ORDINARY MEETING
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The prediction of crack widths in hardened concrete

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Synopsis

A requirement to check the widths of load-induced cracks is now a feature of current British Codes for structural concrete. However, the theoretical background to the procedures given in the Codes has not been published in a readily available and reasonably condensed form. This paper attempts to rectify this situation by presenting the derivation of a theory for the prediction of cracking in hardened concrete. This theory is shown to be a logical development of earlier theories, and is based on the extensive research program carried out at the Cement and Concrete Association over the last 14 years. The theory forms the basis of many Code crack prediction equations, and the derivation of these is discussed.

Introduction

All the current Codes of practice that cover the use of structural concrete—CP 110, BS 5400, and BS 5337[1,2,3]—now include limits on permissible design crack width and formulae for the prediction of design widths. With the exception of the formula given in BS 5337 for the prediction of the widths of cracks induced by early thermal movements, these formulae are of the same form and are based on work carried out at the Cement and Concrete Association. Research on cracking has been in progress at the Cement and Concrete Association for the last 14 years during which time something in excess of 250 reinforced and prestressed members have been tested. This experimental and theoretical work has been published in a number of Cement and Concrete Association Research and Technical Reports[4–10]. However, until now, no condensed statement of the background and derivation of the Code design methods has been published in a form that is readily accessible to the average practising engineer. One of the main objectives of this paper is to rectify this omission.

There are two basic aspects to the problem of cracking in design: the definition of suitable criteria, and the derivation of suitable design methods to ensure that these criteria are met. The first of these, the choice of suitable design crack widths, has been discussed elsewhere[11,12], and will not be considered further in this paper.

Cracks may be dealt with in one of three ways, depending upon the type of cracking involved. They may be avoided, they may be induced to form at prearranged locations where their effects can be dealt with, or they may be permitted to form at random and the reinforcement detailed so that the resulting widths are limited. Crack prediction provisions in Codes obviously only deal with this last approach, though it is worth noting that, where cracking has caused problems in practice, it is usually because large cracks that should have been avoided have been allowed to occur. These could be plastic cracks, which cannot be controlled by reinforcement, or cracks in areas of a structure where stresses were not expected and insufficient reinforcement was provided to produce controlled cracking. Cracking due to loading has rarely been a problem in adequately reinforced members.

At this stage, it needs to be pointed out that, internationally, there is remarkably little agreement on design methods for cracking. If formulae from different national Codes are compared, it is in many cases very difficult to discern any common ground between them. That this lack of agreement goes beyond simply the form of equation used to predict crack widths may be seen from the following example. Fig 1 shows details of a slab that is loaded in flexure to a level that will give a steel stress, calculated on the basis of a cracked section, of 230 N/mm². Formulae from 10 different design documents have been used to calculate design crack widths, and the results are also illustrated in Fig 1. The commonest permissible crack width limits at present are 0·3 mm in mild environments, 0·2 mm in moderate environments, and 0·1 mm in severe environments. It will be seen that the slab in question would be considered unsuitable for any environment by four Codes, suitable for mild environments by one, suitable for mild and moderate environments by three, and suitable for all environments by two. It is hard to understand how such large differences can occur in design calculations for what appears to be quite a normal type of member.

Clearly, a general theory for cracking should be developed, and the first part of this paper attempts to do just that. The method adopted in presenting this theory is to start with a brief history of the development of previous theories. This has been done in order to show that the proposed theory is not a totally new departure, but a logical extension of past thinking.

Development of a theory of cracking

What will be attempted in this section is to trace the development of cracking theory from the Saliger theory of 1936[13] up to the present. It will be suggested that the various theories and the

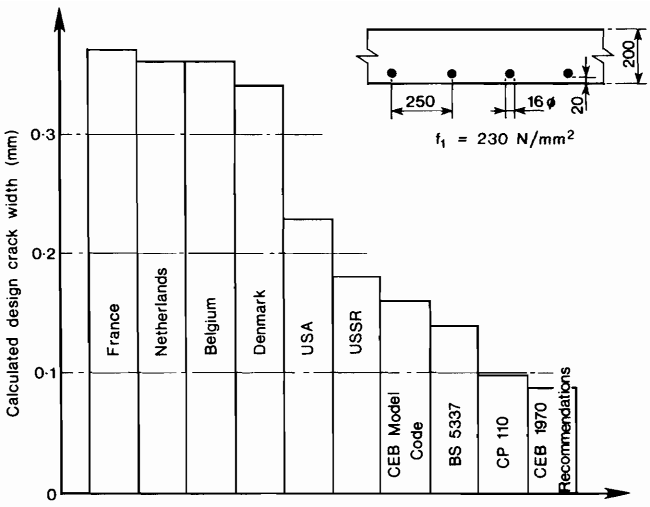


Fig 1. Design crack widths calculated using various regulations for the slab shown

resulting equations for the prediction of crack widths are not totally incompatible, but are mostly partial descriptions of the phenomenon, and that the development of cracking theory follows a natural progression giving a more and more complete picture as more data have become available. This treatment has largely been taken from reference 14.

It should first be made clear that all theories deal with the cracking of hardened concrete; plastic cracking, for example, lies outside their scope. A further condition is that sections should contain sufficient reinforcement to ensure that the steel remains elastic after cracking under the loading considered.

All theories start from the following basic considerations.

1. Consider the situation when the first crack forms in a member. On the surface of the member, the stress in the concrete must be zero at the edge of the crack. With increasing distance away from the crack, the surface stress will increase until, at some distance, S_0, the stress distribution remains unaffected by crack (i.e. the crack affects the stresses only within a distance $\pm S_0$ from the crack). Since the crack has reduced the concrete surface stress to below the tensile strength of the concrete within $\pm S_0$ of the crack, the next crack to form must form outside this region. The minimum distance between cracks is thus S_0. If two cracks form at a distance apart greater than $2S_0$, there will be an area between the cracks where the stress is not affected by either of the cracks and so another crack can form, whereas, if cracks form at a lesser spacing than $2S_0$, the concrete stresses will be reduced over the whole length between the two cracks and another crack will not form. When all the cracks have developed, the maximum spacing will thus be $2S_0$, and the final crack pattern will consist of cracks having some distribution of spacings within the range:

$$S_0 \leqslant S \leqslant 2S_0$$

This argument is illustrated in Fig. 2.

2. The average crack width is given by the average final crack spacing multiplied by the average strain minus the average residual surface strain in the concrete between the cracks:

$$w_m = S_m(\epsilon_m - \epsilon_{cm})$$

Commonly, the strain in the concrete between the cracks, ϵ_{cm}, is ignored. This assumption will normally be reasonable, and results in the relationship:

$$w_m = S_m \epsilon_m \qquad \ldots (1)$$

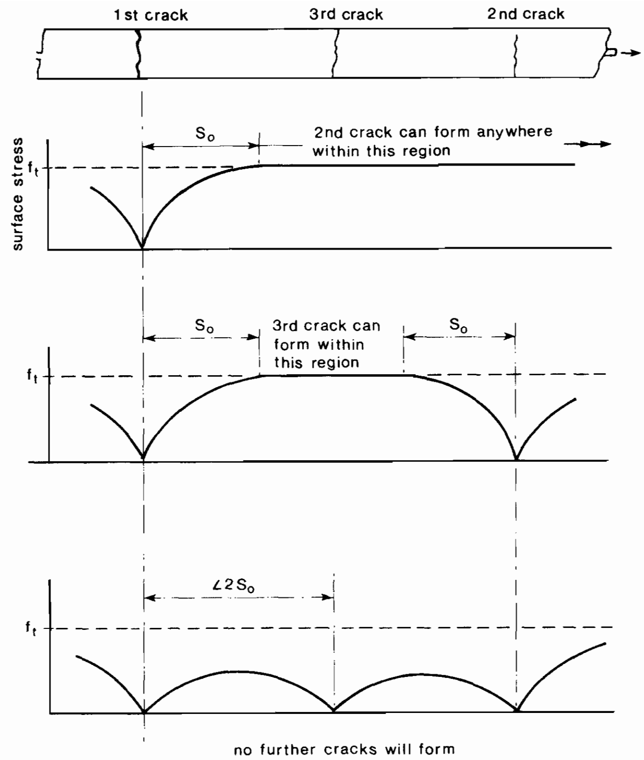


Fig 2. Conditions on the surface of an axially reinforced tension member during the development of cracking

The mean final crack spacing, S_m, has commonly been assumed to be given by $1{\cdot}5S_0$, but there are theoretical reasons for believing that a value of $1{\cdot}33S_0$ is more correct. The problem facing theorists is to develop a means of predicting S_0.

Saliger's theory for the prediction of crack spacing in members subjected to pure tension[13] is based on the arguments set out above, plus the further condition that plane sections remain plane within the concrete. Thus, in a member subjected to pure tension, the stress in the concrete is uniform over the whole concrete section. Compatibility of deformation between the steel and the concrete is not maintained, and it is assumed that there will be relative displacement or slip between the two. Bond failure is thus assumed to occur at each crack. Force will be transferred between the steel and concrete by bond stresses acting at the interface. An additional assumption is made that, since bond failure has occurred at each crack, the distribution of bond stress along the bar between cracks can be taken as a function of the ultimate bond strength. These assumptions lead to the following relationship for S_0:

$$S_0 = k_1 \frac{\phi}{\rho} \frac{f_t}{\tau_{ult}}$$

where

- ϕ is the bar diameter
- ρ is the reinforcement ratio
- f_t is the tensile strength of concrete
- k_1 is a constant, depending upon the shape of the bond stress distribution

It is found that τ_{ult} is directly proportional to f_t for a given bar type, and hence substitution into equation 1 gives the following formula for crack width:

$$w_m = K \frac{\phi}{\rho} \epsilon_m$$

By carrying out tests, K can be found experimentally, with the result that the assumptions concerning the relationship between the minimum, maximum, and average widths and spacings and the shape of the distribution become irrelevant, as does the shape of the bond stress distribution.

The next theoretical approach[4] derived from an assumption exactly opposite to that of Saliger. It was assumed that plane sections did not remain plane and that, at the time the cracks developed, bond failure did not occur, and hence there was no slip. The estimation of the stresses in the concrete in this case is not so simple as for Saliger's approach, but can be done, and it will be found that the distance S_0 between the crack and the point where the stresses remain undisturbed by the crack is roughly equal to the cover. This would be expected from application of the 45° rule: take a line at 45° from the edge of the loaded area (in this case the bar) and the stresses will have evened out by the time the point is reached where this line cuts the surface of the concrete (see Fig 3). This leads to the following equation for crack width:

$$w_m = Kc\epsilon_m \qquad \ldots . (2)$$

This approach proved to be more satisfactory for beams than the bond–slip approach, but still not ideal.

