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LATERAL STABILITY OF REINFORCED CONCRETE BEAMS

By WILLIAM HANSELL and GEORGE WINTER

Some concrete design specifications, including the ACI Code, in various ways limit the distance between lateral supports of beams, presumably to safeguard against lateral buckling. The present investigation is intended to furnish some factual information on which to base such provisions. Ten tests on deep narrow beams have been carried out with unbraced lengths ranging from 28.8 to 86.4 times the beam width. No reduction in strength was observed over this range, showing the absence of lateral buckling. A tentative theory of lateral instability of reinforced concrete beams, including the effects of inelasticity and cracking, is given. It agrees with the tests in showing that present Code provisions are too restrictive, particularly for ordinary steel strengths. Theory indicates that closer lateral supports are required for high strength reinforced beams than for ordinary strength reinforcement.

■ DESIGN SPECIFICATIONS FOR STEEL STRUCTURES contain provisions for safeguarding unbraced beams against lateral buckling. These provisions are usually expressed in terms of slenderness ratios such as L/b or Ld/bt. Lateral buckling of beams, as shown in Fig. 1, involves both lateral bending and torsion. Since torsional rigidity is proportional to the cube of the thickness of a member, it is evident that this rigidity is relatively low for steel members with their comparatively small thicknesses of webs and flanges. It is for this reason that such members, if sufficiently slender, are liable to buckle laterally, and that appropriate measures must be taken in design to prevent such buckling.

In contrast, rectangular beams, such as occur in reinforced concrete structures, possess high torsional rigidity. This would indicate that the danger of lateral buckling is considerably less for such beams than for structural steel shapes. Nevertheless some concrete design codes contain provisions which restrict L/b-ratios, presumably for the same purpose of preventing lateral buckling.

It is the aim of this paper to investigate the lateral buckling of rectangular reinforced concrete beams to furnish some basis for the development of appropriate design provisions. Since the complex nature of reinforced concrete (inhomogeneity, partial cracking, limited elasticity, etc.) makes an entirely

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analytical treatment impossible, resort was had to tests of ten beams, with L/b ratios ranging from 28.8 to 86.4. It was the purpose of these tests to determine the manner in which such large L/b ratios affect flexural capacity. The paper reports the results of these tests, develops an approximate and somewhat tentative buckling formula for unbraced rectangular reinforced concrete beams, and draws conclusions in regard to the design of such members.

Notation

b = width of a rectangular beam

c = distance from the neutral axis of a concrete beam to the extreme compression fiber

d = depth of a concrete beam from center of tension reinforcement to extreme compression fiber

E = modulus of elasticity

E_c = initial tangent modulus of concrete

E_s = modulus of elasticity for steel

E_{sec} = f_c/ϵ_c = secant modulus of concrete pertaining to flexural stress-strain properties and extreme compression fiber stress and strain

E_t = tangent modulus

$\overline{E}$ = reduced modulus, used in inelastic buckling equations

f_c = concrete stress

f_c' = ultimate compressive strength of concrete at 28 days

G = modulus of elasticity in shear

$\overline{G}$ = reduced modulus, used in inelastic buckling equations

h = depth of a homogeneous beam

I_y = moment of inertia of a beam about the minor (vertical) axis of symmetry

K_t = torsional constant of a beam

k = c/d = ratio indicating position of neutral axis

K_3 = ratio of ultimate flexural stress to ultimate compressive stress of concrete

L = distance between points of lateral support

M = bending moment

M_{cr} = critical buckling moment

M_{calc} = predicted ultimate moment

M_{test} = maximum moment observed in beam test

$w(y)$ = width of a beam at a distance y from the neutral axis

y = vertical distance from neutral axis

β, γ = parameters related to the shape of the flexural stress-strain relation of concrete (see Fig. 9)

ϵ = strain

ϵ_c = extreme compression fiber strain in concrete

ϵ_h = extreme fiber strain in homogeneous beam

ϵ_s = strain in tension steel

ϵ_{sy} = strain at yield point of steel

ϵ_o = concrete strain at ultimate stress

ϵ_u = ultimate concrete strain

μ = Poisson's ratio corresponding to elastic values of E and G

σ = stress in homogeneous beam

σ_h = extreme fiber stress in homogeneous beam

BACKGROUND

Little information is available in English literature concerning the lateral stability of reinforced concrete beams. No published reports of lateral instability failures are known to the authors. One experimental study involving tests of three rectangular concrete beams with a constant L/b ratio of 36 is reported.[1] This reference does not indicate whether the beams were restrained from lateral or rotational movements at the load point. Since such restraint has a considerable effect on the stability of slender beams, there is some question concerning conclusions drawn from these test results. Reference 2 gives the results of an analytical study of the stability problem for concrete beams. The author considers both the lateral and torsional rigidities of a beam and concludes that stability regulations which consider only the L/b ratio are basically unsound. However, this brief analysis assumes "uncracked," elastic, homogeneous beams with flexural strength defined by the straight-line theory. The question of whether or not this analysis is conservative, particularly at loads approaching ultimate, is a matter of conjecture since a concrete beam is cracked, inelastic, and nonhomogeneous.

The 1956 edition of the ACI Building Code specifies in Section 704 that: "The clear distance between lateral supports of a beam shall not exceed 32 times the least width of compression flange." It is presumed that this provision was intended to safeguard against lateral instability failures. There is reason to question whether this regulation is appropriate for its presumed purpose, since it is supported by little or no valid factual evidence. It is obvious from experience that the $L/b = 32$ limitation is safe, at least for construction materials and methods of long standing. It is not obvious as to whether this limitation is a necessary and economical solution to the lateral stability problem in reinforced concrete.

A rectangular, homogeneous, elastic beam in pure bending will buckle laterally at the bending moment[3]

$$M_{cr} = \frac{\pi}{L}\sqrt{E\,I_y\,G\,K_t} \quad \ldots\ldots (1)$$

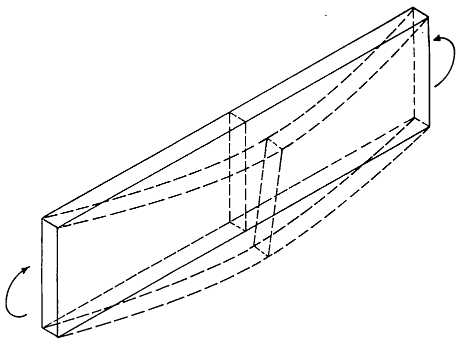


Fig. 1—Lateral buckling

TABLE 1—TEST BEAM DATA

Beam mark	Span, ft	L/b ratio	Stirrup spacing,* in.	Cylinder strength,† f_c', psi
B6	6	28.8	5½	4310
B9	9	43.2	5½	4310
B12	12	57.6	7	4350
B15	15	72.0	7	4215
B18	18	86.4	7	4260

*Uniform along entire span.
†Average of six 6 x 12-in. cylinders tested at 28 to 32 days.

