and then decreasing to 25.26 deg at final yield. It is seen that θ does not coincide with the direction of the normal to principal tension, which is 31.72 deg. Concrete strain perpendicular to the crack direction at ultimate yield is approximately 0.00125, less than the value at first yield when it is 0.00147. Thus, at working loads, crack width can be expected on the order of 0.01 in. (0.254 mm).

Example 2

Three-way isotropic mesh at 10, 70, and 130 deg. It is seen that the element, shown in Fig. 4, is reinforced isotropically. Calculated internal forces are N = (0.5, −0.5, 1) kip/in. (0.088, −0.088, 0.175 kN/mm). Material properties are E_s = 30,000 ksi (206,850 MPa), E_c = 3500 ksi (24,132 MPa), and f_y = 40 ksi (276 MPa), with membrane thickness t = 3 in. (76.2 mm). Bar areas are A_i = t(0.01, 0.01, 0.01) in.²/in. (mm²/mm), corresponding to the three bar directions (10, 70, 130 deg). The solution follows the procedure described in detail in Example 1; the results follow.

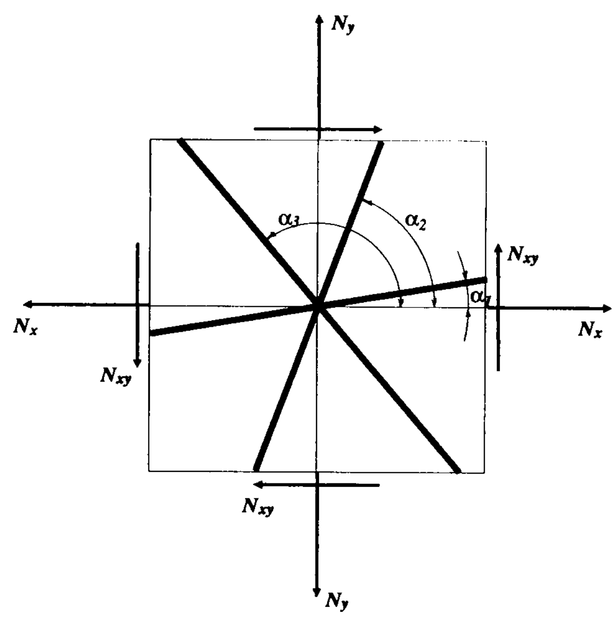

Fig. 4—Example 2—Three-way isotropic mesh at 10, 70, and 130 deg

First yield — Bar 2 (at 70 deg) yields first, and solution of Eq. (8) is

θ = 31.72 deg (normal to principal tension)
ϵ_I = $1.573E$-3, ϵ_{II} = $-1.792E$-4
ϵ_i = (9.006E-3, 1.333E-3, −1.429E-4)
S_i = (0.811, 1.2, −0.129) kip/in.
(0.142, 0.210, −0.023 kN/mm)
C = 1.882 kip/in. (− 0.330 kN/mm)
N = (0.685, −0.685, 1.371) kip/in.
(0.120, −0.120, 0.240 kN/mm)

Bar 3 (at 130 deg) being in compression.

Second yield — Bar 1 yields at the end of the intermediate phase (at 10 deg) and the results are

θ = 27.233 deg (no longer normal to principal tension)
ϵ_I = 2.664E-3, ϵ_{II} = −2.217E-4
ϵ_i = (1.33333E-3, −8.074E-5) (Bars 1 and 3)
S_i = (1.2, −0.0727) kip/in.
(0.210, −0.013 kN/mm) (Bars 1 and 3)
C = −2.327 kip/in. (−0.407 kN/mm)
N = (0.787, −0.787, 1.574) kip/in.
(0.138, −0.138, 0.276 kN/mm)

Again, Bar 3 (at 130 deg) is in compression.

Ultimate strength — At the end of the intermediate phase, Bar 3 (the only bar that has not yet yielded) is in compression, and a further increase in the intensity of forces N results in its gradually changing sign to tension, until it finally reaches yield. Eq. (9) and (3) are used, and the results are

S_i = (1.2, 1.2, 1.2) kip/in.
(0.210, 0.210, 0.210 kN/mm)
θ = 28.22 deg
C = 2.4 kip/in. (0.420 kN/mm)
N = (0.805, −0.805, 1.610) kip/in.
(0.141, −0.141, 0.282 kN/mm)

Note that, even though the mesh of this example is isotropic, crack direction at ultimate yield is not normal to the membrane principal tensile force.

SUMMARY AND CONCLUSIONS

A method for the calculation of the local response of a reinforced concrete membrane subjected to internal in-plane forces obtained through a separate elastic analysis is presented. The equations of the method incorporate the effect of cracking of concrete. Reinforcement may be in the form of a multidirectional mesh with different areas and yield stresses in different directions. The equations represent an extension of previous work by others, primarily by Gupta,[22] applicable to orthogonal two-way reinforcement.

The system of equations is internally statically indeterminate. In addition, it is transcendental, in that it contains trigonometric functions of the crack angle θ. An iterative method is presented for its solution, which permits calculation of the crack angle, strains in concrete and in reinforcement, bar forces, and also the principal force in concrete at any level of internal forces N. The procedure is useful in establishing local ultimate ductile strength of the membrane, and in establishing the nature of the failure mode—brittle or ductile. The designer can readily insure that the failure mode is ductile by increasing thickness of the shell, by increasing concrete cylinder strength, or both.

At ultimate ductile failure all bars are in tension, the calculation of ultimate ductile strength and concrete compression becomes independent of the direction of the crack angle θ, and the solution can be obtained directly from Eq. (9) and (3). For the limiting case of a two-way orthogonal mesh, these equations reduce to those previously derived by Gupta,[21] and predict the same ductile ultimate strength as that obtained by Gupta,[1] by other investigators, and experimentally.[11,17] The procedure is illustrated with the aid of two examples involving membranes reinforced with three-way mesh.

Crack direction at ultimate yield is, in general, not perpendicular to the membrane principal tensile force. This is the case only if the reinforcement consists of a two-way orthogonal and isotropic mesh.

Finally, since the crack direction rotates after first yield has been reached, the width of the crack may be greater at first yield than at ultimate yield, as shown in Example 1.

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