In fact, it is more reasonable to view the 'slip' and 'no-slip' approaches as providing different components of the problem: deformation of the type assumed in the 'no-slip' approach must occur since, locally to a crack, plane sections will not remain plane. This must cause reduced stresses in the surface concrete in the region of the crack. Bond failure or slip will cause a further reduction in stress, increasing the value of S_0. Thus S_0 can be considered to be made up of two components—$S_{0,1}$ which will be the value of S_0 derived from the 'no-slip' approach, and $S_{0,2}$ which will derive from the classical bond failure approach. Ferry-Borges[15] showed that these two components could simply be added together to obtain a crack spacing formula of the type given below:

$$S_m = K_1\, c + K_2 \frac{\phi}{\rho} \qquad \ldots . (3)$$

This, in principle, is the equation given in the 1970 CEB Recommendations[16]. It works well for axially reinforced square-section members subjected to pure tension, of the type for which the theory has been derived.

At this stage, it is necessary to take a more mature look at the concept of bond failure and slip which resulted in the derivation of ϕ/ρ as a prime variable. The picture of the phenomenon assumed in the discussion above is shown schematically in Fig 4(a). It is not difficult to accept that this is the type of behaviour that will occur with plain bars, and it should be noted that, when the early theories were developed, plain bars were the type normally used. However, this is not the way in which sections reinforced with deformed bars behave. Instead of failure occurring along the bar–concrete interface, the distortion of the concrete is accommodated by a series of internal cracks (Fig 4(b)). Clearly, for this type of behaviour, the mathematics of the Saliger bond–slip approach are inapplicable and a different description is required. A study of the nature of cracking around a deformed bar, as revealed by the work of Goto[17] and others, suggests the following stages in the development of a crack.

(a) A crack forms, initially having minimal width at the bar surface.
(b) Further loading causes loss of adhesion adjacent to the crack, transferring load to the ribs of the bar.
(c) Internal cracks form close to the main crack.
(d) Further loading causes more internal cracks to form at successively greater distances from the main crack.

At stage (a), conditions are as described by the 'no-slip' theory, and the stress at the concrete surface will be affected only by the crack within the region ± the cover, c, from the crack. The effect of any events (b), (c), and (d) is to reduce the rate at which force is transferred from the reinforcement to the concrete and hence increase the distance from the crack over which the surface stresses are reduced (i.e. in the terminology used in the derivation of the earlier equations, S_0 increases successively above the minimum value of c as loading increases, causing events (b), (c), and (d) to occur). Further main cracks can form anywhere on the member except within $\pm S_0$ of existing cracks. It has also been explained that no intermediate cracks can form when two cracks are less than $2S_0$ apart.

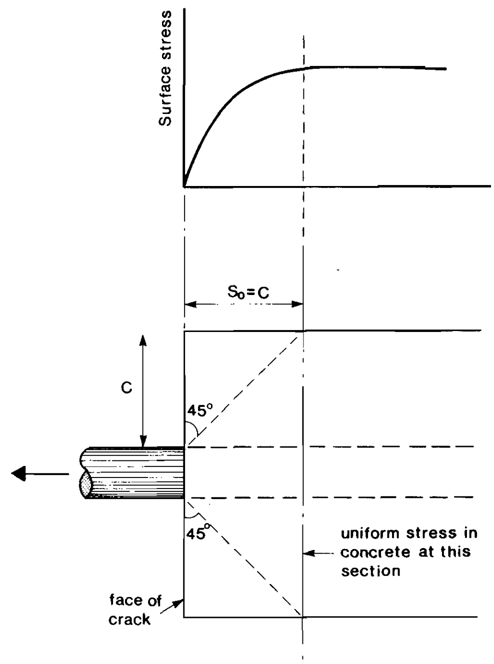
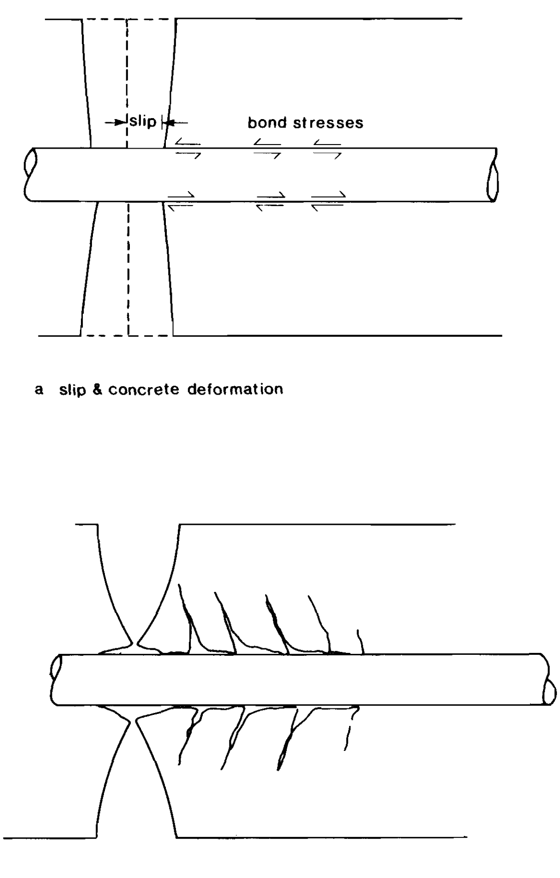


Fig 3. No-slip mechanism of cracking: relationship between C *and* S_0



a slip & concrete deformation

b internal cracking & concrete deformation

Fig 4. Components of crack width

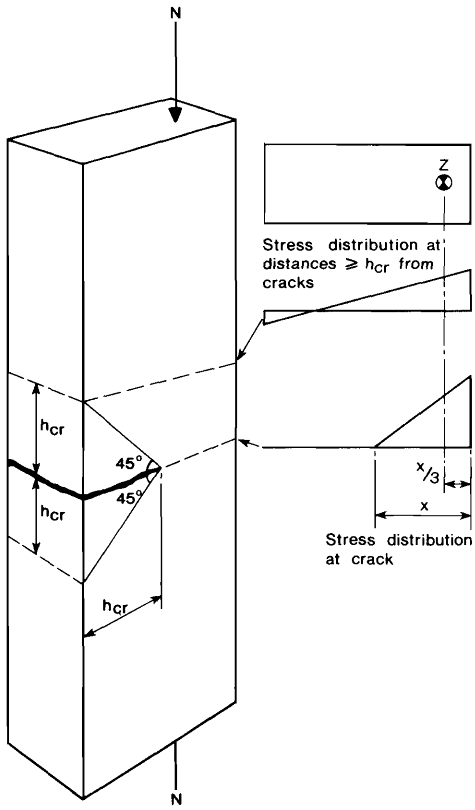


Fig 5. Cracking in an unreinforced member

If, on further loading after a crack has formed, an adjacent crack develops before substantial amounts of internal failure have occurred (events (b), (c), and (d)), it will be able to form close to the previous crack, giving a minimum spacing approaching the cover. However, if substantial internal failure has occurred before the adjacent crack forms, S_0 will be substantially larger than the cover and thus the minimum possible spacing will have increased. The average spacing will be qual to a constant times the cover plus the average increase in S_0 resulting from the average amount of internal failure occurring prior to formation of the adjacent crack. Thus:

$$S_m = K_1\, c + \text{(average influence of internal failure)}$$

It remains to assess the parameters that are likely to control the rate of development of internal failure. Qualitatively, these can be assessed as follows. The maximum force to be transferred between steel and concrete is $f_t A_c$. This has to be transmitted via the concrete around the bar–concrete interface. The strength of this will depend upon the bar diameter; the smaller the diameter, the smaller the area of concrete through which the force must be passed. Hence, stress developed is likely to be proportional to $f_t A_c/\phi$. This is proportional to ϕ/ρ. Assuming that the rate of development of internal failure is proportional to the stress developed, the crack spacing will be given by:

$$S_m = K_1\, c + K_2 \frac{\phi}{\rho}$$

This is identical with Ferry-Borges' equation. Thus it can be seen that the derivation via bond–slip considerations is simply a special case of the above, more general proposition and that the Ferry-Borges type of formula is quite general and independent of the form of the internal failure involved. The parameter ϕ/ρ may be considered as generally defining the stress state in the concrete immediately surrounding the bar rather than the bond stress specifically.

So far, the discussion has been confined to conditions in axially reinforced tension members. It has commonly been assumed that the conditions in the tension zone of a beam could be assumed to be identical with those in pure tension, but this is not the case and it is necessary to introduce a different set of theoretical considerations in order to understand the behaviour of beams.

Consider an unreinforced column subjected to an eccentric load where the eccentricity is large enough to cause part of the section to go into tension (Fig 5). If the load is sufficient, the concrete will crack. This will not result in failure of the column, but merely a redistribution of forces in the region of the crack.

Clearly, this first crack results only in a local disturbance of the stress field. Some distance away from the crack, the stresses remain unaffected, and thus further cracks can be expected.

Roughly, applying the 45° rule, the stress distribution can be expected to be unaffected by the crack at a distance from the crack equal to the height of the crack. Thus, by the same argument as used earlier, the spacing of cracks can eventually be expected to fall within the range:

$$h_{cr} \leqslant S \leqslant 2h_{cr}$$

In this situation, the crack width will be given by the equation:

$$w = K_1\, h_{cr}\, \epsilon_m \qquad \ldots.(4)$$

where h_{cr} is the height of the crack. It can be shown experimentally that this is in fact the case[5].

It might at first appear that further loading above that required to establish this pattern would cause the surface stresses to increase and further intermediate cracks to form. This does not occur because one of two other developments will take place instead. Either the cracks will increase in height, which will reduce the surface stress between the cracks or the cracks will fork, giving cracks roughly parallel to the neutral axis. This latter may occur because tensile stresses perpendicular to the neutral axis exist at the head of the cracks. When the ratio of the crack spacing to the crack height reduces below about 2, these stresses become greater than those on the surface at mid-spacing.