For deep and narrow beams, with sufficient accuracy, $K_t = b^3h/3$. Using this value, the maximum fiber stress in such a beam at buckling is easily computed to be

$$\sigma_{cr} = \frac{\pi \sqrt{E\,G}}{(L/b)} \sqrt{\frac{I_y}{I_x}} \qquad \text{(1a)}$$

Two things are apparent from this equation: The stress at which buckling occurs decreases with increasing slenderness L/b, and this is the reason why design codes have often been formulated in these terms. It is also seen, however, that beams are the more unstable the smaller I_y/I_x, i.e., the narrower and deeper they are. This equally important fact is not reflected in formulas which merely contain L/b.

These simple equations cannot be applied directly to reinforced concrete beams in view of the complex character of the material. For this reason the test program reported below was carried out.

SPECIMENS AND TEST PROCEDURE

Specimens

The experimental program involved the testing to destruction of ten rectangular reinforced concrete beams of identical cross section, main reinforcement, and concrete mix. The beams were loaded at the quarter points on five simple spans of 6, 9, 12, 15, and 18 ft. Two companion beams were tested on each span. Beam specimens are designated by the letter *B* and two numbers. The first number is the simple span in feet and the second denotes one of two companion beams. All beams were 13 in. deep, 2½ in. wide, and used the same tensile reinforcement, one ¾-in. diameter deformed bar 11¼ in. from the extreme compression fiber. These dimensions provided a steel ratio of 1.56 percent. The beam section was purposely made unusually deep and narrow, i.e., more conducive to lateral instability [see Eq. (1a)], than beams normally encountered in practice. This was done to afford every possibility for lateral instability failures. Data concerning the beam specimens and the testing arrangement are included in Table 1 and Fig. 2.

The L/b ratios of the test specimens varied from 28.8 for the 6-ft span to 86.4 for the 18-ft span. All beams except the 6-ft spans violated the requirements of the 1956 ACI Building Code by as much as 270 percent. The 6-ft spans were included in the program to experimentally verify the ultimate flexural capacity without lateral buckling.

The beams were designed to fail in the flexural tension mode if lateral instability did not precede flexural failure. Quarter-point loading was used to create constant moment over an

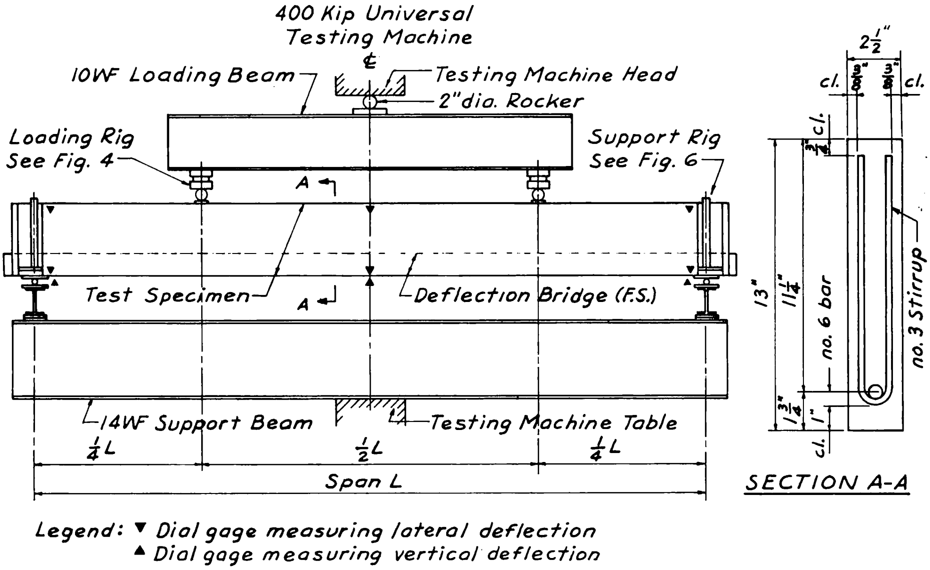


Fig. 2—Testing arrangement layout

appreciable portion of the span. As indicated in Reference 4, loading systems which produce near-maximum moments over sizeable portions of the span tend to increase the danger of lateral instability.* To minimize the possibility of shear failures, liberal web reinforcement was used along the entire span of all beams as indicated in Fig. 2 and Table 1.

Materials

Type I portland cement and intermediate grade deformed steel bars were used. Coarse aggregate was a graded gravel of ⅝-in. maximum size. Six 6 x 12-in. cylinders were cast and cured with each pair of test beams. Average cylinder test results are reported in Table 1. The average cylinder strength was 4290 psi.

The average yield stress and strain of the sharp yielding tensile reinforcement was 43,800 psi and 1.53 mills, respectively. These figures represent the average of four tension tests using standard 0.5-in. diameter specimens machined from the #6 bars.

Testing equipment

The beams were tested in a 400,000-lb capacity universal testing machine which was modified for the beam tests by bolting a steel "support" beam to the table of the machine and by suspending a steel "load" beam from the compression head of the machine. A diagram of the testing arrangement is shown in Fig. 2 and Fig. 3 shows a 12-ft test beam in the machine.

It was essential that the apparatus used to transmit load to the test beam (hereafter termed the loading rig) should provide a minimum of lateral and torsional restraint to the beam. Otherwise the test beam would have been restrained from the lateral and rotational movements associated with lateral instability. As indicated in Reference 5, mere friction may suffice to provide effective lateral bracing and prevent lateral buckling. Fig. 4 shows an assembly drawing of the loaded rig and Fig. 5 shows the rig in place during a beam test. The four rollers in the roller assembly are hollow automotive wrist pins. Keeper plates and roller pins maintain the rollers in a parallel configuration. The load ball is a 1½-in. diameter ball

*Although Reference 4 is primarily concerned with the lateral stability of steel I-beams, the comparison of critical buckling moments for various loading systems when the loads are applied at the centroid is at least qualitatively valid for reinforced concrete beams. See Eq. (48) of Reference 4.

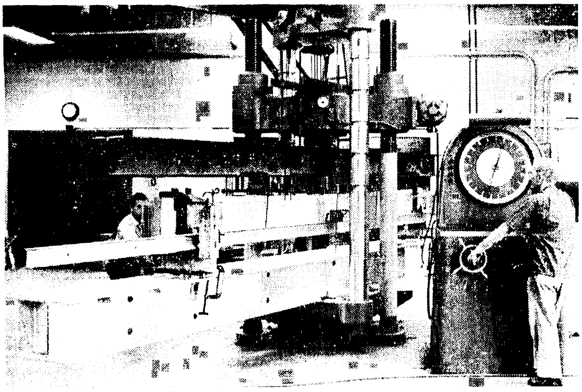
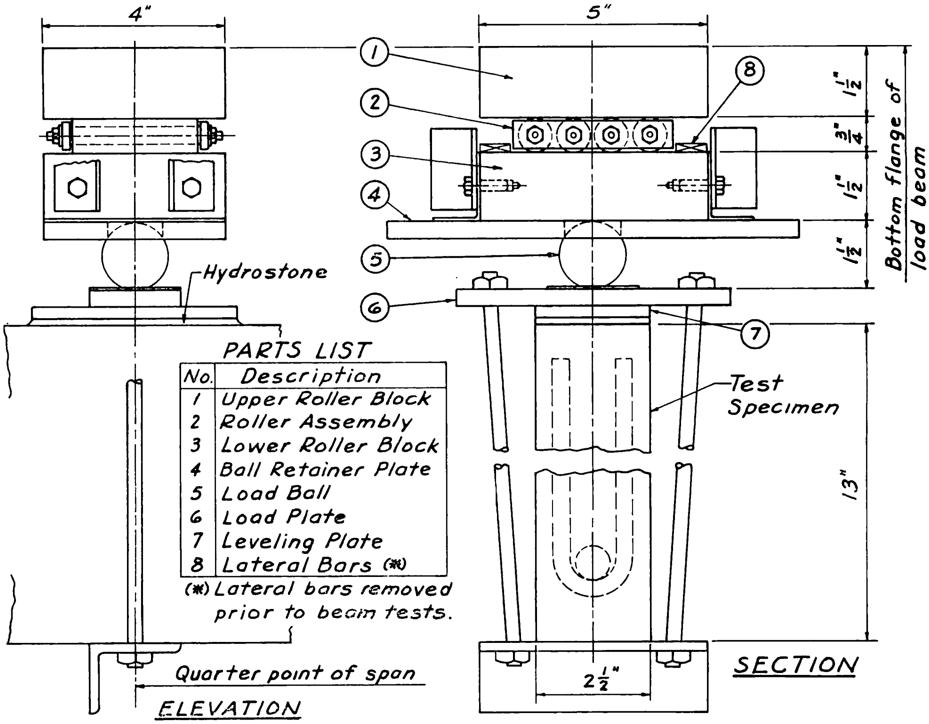


Fig. 3—Beam under test

bearing. All bearing surfaces of the loaded rig were hardened and ground smooth to reduce surface penetration and friction. All rolling surfaces were cleaned and lightly oiled before each test.

During the setup and centering phases of a beam test, the lateral bars were inserted in the positions shown in Fig. 4 (to impart stability), but were removed during the remainder of a test. The only lateral restraint offered by the loading rig (for lateral beam displacements of 1¼ in. or less at the load points) is developed by rolling friction at the roller assembly. Torsional restraint is limited to that developed by point contact between the load ball and



No.	Description
1	Upper Roller Block
2	Roller Assembly
3	Lower Roller Block
4	Ball Retainer Plate
5	Load Ball
6	Load Plate
7	Leveling Plate
8	Lateral Bars (*)

(*) Lateral bars removed prior to beam tests.

Fig. 4—Loading rig

the load plate. The line of action of the load passes through the point of contact between load ball and load plate and moves laterally with the beam.

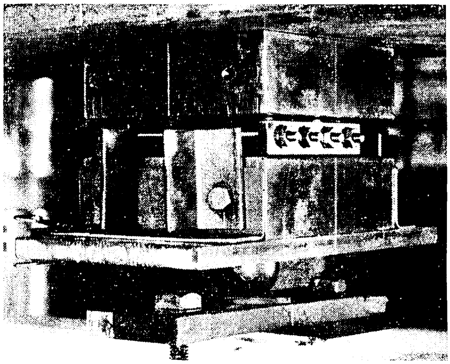


Fig. 5—Loading rig in place

Auxiliary tests* indicated that the maximum lateral restraining force developed by the loading rig did not exceed 0.1 percent of the vertical loads. Except for these negligible restraining forces the test beams were laterally and torsionally unrestrained over their entire simple span.

The apparatus used to support a test beam (hereafter termed the support rig) permitted the beam to rotate about the principal axis of its cross section but prevented rotation about the longitudinal beam axis. This was accomplished as illustrated in Fig. 6 which also shows the idealized support conditions approximated by this rig. Fig. 3 shows the support rig in place during a beam test. Note the vertical rollers extending above the top of the test beam. Vertical and lateral deflections were measured using nine dial gages as indicated in Fig. 2. All dial gages (except the two measuring vertical movement at the supports) were clamped to a steel deflection bridge which was supported on the horizontal rollers at the ends of the simple span. Fig. 3 shows the deflection bridge with dial gages in place.

*These auxiliary tests are described in detail in Reference 6.

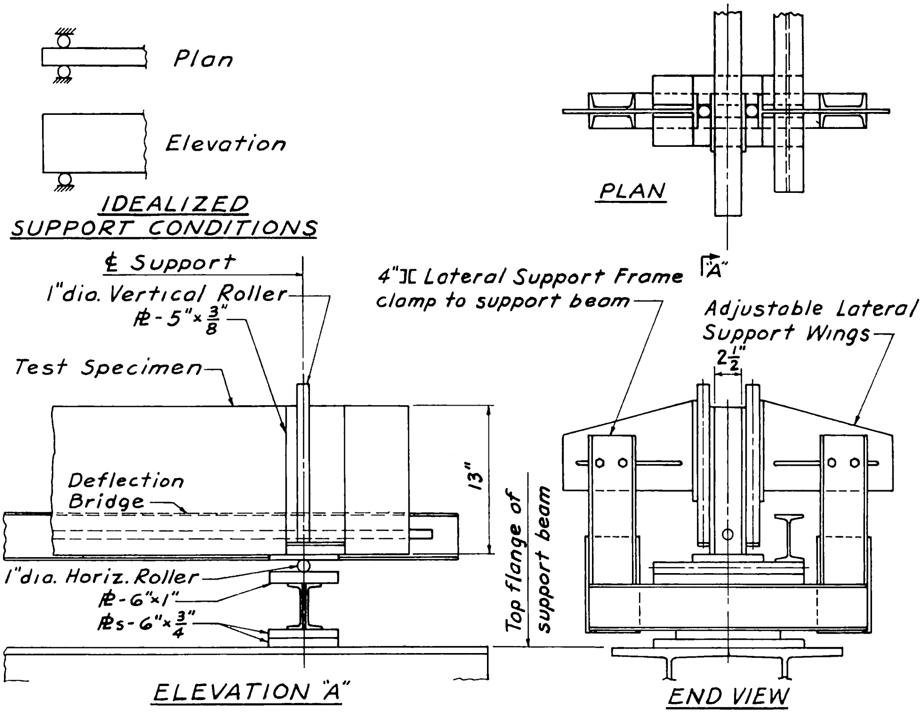


Fig. 6—Support rig

Testing procedure

The beams were measured with a steel scale and calipers and were found to be within ± 1/8 in. of nominal dimensions. A plumb bob, level, and measuring tape were used to accurately locate the beams in the testing machine.

Centering under load was accomplished by minimizing lateral deflection at midspan, under loads not exceeding one-third of the predicted ultimate load.