Thus it is quite possible to obtain a controlled, stable crack pattern without the presence of bonded steel in the section. Fig 6 shows such crack patterns on a series of unreinforced members subjected to axial load and moment[5]. Now consider the effect of adding bonded reinforcement to such a member. It will be seen that the problem differs from the case of pure tension discussed earlier because now the effect of the reinforcement in controlling cracking is being superimposed on an existing stable crack pattern. These two effects will interact to produce the actual pattern obtained at any particular point. That such interaction must occur close to a bar, as well as elsewhere, can be seen by considering a situation where the height of the cracks, h_{cr}, is relatively small (say, for example, 2 to 3 times the cover to the steel) and where the reinforcement ratio is also small. It is perfectly possible, in such a case, for the spacing or width calculated from equation (3) to

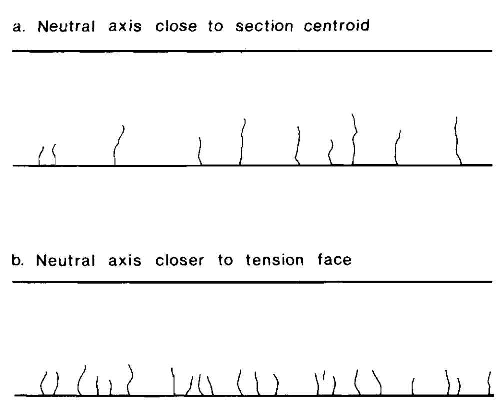


Fig 6. Crack patterns on unreinforced members subjected to combined bending and axial load

exceed that resulting from equation (4). However, the addition of bonded steel cannot worsen crack control—it can only improve it. Thus in this case, equation (3) must be heavily modified by the cracking controlled by the crack height. On the other hand, if h_{cr} is very large, equation (3) will give a width that is much smaller than equation (4), and one would expect the cracking to be dominantly influenced by equation (3), i.e. if a beam is sufficiently deep, conditions in the bottom of the tension zone approach pure tension.

The problem is to decide how equation (3) should be modified to take account of the influence of the type of cracking described by equation (4).

The derivation of K_1 in equations (3) and (4) was identical, and thus one would expect them to have the same value.

In the limiting case, where $h_{cr} = c$, the crack width must be equal to $K_1 h_c$. Hence equation (3) and (4) gives:

$$w = \left(K_1 c + K_2 \frac{\phi}{\rho}\right) \epsilon_m = K_1 h_{cr} \epsilon_m$$

since $h_{cr} = c$, K_2 must be equal to zero. In other words, bond strength, steel percentage, and bar diameter are in this case irrelevant, except in so far as the steel percentage influences the neutral axis depth and hence the value of h_{cr}.

A helpful way of looking at the term $K_2 (\phi/\rho)$ is as follows: K_2 can be considered to have two parts $K_{2,1}$ and $K_{2,2}$, such that:

$K_{2,1}$ defines the probability and extent of internal failure around a crack at the time adjacent cracks form.

$K_{2,1}$ is the value of K_2 that would be obtained from pure tension tests;

$K_{2,2}$ defines the influence that this bond failure will have upon the cracking.

Thus, in the limiting case cited above, $K_{2,2} = 0$, while in pure tension $K_{2,2} = 1$.

It now remains to discover $K_{2,2}$ in more normal circumstances. The general principles, and hence the important variables, can be assessed without difficulty. For any section in flexure, the crack width cannot exceed that given by equation (4), nor will $K_{2,2}$ be less than zero. Hence the width must lie in the range: $K_1 c \epsilon_m \leqslant w \leqslant K_1 h_{cr} \epsilon_m$, and $K_{2,2}$ must take a value that will result in a calculated crack width within this range. Clearly, the smaller the difference is between c and h_{cr}, the smaller must $K_{2,2}$ be to ensure that this condition is met. This result can conveniently be achieved by making $K_{2,2}$ a function of the ratio of these two quantities.

Equation (3) can thus be extended for use in flexural situations to:

$$w = \left(K_1 c + K_{2,1} f_n \left(\frac{c}{h_{cr}}\right)\right) \epsilon_m \qquad \ldots (5)$$

The actual function used to obtain $K_{2,2}$ has to be obtained experimentally, but must have the property that it is effectively zero for $c/h_{cr} = 1$ and approaches 1 as c/h_{cr} decreases towards zero.

Equation (4) was originally derived from consideration of an eccentrically loaded unreinforced column. However, the derivation holds true equally for a reinforced concrete beam. The necessary condition for the applicability of the argument is that equilibrium states should be possible for the section under the loading considered in both a cracked and an uncracked state. This is true for a reinforced concrete beam.

Now consider conditions in the zone where a bar passes across a crack in a member with wide bar spacings. This is illustrated in Fig 7(a). If the 'tooth' between two cracks is considered in isolation, it would look as illustrated in Fig 7(b). If this situation is analysed elastically, the stress distribution in the concrete would appear as shown in Fig 7(c). Effectively, the bar will stress only the concrete in a limited zone around itself. If there is any internal failure, this zone will be even smaller. Thus the direct influence of the reinforcement on cracking can be only local, and the cracking on parts of the member surface beyond this limited zone must be dominantly controlled by the crack height. The form of the concrete stress curve in Fig 7(c) indicates the likely form of interaction

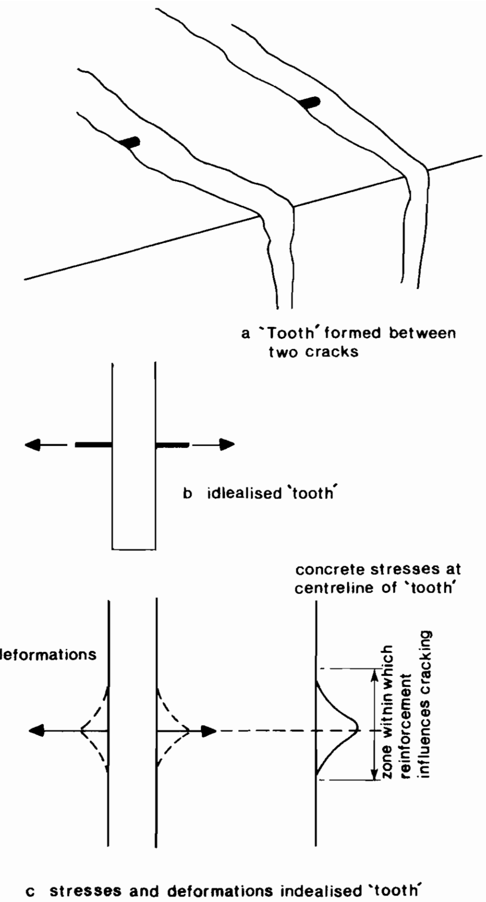


Fig 7. Conditions in a zone between two cracks

between the cracking close to a bar and that well away from a bar. Directly over a bar, equation (5) will hold; as points on the surface further and further away from the bar are considered, equation (4) will be approached asymptotically. It is found experimentally that, if the position on the surface of the member is defined by the quantity a_{cr}, where a_{cr} is the distance from the surface of the nearest longitudinal bar to the point considered, the following hyperbolic relation can be defined.

$$w = \frac{a_{cr}\, w_0\, w_{lim}\, \epsilon_m}{c w_{lim} + (a_{cr} - c)\, w_0} \qquad \ldots (6)$$

where

w is the crack width at point considered
w_0 is the crack width over bar given by equation (3)
w_{lim} is the crack width controlled by crack height (equation (4))
c is the cover

TABLE 1—Values of K_1 *for use in equations (3) and (4)*

Probability of exceedence	K_1
Mean	1·33
20%	1·59
5%	1·86
2%	1·94

This description of the theoretical aspects of crack prediction has been set out in relatively non-mathematical terms in an attempt to make the principles clear. More detailed treatments of the various aspects of cracking theory can be found in the literature (e.g. reference 5). The remaining problem is to obtain values for the coefficients K_1 and K_2 in equations (3) and (4). Note that K_2 in equation (3) is equal to $K_{2,1} f_n (c/h_{cr})$ from equation (5). Values for K_1 for various probabilities of exceedence are given in Table 1.

Values of K_2 can be obtained only empirically, and depend upon how the reinforcement ratio, ρ, is defined. This is no problem for the axially reinforced prism subjected to tension that was used in the derivation of the theory but, for flexural situations or more complex arrangements of reinforcement in tension members, an effective area of concrete surrounding each bar corresponding to an effective axially reinforced prism has to be defined. Many ways of doing this have been proposed, and none is truly satisfactory. Probably the commonest approach is to take an area of concrete surrounding the main steel and having the same centroid as that steel. In cases where the procedure is ambiguous, a result may be obtained by treating each bar separately in this way. Assuming that the reinforcement ratio is calculated on this basis, values for K_2 can be obtained and these are given in Table 2.

TABLE 2—Values of K_2 *for use in equation (3)*

Probability of exceedence	Value of c/h_{cr}					
	0 (Pure tension)	0·1	0·15	0·2	0·25	0·3
Mean	0·08	0·04	0·03	0·02	0·01	0·01
20%	0·12	0·07	0·05	0·04	0·03	0·02
5%	0·20	0·12	0·09	0·07	0·06	0·04
2%	0·28	0·17	0·13	0·10	0·08	0·06

That these values will lead to calculated crack widths that are in good agreement with those obtained experimentally can be seen from Fig 8. This graph uses data from the tests described in references 4, 5, and 6. The way in which the crack width data has been processed is fully described and justified in reference 5 but, briefly, is as follows. A series of lines, parallel to the main reinforcement, were drawn on the surface of the specimens and, at each load stage, each crack was measured where it crossed each of these lines. Each measured width was then divided by the average strain measured along the particular line. Since crack width is proportional to strain, this reduces data from all load stages to a common base. All the resulting values of crack width/strain (w/ϵ) obtained for all lines in geometrically similar locations on the section (for example, all lines directly over bars or all lines midway between bars) were then combined, and the resulting frequency distribution used to obtain values of w/ϵ with various probabilities of being exceeded. In Fig 8, values of w/ϵ with a 5% chance of being exceeded are used, which have been obtained for those points on each member where the largest crack widths would be expected. In most cases, this is midway between bars for the slabs and on the corner of the tension face for the beams. Each single point on the graph may thus derive from many as 70 cracks measured at each of six load stages—about 400 values of w/ϵ. The coefficient of variation obtained from the comparison shown in Fig 8 is 17%, which is a considerable improvement on the performance of other crack formulae. A survey of a number of other formulae is included in reference 14.

Matters not considered in the general derivation

There are a number of areas where the theory outlined in the previous section cannot be applied directly and where further development may be required. These will be looked at very briefly.