When the loading rig was centered, loads were applied in increments of about 10 percent of the predicted ultimate load up to 90 percent of this load. Thereafter, loads were applied in smaller increments at a reduced loading rate until it was evident from the rapid increase of vertical deflections that the flexural steel had yielded. The time between the first and last increment of load averaged about 1½ hr.

TEST RESULTS

Table 2 summarizes the results of the beam tests. In this table, M_{test} is the moment at midspan of the test beams (including dead load moment) corresponding to the maximum observed test load. The values of M_{calc} are the ultimate moments for simple flexural tension failure computed by using Eq. (A1) in the Appendix of the 1956 ACI Building Code. A flexural tension failure is initiated by yielding of the tensile steel. The marked deviation from approximately linear load-deflection behavior which occurs when yielding begins in an underreinforced beam defines the yield point of the beam. The yield point deflections of the test beams are entered in Table 2. This table also gives the initial lateral deflections measured prior to loading.

Inspection of the ratios M_{test}/M_{calc} in Table 2 indicates that slenderness or laterally unsupported length had no effect on the flexural capacity of the test beams. All ten beams in this experimental program failed in the flexural tension mode at practically identical moments, in close agreement with the predictions of ultimate strength theory. They were not weakened by lateral buckling. This was true in spite of the fact that the largest L/b ratio was

TABLE 2—RESULTS OF BEAM TESTS

Beam	B6	B9	B12	B15	B18	Suffix*
L/b ratio	28.8	43.2	57.6	72.0	86.4	1, 2
Observed ultimate moment, M_{test}, kip-in.	216 199	201 205	193 199	192 198	190 196	1 2
Calculated ultimate moment, M_{calc}, kip-in.	196.7	196.7	197.0	195.9	196.2	1, 2
M_{test}/M_{calc}	1.10 1.01	1.02 1.04	0.98 1.01	0.98 1.01	0.97 1.00	1 2
Midspan deflections at yield point, 0.001 in.						
Vertical	192 188	330 330	460 495	825 1005	1015 1080	1 2
Lateral-Top	53 78	33 18	43 515	1260 97	72 500	1 2
Lateral-Bottom	66 82	56 25	14 620	1090 228	150 430	1 2
Initial lateral deflection, 0.01 in.	4 6	8 6	8 11	25 12	13 17	1 2

*Suffix indicates one of two companion beams.

Fig. 7—Beam B18-2 after test

86.6 or three times the smallest, and 2.7 times the largest L/b ratio permitted by Section 704 of the 1956 ACI Building Code. The average value of the ratio M_{test}/M_{calc} was 1.01 indicating an excellent agreement between predicted and observed ultimate flexural capacities. The usual small variations in observed ultimate moments show no significant correlation with span lengths or lateral deflections. These are the most important results of the testing program.

One factor which may have had a minor influence on the results of the beam tests is the rather heavy web reinforcement used in the test specimens. The web reinforcement ratios were 1.60 percent for the 6- and 9-ft beams and 1.26 percent for the 12-, 15-, and 18-ft beams. This reinforcement undoubtedly contributed to the torsional strength of the beams and may have prevented potential torsional failures in those beams which developed large lateral deflections.

Fig. 7 shows Beam B18-2 after completion of its test. The crack pattern for this beam is typical of that for all of the test specimens. Several diagonal tension cracks, due to vertical shearing stresses, developed near the supports of the beams at loads approaching the yield point. Otherwise the crack patterns were characteristic of flexural failures and indicated no evidence of potential torsional failures.

In addition to strength the designer is frequently concerned with deflection characteristics. Fig. 8 shows the moment-deflection behavior of Beam B18-2. The shapes of these deflection curves are typical of the test results regardless of the L/b ratio. It is seen that slight lateral deflection occurred during the first increment of loading and continued to increase, without loss of stability, throughout the test. The presence of a twisting deformation is indicated by the difference between the lateral deflections at the top and at the bottom of the beam.

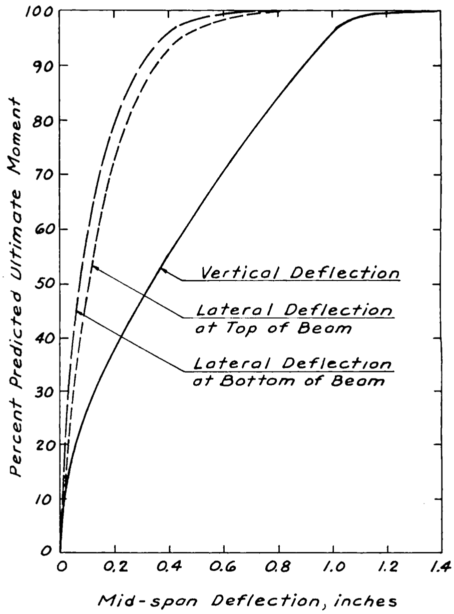


Fig. 8—Deflection behavior of Beam B18-2

This lateral deflection behavior may be ascribed to initial imperfections and to small eccentricities of the applied loads. It had no significant effect on flexural capacity and should not be interpreted as evidence of instability prior to flexural failure.

Table 2 indicates a considerable range of lateral deflections at the yield point. There appears to be little more than a qualitative correlation between lateral deflection under load, and the L/b ratio or initial lateral deflection. It may be noted that seven of the beams exhibited larger lateral deflections at the bottom than at the top and that lateral deflections exceeded vertical deflections in two tests.

INELASTIC BUCKLING ANALYSIS

The test results above constitute the main contribution of this paper. In an attempt to generalize this somewhat limited evidence the following approximate buckling analysis is presented.

Many factors affect the lateral stability of a reinforced concrete beam. One is the contribution of longitudinal steel reinforcement to the lateral and torsional rigidities of the beam. Others concern the effects of the inelastic stress-strain properties of concrete, cracking, and shear reinforcement. The effects of creep are a factor in stability studies of concrete members subjected to sustained loading. Finally, the location of loads and the conditions of end support have a considerable influence on the lateral stability of beams. It is evident that a rigorous analysis of this problem, considering all of the above factors, would be cumbersome if not impossible. It is the purpose of this section to propose conservative simplifying assumptions leading to an approximate expression for the critical buckling moment of a rectangular reinforced concrete beam subjected to uniform bending.

The buckling moment for an elastic, homogeneous beam was given in Eq. (1). This relation is analagous to the Euler formula for elastic column buckling. Obviously, Eq. (1) cannot be applied directly to a concrete beam since such a beam is neither homogeneous nor elastic. It is, therefore, necessary to establish the conversion from an elastic to an inelastic relation. This is done by replacing E and G by appropriate values in the inelastic range in analogy with the Engesser-Shanley tangent modulus theory for inelastic column buckling. This theory states that the Euler column formula predicts bifurcation of equilibrium in the inelastic range if the tangent modulus is used in place of the elastic modulus. (For all practical purposes, the load at which bifurcation of equilibrium occurs is identical with the load causing buckling.)