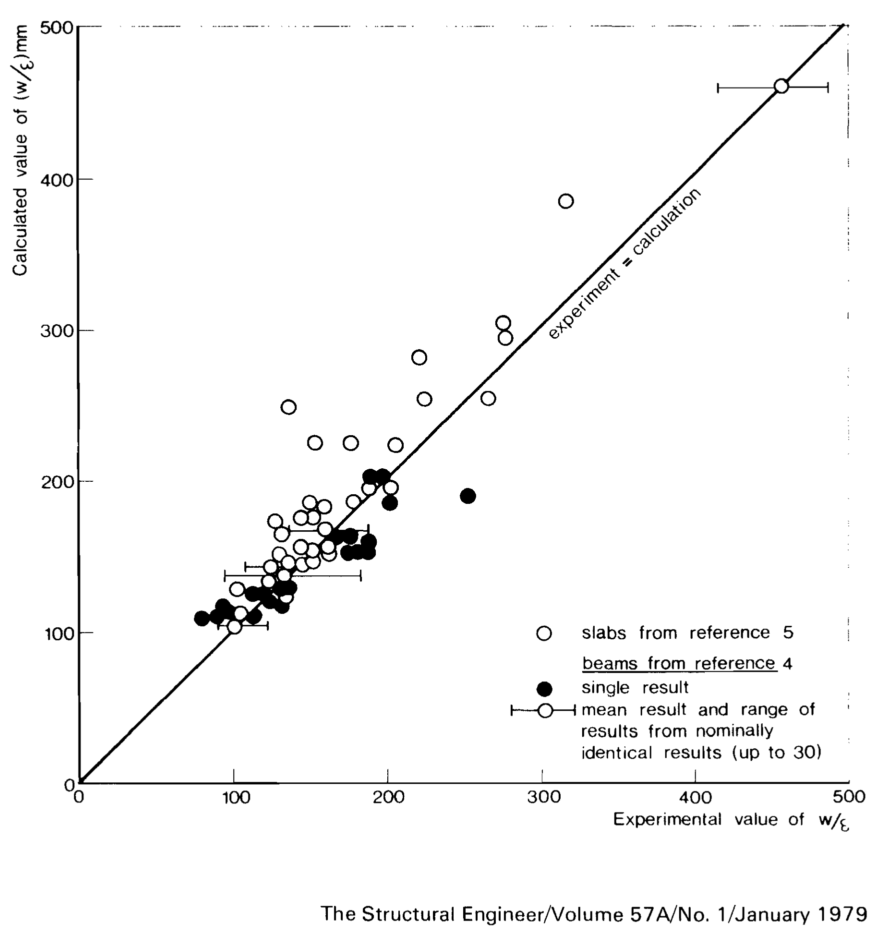
1. Pure tension in walls or slabs

In pure tension, w_{lim} becomes infinite and equation (6) reduces to:

$$w = \frac{a_{cr}}{c} w_0 \epsilon_m$$

Fig 9 shows the variation in crack width which the formula predicts over the surface of a wide member subjected to pure tension. It will be seen that, midway between the bars, the equation results in a cusp where the lines, drawn from the bars on either side, intersect. It is highly unlikely that such behaviour actually occurs and, in fact, there is some experimental evidence to suggest the contrary[18]. Some variation in width such as that shown by the broken line in the Fig is what would be expected. Theoretically, in the same way as one expects the cracking controlled by the crack height to inter-

Fig. 8. Comparison of experimental and calculated values of crack width/strain (w/ϵ) with a 5% chance of exceedence



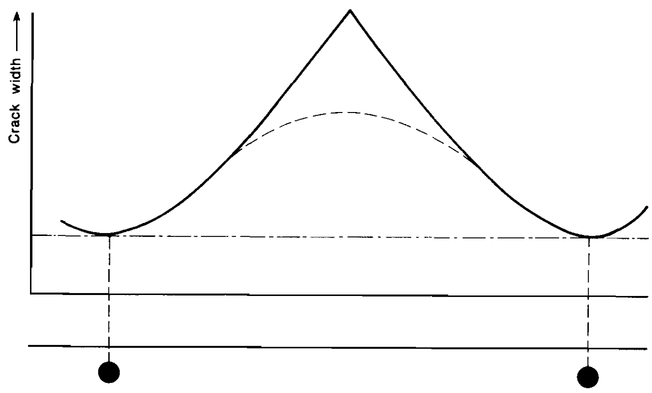


Fig 9. Variations in crack width in a member subjected to pure tension

act with the cracks controlled by the bars, it is reasonable to expect that the cracks controlled by one bar will interact with those controlled by an adjacent bar. However, the interaction of the variables involved is likely to be complex. Unfortunately, there are not enough test results available to allow any modification to be proposed to the formulae for this case.

2. Cracking where the principal tension is not parallel to the bars

This occurs in some types of slab and also in areas of higher shear in beams.

The situation is solid slabs has been investigated in detail by Clark[9], who has concluded that the theory remains valid and that the equations can be applied, provided that the reinforcement ratio is modified to allow for the effective area of steel acting in the direction of the principal tension.

The problem of the prediction of shear cracking has not yet been resolved, but is currently the subject of testing in a number of laboratories.

3. The influence of transverse bars

Transverse bars can act as crack formers, and it is clear that, where a crack could be expected to form roughly in the region of a transverse bar, the crack will almost certainly form along the line of that bar.

In some circumstances, the crack-forming influence can be so strong that the crack formation, and hence the crack spacings and widths, are entirely controlled by the spacings of the transverse bars. In these circumstances, the formulae become irrelevant. The problem is that it is not clear in exactly what circumstances this will happen, though it seems to be relatively unlikely in normal reinforced concrete structures.

4. The influence of the surface strain in the concrete between cracks

It was mentioned earlier that the rigorous formulation of the relationship between final crack spacing and crack widths was:

$$w_m = S_m(\epsilon_m - \epsilon_{cm})$$

and that, normally, ϵ_{cm} could be ignored. There are, however, situations where this may lead to problems. For example, Clark[10] found that, in models, concrete exhibited a higher effective tensile strain capacity. The term ϵ_{cm} was not negligible, and allowance had to be made for this if the model results were to be used to predict prototype cracking.

5. Bars other than deformed bars

The coefficients given in Table 2 refer to deformed bars. The behaviour of plain bars or smooth-drawn wires as used in prestressing is somewhat different, and the internal failure will more commonly be slip rather than internal cracking. This should not influence the basic formulae, but will require different values for K_2.

6. Prestressing

The basic principles clearly apply to prestressed concrete. However, there is a problem in choosing a suitable crack height, h_{cr}. In reinforced members with normal levels of reinforcement, the crack will immediately form to a level close to the neutral axis calculated on the basis of a cracked section, and will not increase in height much thereafter. This is not necessarily so with partially prestressed members, where a steady increase in crack height with moment is possible. It will be safe to assume a crack height equal to the depths of the tension zone under the load considered, but tests show that rigorous treatment of the problem is somewhat more complex than this. This is discussed in reference 8.

7. Early thermal movements

Much work has been done on this problem by Hughes[19], who has come to rather different conclusions about cracking in this case. It may be that, where the movements occur at a very early age, the very different properties of the concrete invalidate the basis of the theory outlined above.

Estimation of strains

So far, only the estimation of the final crack spacing has been discussed, and problems associated with estimating the average strain, ϵ_m, have been ignored. Obviously, the accuracy with which crack widths can be predicted depends as much upon the accuracy with which ϵ_m can be estimated as it does upon the accuracy of estimation of the final crack spacing. A maximum value for the strain can be calculated on the basis of a cracked section, and a number of Codes use this value. However, this can provide a very substantial overestimate of the strains since, even after cracking, the concrete between the cracks carries considerable stress and effectively increases the stiffness. It has been found that this effect, commonly referred to as tension stiffening, can conveniently be dealt with by calculating the strain on the basis of a cracked section and then subtracting an appropriate tension stiffening allowance. A number of empirical equations have been developed for the prediction of this allowance, and a reasonable general format for such an equation is:

$$\Delta\epsilon = K\frac{f_t f_{scr}}{E_s \rho f_1} \qquad \ldots.(7)$$

where

- $\Delta\epsilon$ is the tension stiffening correction at the level of the reinforcement
- f_t is the tensile strength of concrete
- f_1 is the steel stress under load considered, calculated on the basis of a cracked section
- f_{scr} is the steel stress at the cracking, calculated on the basis of a cracked section
- ρ is the reinforcement ratio
- E_s is Young's modulus for steel
- K is a constant that depends upon bar type and the way in which ρ is calculated

Under sustained loading or repeated loading, tension stiffening decreases, and it is by no means clear just how much, if any, tension stiffening it is reasonable to include in design. Prudence might suggest ignoring tension stiffening. However, if this were done, calculations of deflection and cracking would indicate that much current construction, which experience shows to be satisfactory, was apparently unsatisfactory.

Development of design procedures in CP 110

Clearly, the use of equations (4), (5), and (6) in design was unpractical, and considerable simplification was required before an equation suitable for inclusion in CP 110 was possible. Also, some thought had to be given to what should be predicted.

It had been decided that serviceability conditions (cracking and deflections) should be checked under the characteristic loads (i.e. with $\gamma_s = 1{\cdot}0$). It was recognised that, from the point of view of cracking, this was not strictly logical. The characteristic load is nominally one that has a 5% chance of occurring during the structure's life and, clearly, a crack that has this very low probability of occurrence is unlikely to appear often enough, or for long enough, either to pose a corrosion risk or to impair appearance seriously. However, it was felt that to require calculations for loads other than characteristic would introduce unacceptable extra complications into the design process. Thus, instead of designing for the characteristic crack width under lower than characteristic loads, as is done in the CEB Recommendations[16], it was decided to design under the characteristic loads for a crack width with a probability of occurrence higher than characteristic. It was therefore decided to formulate the crack width equation so that it predicted a width with a 20% chance of being exceeded rather than the characteristic value of 5%.

The basic equations for the prediction of crack widths given in the previous section were accepted, but had to be simplified for design use. Firstly, the crack height, h_{cr}, was assumed to be proportional to $(h-x)$.

This simplified the equation for w_{lim} with a 20% chance of exceedence to:

$$w_{lim} = 1 \cdot 5\,(h-x)\,\epsilon_m$$

Secondly, the equation for estimating the cracking directly over a bar (equation (3)) was more drastically simplified to:

$$w_0 = 3c\epsilon_m$$

The justification for this is that, with increasing distance from a point directly over a bar, the width approaches w_{lim} and the influence of w_0 decreases. The limiting condition in design will almost always be the width at maximum distance from a bar and rarely, if ever, the width over the bar. This being so, the influence of w_0 upon the critical design width will be at a minimum, and a fairly gross approximation can be adopted without seriously compromising the overall accuracy of the method.