Correspondingly, in the inelastic range Eq. (1) for the buckling moment may be written in the form

$$M_{cr} = \frac{\overline{E}}{\sqrt{2\,(1+\mu)}} \times \frac{\pi}{L}\sqrt{I_y\,K_t} \qquad \text{(2)}$$

in which μ is Poisson's ratio and $\bar{E}$ is a reduced modulus related to (a) the shape of the cross section, (b) the stress-strain properties of the material, and (c) the extreme (compression) fiber strain caused by M_{cr}. The transition from Eq. (1) to Eq. (2) and the assumptions involved are outlined in the appendix, where it is shown that for a rectangular beam $\bar{E}$ equals the secant modulus, E_{sec}, corresponding to the extreme (compression) fiber strain. It is pertinent to note that M_{cr} and E_{sec} are both functions of the extreme compression fiber strain.

It remains to evaluate the geometric properties I_y and K_t in Eq. (2). Assume temporarily that the beam has an effective depth h. Then considering a rectangular section with $b < h$,* I_y and K_t may be written

$$I_y = \frac{1}{12} b^3 h \qquad \text{(3a)}$$

$$K_t = \tfrac{1}{3} b^3 h \, (3\lambda) \qquad b < h \qquad \text{(3b)}$$

where λ is a function of the ratio h/b. For practical purposes one may take

$$\sqrt{3\lambda} = 1 - 0.35 \frac{b}{h} \qquad \text{(4)}$$

and so obtain the result

$$M_{cr} = \frac{\pi}{6\sqrt{2(1+\mu)}} \times E_{sec} \times \frac{b^3 h}{L} \left(1 - 0.35 \frac{b}{h}\right) \qquad \text{(5)}$$

One may conservatively compute the lateral and torsional rigidities of a cracked concrete beam by considering only the compression area of the concrete.

This assumption neglects any contribution of the longitudinal and shear reinforcement, and of the concrete below the neutral axis. The longitudinal reinforcement is implicitly considered only in that it affects the location of that axis. Using this assumption, I_y and K_t are given by Eq. (3a) and (3b) if h is taken as the depth c of the compression area. Then with the substitution $h = c = kd$, Eq. (5) may be written

$$M_{cr} = \frac{\pi}{6\sqrt{2(1+\mu)}} \times \frac{E_{sec}}{L} b^3 d \left(k - 0.35 \frac{b}{d}\right) \qquad k > \frac{b}{d}$$

and with $\mu = 0.16$ this becomes[7]

$$M_{cr} = 0.34 \frac{E_{sec}}{L} b^3 d \left(k - 0.35 \frac{b}{d}\right) \qquad k > \frac{b}{d} \qquad \text{(6)}$$

*In later developments pertaining to reinforced concrete beams it will be seen that h may be less than b (kd less than b), in which case b and h should be reversed in Eq. (3b) and (4).

It is proposed that Eq. (6) be used as an estimate for the critical buckling moment of an initially straight rectangular reinforced concrete beam subjected to uniform bending over a span L.

It should be noted that Eq. (6) is a conservative approximation in that it neglects the increase in the rigidities of a beam due to reinforcement and the uncracked concrete below the neutral axis of the beam. In regard to torsion, Cowan concludes that longitudinal reinforcement increases the torsional stiffness of reinforced concrete beams, but that "this increase is generally too small to be considered in practice" (Reference 8, p. 18). In regard to lateral bending, unless the longitudinal reinforcement is purposely placed to resist lateral bending (i.e., distributed along the sides of the beams), its contribution to the lateral rigidity of the beam is limited, particularly in slender beams which have a small width to depth ratio.

Ernst has investigated the torsional properties of reinforced concrete beams with variable quantities of shear reinforcement.[9] His experimental results indicate that for small torsional moments, the torsional behavior of a reinforced concrete beam is not significantly affected by shear reinforcement. This statement applies to torsional moments less than those causing diagonal tension cracks. One is therefore justified in neglecting reinforcing steel in the rigidity calculations leading to Eq. (6), at least as an approximation.

It is pertinent to distinguish between buckling of an "ideal" (initially straight, concentrically loaded, free from imperfections) beam and instability of a "real" (initially bowed, eccentrically loaded, imperfect) beam. Eq. (6) gives critical buckling moments for an ideal beam subjected to pure bending. A real beam becomes unstable at loads somewhat less than an identically loaded ideal beam. The application of loads above the shear center of a beam reduces the critical buckling load while loads applied below the shear center have an opposite effect. A rigorous analysis of the influence of initial bow, eccentric loading, and loads applied above the shear center on the stability of reinforced concrete beams is beyond the scope of this discussion.

Creep would appear to have a twofold effect on the stability of reinforced concrete beams subjected to sustained loads. Lateral deflections increase due to creep. In addition the secant modulus E_{sec} decreases as creep strains increase. Each of these effects would impair the stability of the beam. In considering the stability of reinforced concrete arches subjected to sustained loads, it has been suggested that creep effects may be approximated by using a reduced value of the concrete modulus.[10] It seems reasonable to use a similar procedure for concrete beams without compression reinforcement. Washa and Fluck have reported significantly reduced creep deflections for beams with compression reinforcement.[11] It is therefore reasonable to predict smaller reductions in buckling moments due to creep for compression reinforced beams than for similar beams without compression steel.

Finally, Eq. (6) applies specifically to beams subjected to a constant bending moment over the full span L and simply supported at the ends of the span. A discussion of other loading and support conditions is omitted here for brevity. However, it may be stated that the loading and support conditions on which Eq. (6) is based are more severe from the standpoint of stability than most conditions encountered in practice.[4] This statement is true for both elastic and inelastic stress-strain conditions. In fact, if one considers the inverse variations of flexural rigidity with bending moments in the inelastic range, it is evident that a beam is more stable than calculations based on the minimum rigidity at a point of maximum moment would indicate.

It may be concluded that, under most conditions, a concrete beam will remain stable if the maximum bending moment in the beam is less than the critical buckling moment given by Eq. (6). The reduction in the stability of a beam due to initial bow, eccentric loading, and loads applied above the shear center is probably cancelled approximately by the fact that Eq. (6) is conservative for most loading and support conditions met in practice.

APPLICATION OF BUCKLING ANALYSIS

For test verification and possible design application it is desirable to construct curves of laterally unsupported span length versus critical buckling moments. This section discusses the construction of such a curve for the beam section and material properties of the previously reported tests.

As indicated, the values of M_{cr} and E_{sec} in Eq. (6) correspond to the same value of the extreme compression fiber strain ϵ_c. This implies that to use Eq. (6) one must first establish the relation between M and E_{se} throughout the entire loading range from zero to ultimate, since buckling may occur at any load in this range, depending on unsupported length. Both the area under, and the slope of, the stress-strain curve used to establish this relation must conform as closely as possible to actual stress distributions in concrete beams since this curve determines both M and E_{sec}. Thus simplified stress blocks such as a trapezoid or rectangle are not suitable for buckling calculations.