It can be shown, by application of equation (3) to typical situations, that the most important variable controlling the crack width near a bar is the cover and that the influence of ϕ/ρ in flexural situations is usually secondary. Hence, it seems reasonable to neglect the ϕ/ρ term in equation (3) and increase the coefficient K_1 to allow for this.

If the simplified equations given above for w_0 and w_{lim} are substituted into equation (6), and the result is rearranged, the CP 110 equation will result:

$$\text{Design width} = \frac{3a_{cr}\epsilon_m}{\left(1 + 2\,\dfrac{(a_{cr}-c)}{(h-x)}\right)}$$

Drastic simplifications have also been made to the tension stiffening equation (equation (7)). Firstly, it has been assumed that $f_t f_{scr}/E_s$ is roughly equal to $0 \cdot 7 \times 10^{-3}$, and secondly that f_1 can be taken as $0 \cdot 58 f_y$. This results in the relationship:

$$\Delta\epsilon = \frac{1 \cdot 2bh}{A_s f_y} \times 10^{-3}$$

This gives the correction at the tension face, and further adjustment is required to reduce this figure linearly to zero at the neutral axis. This gives the final formula given in appendix A of CP 110:

$$\epsilon_m = \epsilon_1 - \frac{1 \cdot 2 b_t h(a'-x)}{A_s f_y (h-x)} \times 10^{-3}$$

where

- ϵ_m is the average strain at level where cracking is being considered
- ϵ_1 is the strain at level considered, calculated on the basis of a cracked section
- b_t is the breadth of the section at the steel level
- h is the overall depth
- x is the neutral axis depth
- A_s is the tension steel
- a' is the distance from compression face to point where cracking is being considered
- f_y is the characteristic steel strength

Even these formulae are considered too complicated for general use, and so they have been used to derive a series of 'deemed to satisfy' rules for bar spacing which should ensure that cracking is not serious in normal members. These are given in clause 3.11.8.2 of CP 110, and their derivation is dealt with in the handbook to the Code[20].

The Code for water retaining structures, BS 5337, employs slightly different versions of these formulae. A design width with a 5% probability is aimed for rather than the 20% in CP 110. Further, the assumption used in CP 110 that $f_1 = 0 \cdot 58\, f_y$ is not used in BS 5337, as it is definitely inapplicable in many water retaining structures where steel stresses can be relatively low. With hindsight, it can be seen that it would have been better not to have introduced this assumption in CP 110 either. It decreases the generality of the equation for tension stiffening without giving any real simplification, since f_1 has to be calculated anyway.

Conclusions

This paper has attempted to describe, in relatively non-mathematical terms, a theory for the cracking of hardened concrete. The approach used has been to describe the historical development of cracking theory, in order to indicate a continuity in the development of ideas and to suggest that the theory, as finally developed, is a logical development of earlier theories. However, an attempt will here be made to express in a different manner the basic principles involved in the final theory.

The cracking at any point on the tension zone of a member is the result of an interaction between two basic crack patterns:

1. A crack pattern controlled by the initial height of the cracks

The only influence that reinforcement has upon this pattern is in controlling the crack height. The crack widths and spacings produced by this type of cracking are proportional to the initial crack height, h_{cr}. Thus:

$$w_{lim} = K_1\, h_{cr}\, \epsilon_m$$

2. A crack pattern controlled by the proximity of the reinforcement

This pattern will depend upon the cover to the reinforcement, the bar diameter, the steel percentage related to an area of concrete immediately surrounding the bars, and the bond qualities of the steel. This cracking can be predicted by using a relationship of the form:

$$w_0 = \left(K_1\, c + K_2\, \frac{\phi}{\rho}\right)\epsilon_m$$

This cracking will occur in axially reinforced tension members.

In the above relationships:

- h_{cr} is the crack height
- ϵ_m is the average strain
- c is the cover
- ϕ is the bar diameter
- ρ is the effective reinforcement ratio
- K_1 and K_2 are constants

In flexure the second of these patterns dominates at points on the member surface directly over a bar, except that K_2 will decrease as the ratio of cover to crack height increases. With increasing distance from a bar, the cracking approaches w_{lim} asymptotically. The interaction is described by the relation:

$$w = \frac{a_{cr}\, w_0\, w_{lim}\, \epsilon_m}{c w_{lim} + (a_{cr}-c)\, w_0}$$

where w is the crack width at the point considered, and a_{cr} is the distance from the point considered to the surface of the nearest bar.

The paper has shown that the formulae in current Codes of practice are derived by simplification from these equations.

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Discussion

The prediction of crack widths in hardened concrete

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This paper was presented at a meeting of The Institution of Structural Engineers, 11 Upper Belgrave Street, London SW1X 8BH, on 22 March 1979, with Professor Sir Alan Harris, CBE, BSc(Eng), FEng, FIStructE, FICE (President), in the Chair, and published in The Structural Engineer, *Vol. 57A, No. 1, January 1979, p.9.*

Professor B. P. Hughes (F) (University of Birmingham): I should like to congratulate the author on a most interesting and useful paper, and for his presentation of the theoretical background to the crack width predictions given in the current British Codes of Practice. Dr. Beeby has asked, 'Is the picture reasonable and right, and what limitations has it?'

At one extreme there is cracking in flexure, where a strain gradient is present in the mature concrete—a case with which Dr. Beeby has been very much concerned. At the other extreme, Dr. Beeby also referred to cracking in immature concrete in direct tension due to early thermal contraction. However, I think that it is in the intermediate situation of direct tension in mature concrete, which Dr. Beeby has also covered in his paper, that the prediction of crack widths is most difficult.

Coming specifically to limitations, Dr. Beeby has already stated that many formulae have been proposed for crack widths in mature concrete. However, I should like to refer to one further investigation—that by Illston and Stevens[1]—and to two of their conclusions. First, although there is little slip initially between the concrete and the steel, they showed that crack widths can subsequently widen fairly extensively under prolonged loading and that slipping between the concrete and steel does occur. Second, they recommended that it was preferable to take distances to the centre of the bar rather than to the perimeter. As far as applications in current British Codes are concerned, this would seem to be a very reasonable simplification, since it is preferable to working out distances to the bar surface. If this has some technical merit as well, I would recommend that the simplification of measuring distances to the bar centres be considered when CP110 is revised.

Dr. Beeby: Professor Hughes' suggestion that the distance to the centre of the bar should be used, rather than that to the surface, is interesting. I feel myself that it is more logical to work to the surface of the bar, since the phenomenon begins on the surface not at the centre of the bar. From a practical point of view it may be that there is not too much difference, and perhaps a formula that used the bar centre would be almost as good. However, as to simplifying Codes, the complete crack prediction formula appears only in the appendix, and I do not see that people would use it very often. My experience is that people will normally work by simply using the deemed-to-satisfy bar spacings and that crack widths are very seldom calculated. If the calculation is going to be carried out only on rare occasions, minor simplifications are probably of no great value.

Dr. A. D. Edwards (Imperial College of Science and Technology): I should like to congratulate Dr. Beeby on his fine exposition of the prediction of crack widths in hardened concrete. He has given us a detailed insight into the history and development of crack width formulae, and has also described some effects of path dependence, i.e. strain history. I should like to examine this further with the aid of some theoretical results[2,3] obtained at Imperial College.

Most crack width experiments are so conducted that the load increases monotonically between periods of constant or slight decreasing load, during which the observations are made. Until the stable pattern is achieved, the initiation of cracks must be progressive.

A detailed analysis of the post-cracking behaviour of structural concrete could not be made until the advent, in the late 1960s, of large storage computers. Typical of the results obtained by finite elements at this time are overall crack patterns and load deflection characteristics. However, before analytical parametric studies can be carried out with confidence, a more sensitive comparison of predicted and experimental results must be made. Experimental strain fields are not easily come by, but there is an ample fund of crack widths. We concentrated on 2- and quasi 3-dimensional analysis. Included in the latter was the prediction of internal cracks as found by Goto (Fig 4 in the paper). This was carried out using axisymmetric elements. Fig 1 shows the internal crack pattern of a cylindrical concrete specimen, reinforced axially, subjected to loading by pulling the reinforcement. As ever, symmetry is taken into account and only one-quarter of a longitudinal cross-section is shown.

One of the beam tests that we chose to simulate 2-dimensionally was a partially prestressed I-beam tested Desai[4] at Leeds. It was subjected to 2-point loading. In order to contain computer time, one has to use relatively large increments of load and, in our analysis at a load of 11·7 kN, all the elements in the bottom of the flexural span were above the assumed failure stress. However, in an attempt to reproduce what would happen under monotonically increasing load we allowed only the most highly stressed element to crack during each iteration. Thus the first crack appeared near the load point; the second did not appear until the first had crossed the steel, while the third did not appear until the second had crossed the steel, as shown in Fig 2. Proceeding in a similar manner, the horizontal crack widths in the flexural span at the service load of 20 kN were obtained. The four cracks marked 'I' (Fig 3) initiated not at the surface, but adjacent to the steel, and propagated in both directions. Another four cracks, marked 'X', have a smaller width at this load than previously. The maximum crack width is often at mid-height. The crack width at steel level of the shorter cracks can be of the same order as the maximum crack width at the steel level of the longer cracks. These last two observations are similar to those made by Dr. Beeby when reporting tests on reinforced concrete specimens in reference 6 of the paper.

Fig 4 compares the theoretical and experimental crack widths obtained at different steel stress levels. The experimental stress is calculated according to normal beam theory. The analytical steel stresses and crack widths are reported for both the top and bottom of the bar. The analysis took into account only deterministic material and steel–concrete interface characteristics but, because of the type of loading and the progressive nature of cracking, the spread of the calculated crack widths is similar to the spread of the experimental ones. That is not to say that we discount the inherent random nature of cracking—we are only too conscious of the simplifications made in our analysis.

In conclusion, I should like to ask Dr. Beeby whether he has witnessed, in his many tests, crack propagation similar to that shown in Fig 2, and whether he has seen such a large percentage of cracks closing in any one test as suggested by Fig 3.

Dr. Beeby: I am very interested to see the close qualitative agreement which Dr. Edwards seems to have obtained between his finite element modelling and the type of behaviour observed in practice. This certainly indicates that his approach may have great potential as a means of understanding just what occurs in the region of a crack.

Dr. Edwards has asked two specific questions. Firstly, is observed crack propagation similar to that shown in Fig 2? There are two aspects of crack propagation which may be noted.