The M-E_{sec} relation may be established using a modification of Stüssi's flexural theory. Since this theory has been discussed at length by previous authors, notably Hognestad and his associates,[12,13] only the results of its application will be given here. The flexural stress-strain relation shown in Fig. 9 and the materials properties in Table 3 were used as a basis for estab-

TABLE 3—MATERIALS PROPERTIES USED IN BUCKLING CALCULATIONS

Properties of concrete
$f_c' = 4290$ psi
$K_3 = 0.85$
$\beta = 2$
$\gamma = 0.25$
$\epsilon_u = 3.8$ mills
$\epsilon_o = 1.9$ mills
$E_c = 3.83 \times 10^6$ psi
Properties of steel
$f_{sy} = 43{,}800$ psi
$\epsilon_{sy} = 1.53$ mills
$E_s = 28.8 \times 10^6$ psi

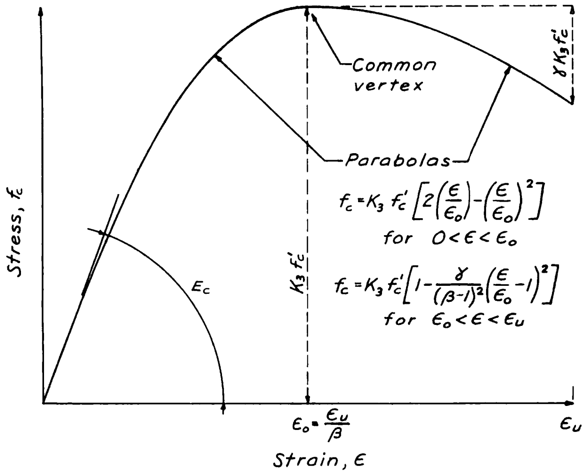


Fig. 9—Assumed stress-strain relation for concrete in flexure

lishing the M-E_{sec} relation. The assumed flexural stress-strain curve and the particular values of the parameters β and γ which determine the shape of this curve were chosen to approximate the measured flexural stress distributions reported in Reference 12.

Using Stüssi's flexural theory, data for the moment and steel strain curves in Fig. 10 were calculated. These curves assume that the steel remains elastic up to the yield point. As indicated in Fig. 10 a yield strain of 1.53 mills for the tension steel in the test beams corresponds in this analysis to a yield point bending moment of 183 kip-in. which is 8 percent on the conservative side compared with observed test beam behavior (see Table 2). Portions of the moment curve in Fig. 10 above 183 kip-in. correspond to larger steel yield strengths than those used in the testing program. A secant modulus curve based on the assumed flexural stress-strain relation is also included in Fig. 10.

Having simultaneous values of E_{sec} and M from Fig. 10 one may use Eq. (6) to estimate critical buckling moments of the test beams.

With $k = 0.382$ (a conservative and minimum value prior to tensile steel yielding), $b = 2.5$ in., $d = 11.25$ in., this equation reduces to

$$M_{cr} = 1.51 \frac{E_{sec}}{L} \qquad (7)$$

in which the following units apply: M_{cr} = kip-in., E_{sec} = kips per sq in. and L = ft. Eq. (7) was used to plot the buckling curve shown in Fig. 11. This curve gives the relation between the buckling moment and the laterally unsupported span length which would cause buckling of the test beams according to Eq. (6), assuming elastic steel behavior. Also shown is the average yield point bending moment M = 183 kip-in. of the test beams computed from the assumed stress-strain relations as shown on Fig. 10. The maximum unbraced test span was 18 ft while M = 183 kip-in. is seen to correspond to L = 22.2 ft. Thus, Fig. 11 indicates that none of the test beams were sufficiently long to produce instability failure prior to reaching the yield point, which agrees with the test observations. Assuming the validity of the buckling curve, the minimum unbraced span length at which buckling would precede steel yielding for the test beams is 22.2 ft, corresponding to an L/b ratio of 106.

Some observations concerning the stability provisions of the current ACI Building Code (ACI 318-56) may be made from Fig. 10 and 11. The curves in these figures indicate that the limiting L/b ratio of 32 stipulated by Sec-