- —In practice, cracks develop to some considerable length before the next crack forms. There is commonly a degree of instability at a section when a crack forms, and it is necessary for the crack to develop some way before equilibrium between internal and external forces can be re-established. This initial crack height is smaller for prestressed beams than for reinforced sections.
- —Dr. Edwards' cracks develop sequentially from the support. Under uniform bending, I do not believe I have observed this. That it occurs in Dr. Edwards' analyses, and not in practice, is possibly due to there being quite large random variations in tensile strengths from point to point along the member in practice.

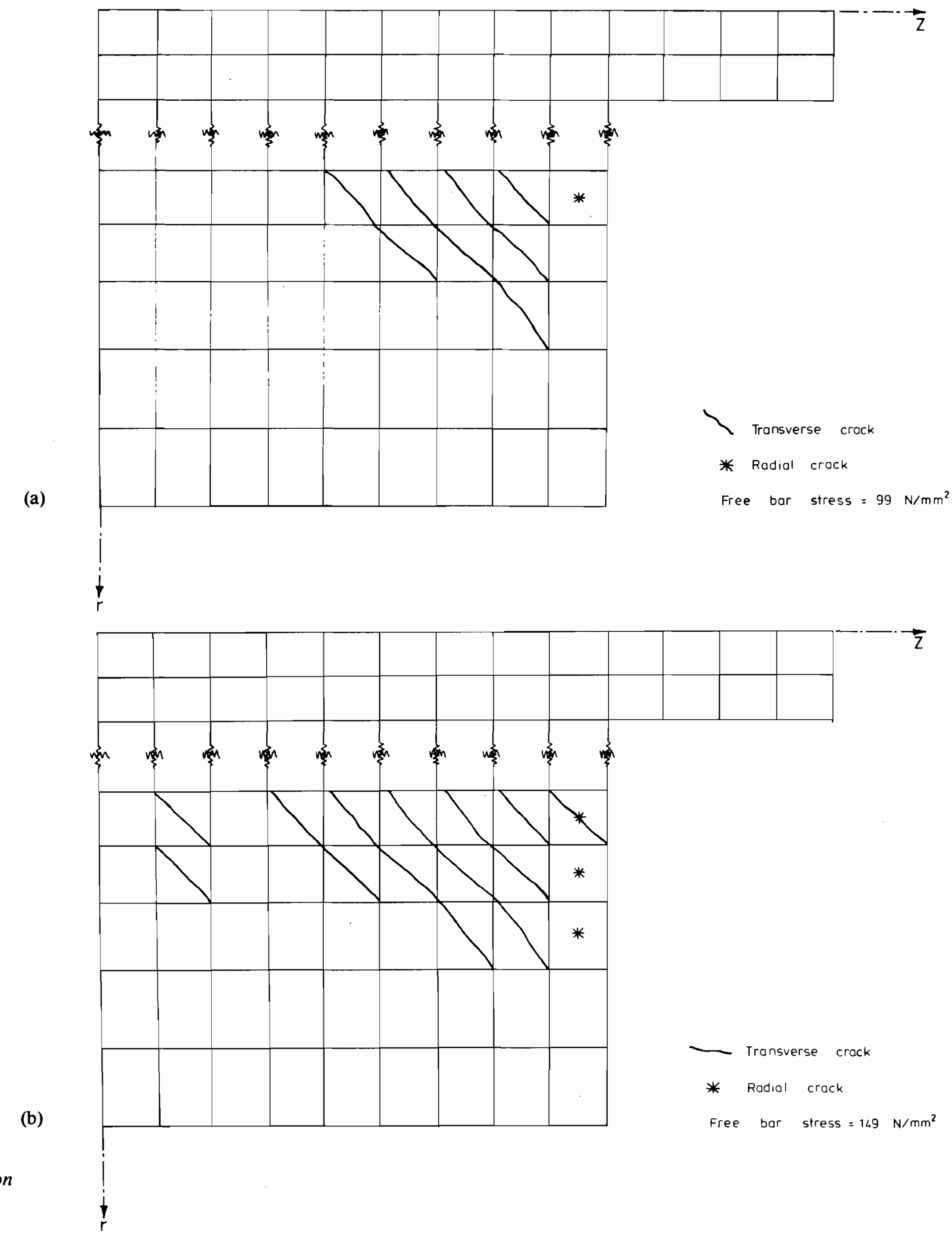

Fig 1. Formation and propagation of internal cracks in reinforced concrete tension specimens

The second question relates to the closure of some cracks with increase in load. I have rarely observed cracks actually closing, but it is not uncommon for a crack to open to some small width and then remain at that width while those around increase with increasing load. I suspect that, in practice, the act of cracking dislodges small particles into the crack which inhibit complete closure. Thus, I do not think that the analysis is really giving results that contradict the experimental evidence.

Dr. Edwards makes the point that there did not appear to be any great difference in width between long cracks and short cracks at the steel level and also that the maximum width was often well above the steel level. Both these conclusions agree with our observations.

Mr. J. D. Peacock (F) (Bison Concrete Ltd.): It is time a contractor had something to say, to balance the academic discussion so far. I am grateful to Dr. Beeby for giving me an understanding of crack propagation. It is a pity that Dr. Edwards does not entirely agree—perhaps if Drs. Edwards and Beeby could get together and produce a unified theory, I could then have a better understanding!

There is a practical need to be able to predict crack widths. This arises because the use of Code span/depth ratios (prior to CP 110) is, I believe, responsible for much of the cracking that has occurred, and we have to consider whether this cracking is reasonable by using the information available today.

CP 110 gives advice on the calculation of crack widths, but does not contain all the information that is necessary for the calculation to be completed. It would therefore be better to omit this advice from the Code.

Bearing in mind the number of cracks that have to be investigated, it is good to know that the subject can now be openly discussed.

Dr. Beeby: I am not absolutely certain as to your precise problem with the formula in the Code. Certainly, at the moment there appears to be a practical need for predicting crack widths, and certainly all three current British structural concrete Codes contain formulae.

Dr. L. A. Clark (M) (University of Birmingham): I would first like to say how much I welcome this paper because, up until now, there has not been a paper that summarises all the work that Dr. Beeby has done on crack control. I think that this is a great pity because a lot of his work has been misunderstood in some circles. I am sure that the paper we have heard tonight will redress this situation.

I should like to ask Dr. Beeby to explain his crack-data collecting technique, which I know to be misunderstood by many.

Dr. Beeby asked us two questions. First, 'is the theory reasonable?' As a former colleague, and collaborator, I am virtually honour bound to say 'yes'! However, I do firmly believe that it is a reasonable theory, since it is both elegant and simple. It is the simplicity of the theory that reinforces my faith in it.

The second question was, 'what are the limitations?' In his presentation, and in the paper, Dr. Beeby has concentrated on cracks that cross the reinforcing bars at right angles. The theory that he has presented is limited in that it is applicable to that situation, but not necessarily applicable to the situation where the cracks cross the reinforcing bars at an angle. This situation can occur in a number of structures. It can occur in skew slab bridges, where the principal moment directions vary widely over the surface of the slab and since, at the slab surface, the cracks tend to form normal to the principal moment directions, it is obvious that situations arise where the cracks are not normal to the reinforcing bars. Skew cracking can also occur in regions of high shear, such as deep beams.

Dr. Beeby has indicated in his paper that I have attempted to extend his work to cover the skew cracking situations, and I should now like to summarise the main conclusions of this work. Firstly, at the service load, it is reasonable to assume that the cracks form normal to the principal stress directions. The second point is that it is necessary to carry out the crack width calculations in the direction that is normal to the crack. It is thus required to resolve all the reinforcement into a direction that is normal to the crack, and this is done by multiplying the individual steel areas per unit length by the fourth power of the cosine of their respective orientations to the normal. This effective steel area is then used to calculate the neutral axis depth, the crack height, the strain, and the tension stiffening effect; all of which are parameters in Dr. Beeby's crack width formulae. Dr. Beeby's equations are applied to each set of bars individually, and the distance from the bar to the point at which the crack width is required to be known is measured normal to the bar, rather than along the crack. The fourth power resolution referred to above is derived approximately as follows.

Consider a set of reinforcing bars having an area per unit length of A_i and an elastic modulus of E crossing a crack at an angle α_i to the normal to the crack. The steel force per unit length is given by $F_i = A_i f_i$, where f_i is the steel stress. The component normal to the crack of this force per unit length is $F_n = F_i \cos^2 \alpha_i$. The steel strain is $\varepsilon_i = f_i/E$, and in order to develop this strain a larger strain (ε_n) must occur normal to the crack. Ignoring any transverse or shear strain, to simplify the presentation, $\varepsilon_n = \varepsilon_i \sec^2 \alpha_i$. Hence, the stiffness normal to the crack per unit length $= F_n/\varepsilon_n = EA_i \cos^4 \alpha_i$. The stiffness can also be defined as E multiplied by an effective steel area (A_n) per unit length; hence, $A_n = A_i \cos^4 \alpha_i$.

The next point I should like to mention is crack control in the new Code of Practice for Bridges (BS 5400), where the clauses are a little more complicated than those in CP 110. For example, different formulae for beams and slabs are given, but also tension stiffening is ignored when calculating the strains in beams because it is envisaged that beams in bridges are heavily reinforced (in which case the tension stiffening is small and can be ignored), whereas slabs are likely to be more lightly reinforced and tension stiffening is then taken into account.

In addition, there are a number of deemed-to-satisfy rules which mean that a calculation is not required for all structures, e.g. slab bridges.