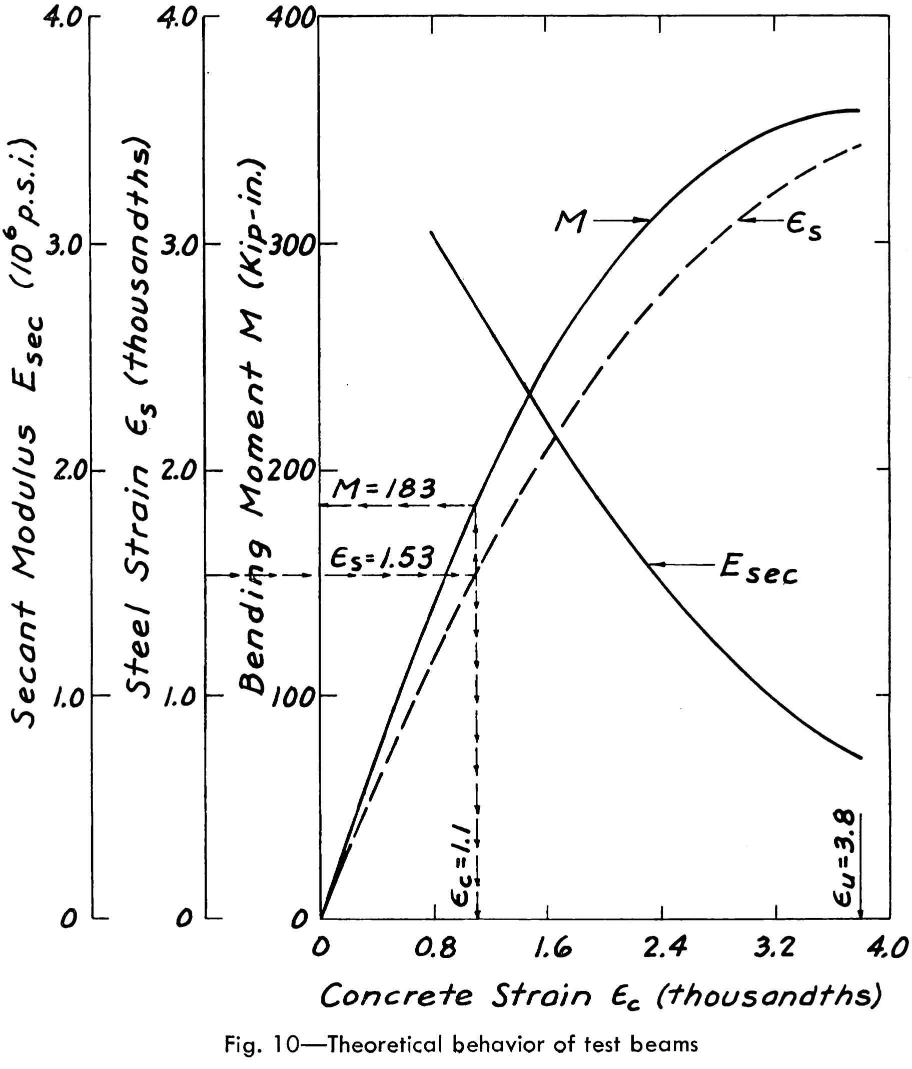


Fig. 10—Theoretical behavior of test beams

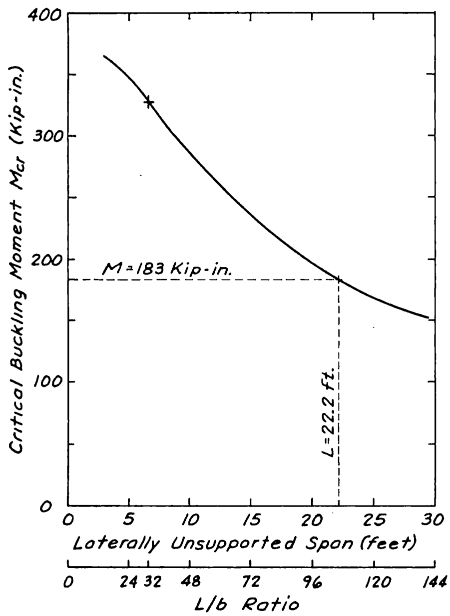


Fig. 11—Theoretical critical buckling moments for test beams

tion 704 of the Code is devoid of any significance for ordinary strengths of steel and concrete, even for beams of the unrealistically extreme cross-sectional dimensions which were used in the tests. It is seen from Fig. 11, on the other hand, that $L/b = 32$ would correspond to a critical moment of 328 kip-in. For the test beams Fig. 10 shows that this moment would be reached only if a steel strain of 2.93 mills were developed elastically, i.e., at an elastic steel stress of about 85,000 psi, which is possible only with high strength steels. This illustrates the fact that the higher concrete strains which can be developed prior to steel yielding with high strength reinforcements can lead to a significant reduction of E_{sec} and a corresponding reduction of the L/b ratio at which lateral buckling can occur.

Correspondingly, it appears justified to liberalize considerably the present limitation of $L/b = 32$ in the ACI Code when applied to beams reinforced with steel of 40,000 to 60,000 psi yield point. At the same time, since reinforcement with yield values of the order of 80,000 psi is in the process of becoming available, it would appear desirable to impose more stringent limitations on unbraced length for beams reinforced with such high strength steels.

It should be emphasized that the theoretical results of this investigation on which the above remarks are based are limited in scope and have not been experimentally verified, except to the extent that they agree with the fact that no buckling failures were obtained in any of the tests up to $L/b = 86$.

SUMMARY AND CONCLUSIONS

Ten slender beams with d/b ratios of 4.5 and L/b ratios from 28.8 to 86.4 were tested using specially designed loading rigs which eliminated any significant lateral and torsional restraint at the load points. The beams were loaded at the quarter points and were laterally unrestrained over their full span between supports. All of the beams failed in simple, vertical bending due to yielding of the tension steel, at loads in close agreement with the predictions of ultimate strength theory. There was no evidence of any reduction in strength due to laterally unsupported span length even though the largest L/b ratios were 2.7 times as large as permitted by the limitations of the current ACI Building Code (ACI 318-56).

Since the test beams were dimensionally more extreme, i.e., more conducive to lateral buckling, than beams likely to occur in practice, it is concluded that the L/b limitations of Section 704 of the ACI Building Code are excessively conservative when applied to realistically dimensioned laterally unbraced beams reinforced with ordinary strength steel. However, it was shown that more stringent L/b limitations are advisable when high strength steel is used than for ordinary reinforcement.

An approximate analytical method was suggested for estimating the critical buckling moment of a simply supported, initially straight, concentrically loaded, rectangular, reinforced concrete beam. It is believed that this method gives conservative estimates for most conditions met in practice. The proposed buckling theory was used to predict the critical buckling moments of the test beams. The theory correctly predicted that none of the test beams would buckle prior to yielding of the tension steel.

It should be noted that the testing program involved only short time loading, one beam cross section, one steel ratio and steel strength, one loading system, and one concrete strength. Further studies are desirable to verify the general applicability of the proposed buckling relation.

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APPENDIX—INELASTIC BUCKLING MODULI

An analytical investigation of the lateral stability of reinforced concrete beams must consider the inelastic stress-strain properties of concrete. In particular, one must determine what values of the elastic constants E and G apply in the inelastic case. It is the purpose of the appendix to consider this problem. For simplicity, lateral buckling will be considered without particular reference to reinforced concrete. It will be assumed that all members are homogeneous, initially straight, and free from imperfections, and that all loads are applied concentrically.

Consider a slender beam with constant symmetrical cross section bent about its major principal axis (X axis) by equal couples applied at the ends of the beam [Fig. A-1(a)]. The ends of the beam are supported so as to prevent rotation of the end sections about the longitudinal beam axis (Z axis). Otherwise the beam is unrestrained. The beam material is assumed to have equal stress-strain properties in tension and compression [Fig. A-1(b)]. The stress-strain relation is assumed to be inelastic and is represented in the general form $\sigma = g(\epsilon)$ where σ and ϵ denote stress and strain respectively. The tangent modulus E_t is then defined as

$$E_t = \frac{d\sigma}{d\epsilon} \qquad \text{(A1)}$$

Plane sections are assumed to remain plane during bending and the stress-strain properties of the material for axial loading are assumed to be valid for flexure. The symmetrical cross section of the beam is assumed to have the general shape indicated in Fig. A-1(c) where $w(y)$ denotes the width of the beam at a distance y from the horizontal centroidal axis and h is the depth of the beam.

Incipient lateral buckling occurs in the inelastic stress range at a moment M_{cr} of

$$M_{cr} = \frac{\pi}{L} \sqrt{\overline{E}I_y\overline{G}K_t} \quad \text{(A2)}$$

where $\overline{E}$ and $\overline{G}$ represent reduced moduli which are related to the stress-strain curve and the strains caused by M_{cr}. The problem of inelastic buckling is reduced to evaluating $\overline{E}$ and $\overline{G}$.

In discussing the lateral stability of metal I-beams stressed beyond the proportional limit, Timoshenko[3] and Bleich[14] have proposed that the tangent modulus corresponding to the maximum extreme fiber stress be used for $\overline{E}$, i.e.

$$\overline{E} = E_t \quad \text{(A3)}$$

and that $\overline{G}$ may be approximated by

$$\overline{G} = \frac{\overline{E}}{E} G \quad \text{(A4)}$$

In a recent series of tests of aluminum I-beams, Clark and Jombock[15] found that Eq. (A2), (A3), and (A4) gave reasonably accurate estimates of observed buckling moments (for uniform bending). However, one is not justified in using Eq. (A3) for rectangular sections since their flexural strength is not derived from material concentrated in "flanges." The following discussion would establish Eq. (A3) as a special case of a more general expression for $\overline{E}$.