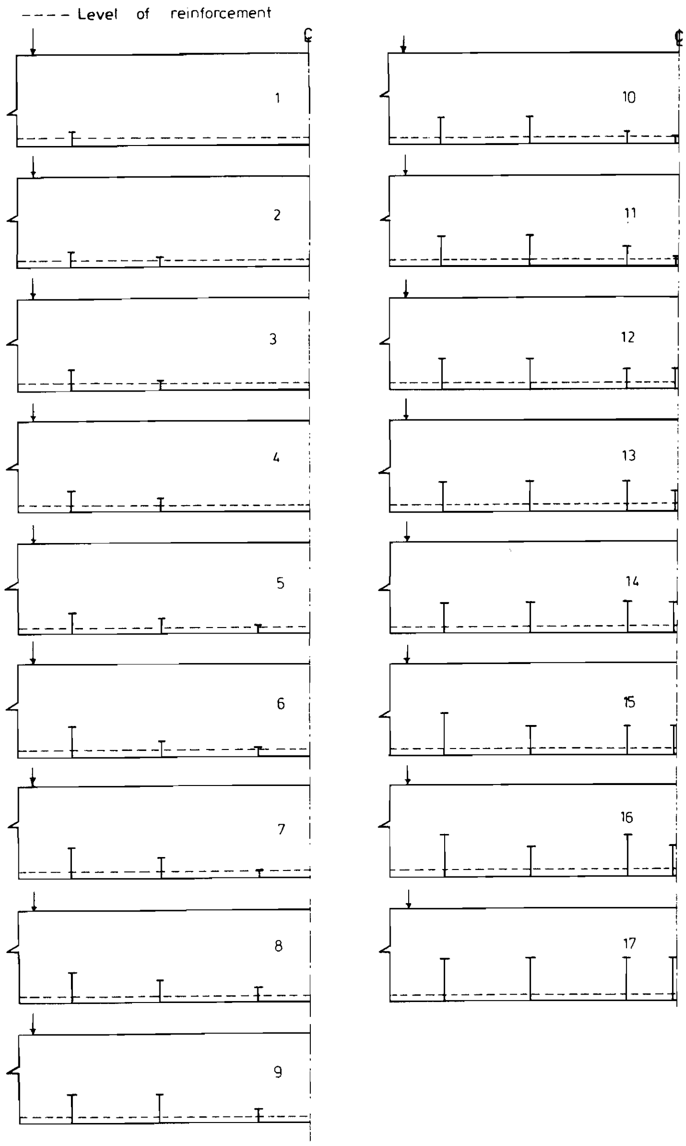

Fig 2. Sequence of cracking of the first flexural cracks at 11·70 kN (cracking load)

Finally, with regard to the question asked by Dr. Edwards concerning whether Dr. Beeby had noticed cracks closing up at later stages of cracking—in some of my tests on skew slabs, which have rather complex stress fields, I have noticed that some cracks do close, but this is compensated by the fact that other cracks are opening somewhere else.

Dr. Beeby: Dr. Clark has asked for a detailed explanation of the crack-data collecting technique used in the C&CA tests. Briefly, this was as follows. All the tests with which I was involved were on sections of beam

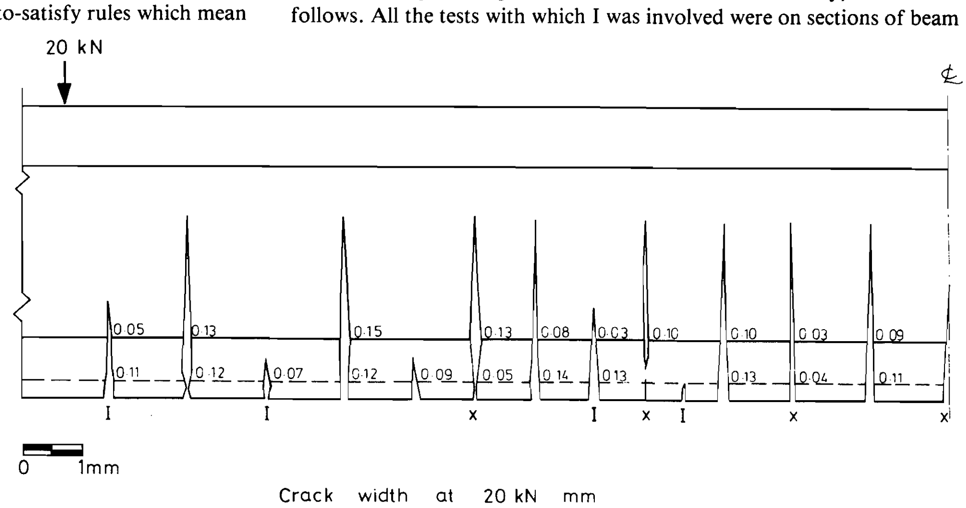

Fig 3. Crack width at 20 kN mm

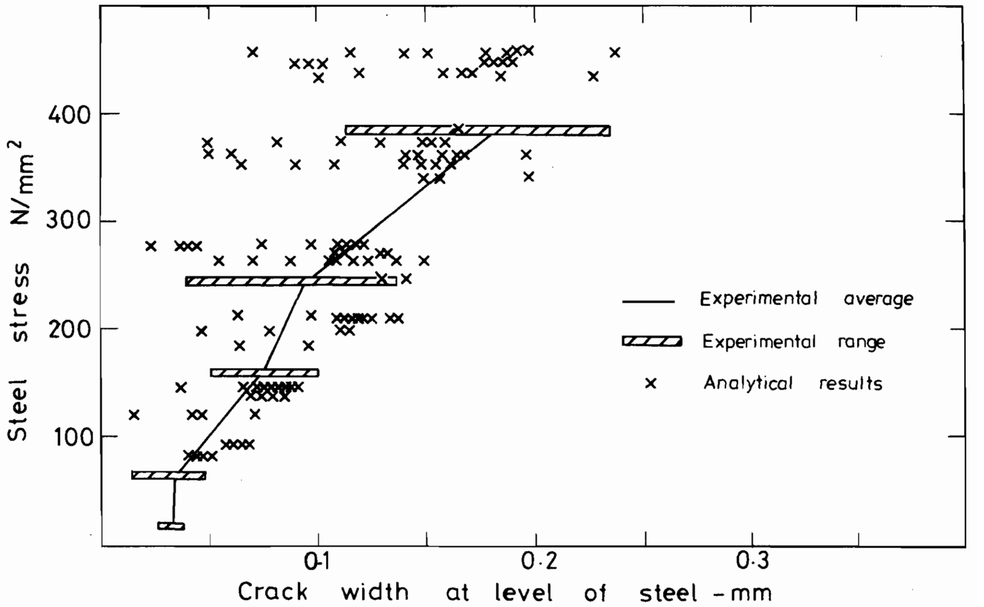

Fig 4. Crack width at level of steel (mm)

under uniform bending or members subjected to uniform tension. In all cases the bars were parallel to the principal tensile stress. Grid lines were drawn on the surface of the specimens parallel to the reinforcing bars. These lines might be on the surface directly over a bar or offset from the line of the bars. The tests were usually organised to give about seven load stages between cracking and failure. At each load stage, all the visible cracks crossing each grid line were measured, and the average strain along each grid line was obtained using a 'Demec' gauge. The crack width was assessed as the opening of the crack parallel to the grid line. Where the crack surfaces were too rough to allow easy measurement directly on the grid line, a search for a better spot was permitted within the region of about ± 1 cm of the line. It was found very early on in the investigation that the mean crack width, or maximum width or the width exceeded by any given percentage of the results was directly proportional to strain. This meant that all the widths obtained on any particular grid line during a test could be normalised by dividing each width by the average strain appropriate to the load stage at which it was measured. All the crack widths measured on a particular line could then be lumped together and be treated as a single population. This distribution was then used to define values of (crack width/strain) with specified chances of being exceeded (usually the mean and the values with 2 %, 5 %, 10 %, and 20 % chances of being exceeded). The final result from each test was thus a series of figures defining the distribution of the parameter (crack width/strain) for each grid line along which cracks were measured.

A feature of this procedure which has led to some discussion is the definition of crack width as the opening parallel to the grid line. This is illustrated for an idealised crack in Fig 5. It will be seen that the alternative definition of crack width as the opening measured perpendicular to the sides of the crack is largely meaningless. Furthermore, the strains are measured parallel to the grid line, and it seems logical to relate the strain to the crack opening in the same direction. Indeed, it is hard to see how any other approach could be expected to lead to meaningful results.

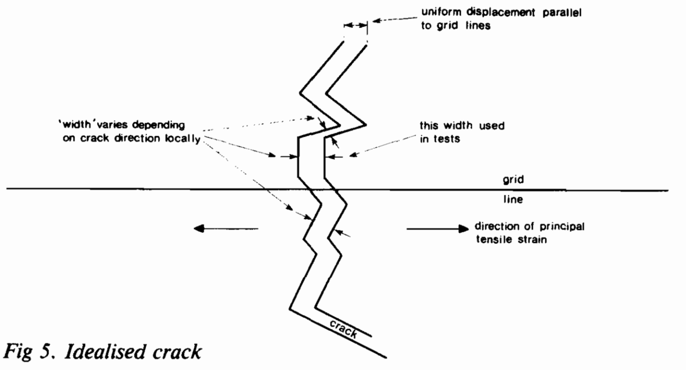

Fig 5. Idealised crack

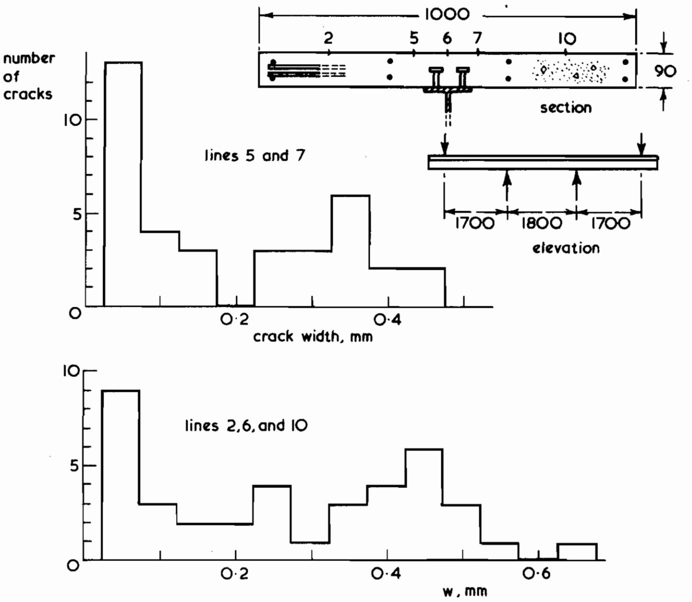

Fig 6. Crack widths in beam UC6, load stage 9

Professor R. P. Johnson (F) (University of Warwick): Dr. Beeby and his colleagues are to be congratulated on the tenacity with which they have studied this awkward subject over 14 years, and the clarity with which their conclusions have been summarised for us today.

We now have a good working theory for the problems they studied; but it is still an empirical theory. It applies only to fully developed crack patterns in reinforced concrete members, and may not be correct for other situations. One of these is the top flanges of continuous T-beams in tension over internal supports. The C&CA did very few tests, if any, on slabs acting in this way. The corresponding problem in continuous steel – concrete composite beams has been studied at the University of Warwick for the last 6 years, by means of tests on slab flanges in both uniaxial and biaxial tension, and by finite element analyses. This work is being prepared for publication, so I refer now only to some aspects of it that are relevant to Dr. Beeby's paper.

The first is that the design equations are intended to give 'crack widths that have a 20 % chance of being exceeded'. The meaning of this in relation to sets of test data is clear enough, but what does it mean in relation to a real cracked slab on which no measurements have been taken—the normal situation in practice? Most research has been done on regions of uniform mean strain, and these are rare in practice. Most grid lines for crack measurement run above bars or midway between them; measurements are rarely taken in the regions where lie the ends of the short cracks that form above bars, or where strain gradients are high.