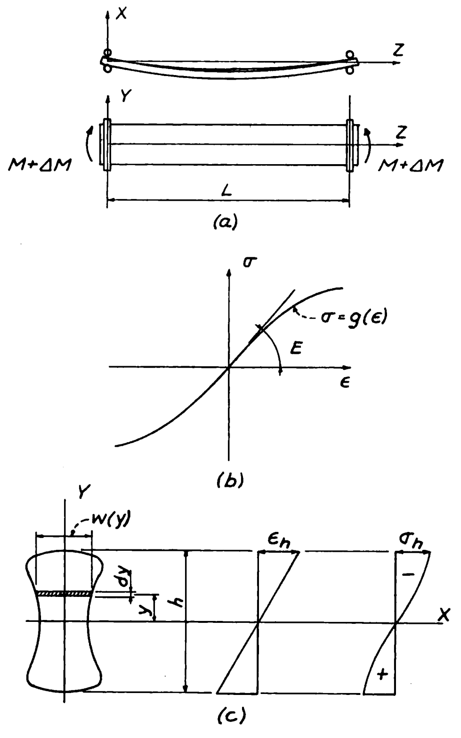

Fig. A-1—Inelastic lateral buckling

The product $\overline{E}I_y$ in Eq. (A2) is the lateral flexural rigidity of the beam at the instant of incipient buckling. It is assumed that, according to Shanley, bifurcation of equilibrium is a convenient and only slightly conservative criterion for incipient buckling. One is then led to consider the lateral flexural rigidity of the beam of Fig. A-1 at the instant when bifurcation of equilibrium occurs. Before evaluating $\overline{E}I_y$ it is pertinent to describe in detail the relations between the stresses and strains involved in bifurcation of equilibrium.

Let M represent the largest moment the beam can resist without deflecting laterally. By analogy with the tangent modulus theory, a small increase in bending moment from M to $M + \Delta M$ causes a small but stable rotation and lateral deflection of the beam. As ΔM approaches zero, the straight and deflected forms of the beam approach coincidence so that M is the moment at the instant when bifurcation of equilibrium occurs.

Again by analogy with the tangent modulus theory, the moment ΔM causes the strains at every point in the beam to increase in absolute value.* If $\Delta\epsilon$ represents the increase in strain at any beam fiber due to the moment ΔM then the increase in stress $\Delta\sigma$ at this fiber is approximately $\Delta\sigma = E_t\,\Delta\epsilon$ where E_t corresponds to the strain at the fiber caused by the moment M. When bifurcation occurs the stresses $\Delta\sigma$ and strains $\Delta\epsilon$ vary across the width of the section. The resistance of the beam to lateral deflection is associated with these variations in $\Delta\sigma$ and $\Delta\epsilon$.

One is now in a position to evaluate the lateral flexural rigidity of the beam, $\bar{E}I_y$, at the instant of bifurcation of equilibrium. Consider an element of area in the beam cross section [Fig. A-1(c)] of width $w\,(y)$ and height dy at a distance y from the neutral axis of the beam. The contribution of this element to the lateral flexural rigidity of the beam may be expressed as

$$d(\bar{E}I_y) = E_o\left[\frac{1}{12}\,w^3(y)\right]dy$$

where E_o is the modulus relating the stresses and strains associated with lateral deflection. The preceding discussion indicates that $E_o = E_t$ so that

$$\bar{E}I_y = \frac{1}{12}\times 2\int_0^{h/2} E_t w^3(y)dy$$

is the lateral flexural rigidity of the beam associated with bifurcation of equilibrium.

To evaluate $\bar{E}$, one may express I_y as

$$I_y = \int_A x^2 dA$$

$$= 4\int_0^{\frac{1}{2}h}\int_0^{w(y)/2} x^2 dx dy$$

$$I_y = \tfrac{1}{6}\int_0^{h/2} w^3(y)dy$$

and so obtain the result

$$\bar{E} = \frac{\displaystyle\int_0^{h/2} E_t\, w^3(y)dy}{\displaystyle\int_0^{h/2} w^3(y)dy} \qquad \text{(A5)}$$

Eq. (A5) is a general expression for the reduced modulus $\bar{E}$ and is based on an analogy with the tangent modulus theory for columns. It is seen that the shape of the cross section has an important influence on $\bar{E}$. For instance, neglecting the thin web, Eq. (A5) gives $\bar{E} = E_t$ for an I-section, in agreement with Eq. (A3). For the rectangular section, one obtains from Eq. (A5)

*If the lateral deflection is small but finite, this statement is not strictly valid for beam fibers in a small band near the neutral axis. Strain reversal will occur in this band but the stress-strain curve is nearly elastic for the small strains involved. Therefore strain reversal near the neutral axis will have only a minor effect on the lateral flexural rigidity.

$$\overline{E} = \frac{b^3 \int_0^{h/2} E_t dy}{b^3 \int_0^{h/2} dy}$$

and with the substitutions

$$y = \frac{1}{2} h \frac{\epsilon}{\epsilon_h}$$

$$dy = \frac{1}{2} \frac{h}{\epsilon_h} d\epsilon$$

derived from the strain diagram in Fig. A-1(c) it follows that

$$\overline{E} = \frac{1}{\epsilon_h} \int_0^{\epsilon_h} E_t \, d\epsilon$$

Finally from Eq. (A1) one obtains

$$\overline{E} = \frac{1}{\epsilon_h} \int_0^{\epsilon_h} \frac{d\sigma}{d\epsilon} d\epsilon$$

$$= \frac{1}{\epsilon_h} \int_0^{\sigma_h} d\sigma$$

$$\overline{E} = \frac{\sigma_h}{\epsilon_h} = E_{sec}$$

where σ_h and ϵ_h are the extreme fiber stresses and strains respectively. This interesting result indicates that the modulus for lateral buckling of a rectangular section is the secant modulus E_{sec} corresponding to the extreme fiber strain.

No direct method of evaluating the reduced shear modulus is evident to the writers. One must therefore resort to somewhat arbitrary approximations, such as Eq. (A4). Substitution of this relation in Eq. (A2) gives

$$M_{cr} = \overline{E} \sqrt{\frac{G}{E}} \times \frac{\pi}{L} \sqrt{I_y K_t}$$

Using the known relation

$$\frac{G}{E} = \frac{1}{2(1 + \mu)}$$

where μ is Poisson's ratio, one obtains Eq. (2) as an approximate expression for the critical inelastic buckling moment of a beam subjected to uniform bending. The reduced modulus $\overline{E}$ in this equation may be evaluated for a given beam section using Eq. (A5) if the stress-strain curve and the extreme fiber strain are known. It is equal to E_{sec} for rectangular, and to E_t for I-shaped beams. Note that according to the assumptions made here, μ corresponds to elastic values and does not vary with the extreme fiber strain.

The prime objective of the preceding study has been to show the relation between the shape of the beam section and the reduced modulus. No claim is made to a rigorous or exact solution of the problem of inelastic lateral buckling. In essence the study consists of a rational application of the tangent modulus theory to this problem.

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