The number of wide cracks in a region is easily found, but the number of narrow cracks depends on how powerful one's microscope is. It therefore seems illogical to define limiting crack width in a way that implies that the total number of cracks is known. To illustrate this, Fig 6 shows histograms of measured widths for a composite beam loaded in uniform negative bending over a length of 1·8 m. The elevation and dimensions of the concrete top flange are shown inset. The results show that many more cracks measured as 0·05 mm wide occur on lines 5 and 7 (in the region where short cracks over bars normally end) than on lines 2, 6, and 10 (midway between bars). The histograms do not show the cracks less than about 0·03 mm wide, which would not have been recorded. The number of narrow cracks is uncertain, and may have no effect on the widths of the widest cracks. Statistical methods based on whole populations of recorded crack widths may be unreliable when applied to bimodal distributions of this type. Our studies of other methods have not yet led to anything significantly better. This problem is one of several that introduce errors into the whole process of crack-width control.

The results of tests by R. W. Allison and P. Ogunronbi on seven beams like that shown inset in Fig 6 are shown in Fig 7. Each point gives the mean crack slope for all cracks on a grid line at all load stages, and represents 50 to 100 measurements of crack width. Test and theory agree well on average, but the ratios test/theory for the points have a coefficient of variation exceeding 30 %. One reason is that crack slope w/ε_m diminishes as mean strain ε_m increases, whereas the theory assumes that w is proportional to ε_m.

Is the theory good enough? That depends on how accurately one needs to predict crack widths—another subject altogether!

The last comment relates to Fig 9 on p.15 of the paper. The author points out that the theory (full line) is likely to overestimate widths of cracks at points midway between bars; the dashed line is likely to be more accurate.

Dr. C. Arnaouti has studied this question by means of finite element elastic analyses of concrete prisms 1 m long, loaded by applying tension to two longitudinal bars. Half a prism is shown in plan and elevation on the left side of Fig 8. Its free ends represent cracks. The variation in the width of each crack with distance from the bar can be deduced from the deformed shape of the end surface (assuming that there is no bond slip). The dashed curve extends midway to the next bar; the full curve to a free edge. We are discussing the difference between the shapes of the two curves, which can be shown by reflecting one of them about the axis of the bar, to lie above the other. Pairs of such curves are shown in Fig 8 for four values of the width b. (As drawn, the lower ends of each pair of curves coincide; for comparisons at constant tension in the bar, the dashed curves should be moved upwards, so that the upper ends of each pair coincide.) The curves show that for specimens of this shape the theory (full line) overestimates crack widths midway between bars by 16 %, which confirms Dr. Beeby's prediction.

Dr. Arnaouti has also completed five tests on cruciform composite girders in biaxial bending tension. The detailed conclusions are too complex to give here; but the main result is that, where crack widths are controlled in design to a low value, such as 0·1 mm, actual mean widths can be up to double those predicted. However, the error diminishes as design width increases, and is negligible for a design width of 0·3 mm.

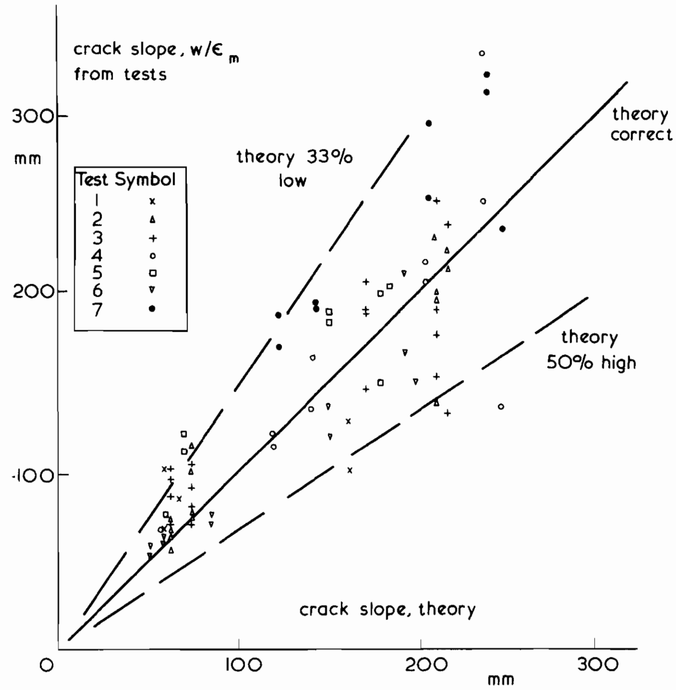

Fig 7. Crack widths in seven composite beams under negative moment, compared with prediction by the C & CA method (Report 42.468, June 1972) for members in pure tension

Dr. Beeby: Professor Johnson asks—what does a 20 % chance of exceedance mean in practical situations? Roughly, that, if you calculate a crack width for a particular point, then, if a crack forms at that point, there is a 20 % chance that it will be greater than the calculated value and an 80 % chance that it is less than, or equal to, the calculated value. From the design point of view I see no major problem in this type of approach, though there will be differing views as to whether the specified level of exceedance should be 20 % or some other value, possibly 5 %. If tests are carried out on members where there are rapidly changing strains, then certainly difficulties will arise in interpreting the results in any logical way, and I sympathise with Professor Johnson's problems.

Having said this, it does appear that there are significant differences

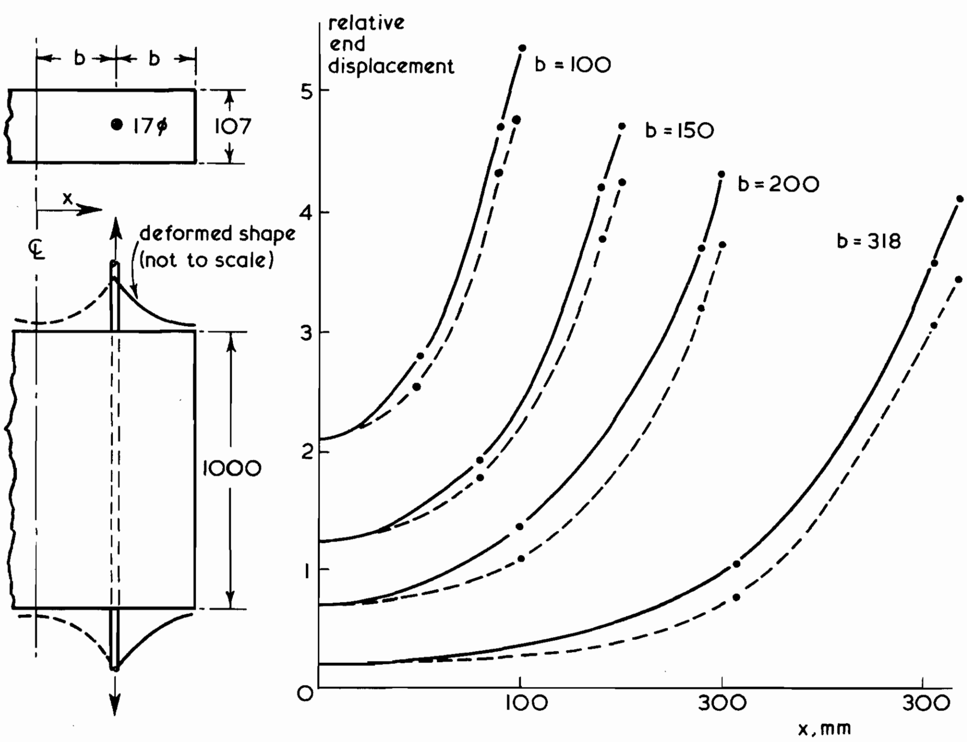

Fig 8. Finite element studies of crack width

between what Professor Johnson and his co-workers have found and what was found in the C&CA tests. Professor Johnson notes that the number of cracks found depended on the power of the microscope used and that the experimental results were thus incomplete. This would result in the assessment of a crack exceeded by 20 % of the cracks being an unreal exercise, since the total number of cracks was not known. This observation is at variance with what we found—on very few tests did searching with a magnifying glass, or even microscope, reveal any significant increase in the number of cracks discovered over the number found with the unaided eye. This did not seem too surprising to us. When a crack forms, stress is shed from a considerable volume of concrete around the crack. The reduction of strain in this concrete, together with the deformations necessary to permit a redistribution of the internal forces so that equilibrium is maintained, leads to an immediate substantial opening of the crack. This is rarely less than 0·01 mm in reinforced concrete sections with practical dimensions, and this width can be discerned by the practiced eye on a good surface.

Professor Johnson also states that he and his co-workers found that crack width was not proportional to strain. There can be little doubt that, in reinforced concrete, crack width is proportional to average strain to within very close limits.

The reasons for these differences between the results obtained at Warwick for composite members and our results from reinforced members unfortunately remain unclear. We are currently engaged on a series of tension tests on large slabs which might be expected to compare with the flanges of composite beams. It will be interesting to compare results in due course. So far, it appears that the differences in crack widths and spacings over the bars compared with those between the bars are less than the formulae in the paper suggest and that there are, as a consequence, fewer small cracks. This agrees with the suggestion I made in the paper, and also with Dr. Arnaouti's finite element studies.

Dr. Paul Regan (Polytechnic of Central London): We were asked to suggest limitations to the work, and I can suggest two.

I think that one arises in members subjected to axial tension. The purely geometrical method of predicting crack spacings, and thence widths, leads to the calculated spacings between bars becoming large. We have recently tested specimens in tension with bar spacings of only 200 mm. Using the formula in the paper, the spacings and widths of cracks between bars should be almost three times those over the bars. In fact, we found no real difference in the average spacing and only about 20 % differences in widths.

In the case of some German work[5] crack spacings predicted from the paper are in error by factors of up to 10 for large bar spacings. I believe the reason is that the purely geometrical approach ignores the significance of bond conditions. If the same tests are analysed by relating crack spacings to the sums of the perimeters of the bars the correlation is almost perfect.

The other area where the failure to treat bond conditions can lead to trouble is that of lightly reinforced members, particularly slabs. Dr. Beeby showed that discrepancies between different theories become much greater for slabs than for beams. We have made quite a number of tests of slabs with steel ratios down to 0·4 %. Taking the paper as a starting point, it is interesting to work out what bond stresses would have to have been developed in order to produce the predicted crack spacings near the bars. Using Dr. Beeby's 45° angle of spread of stress, for very ordinary slabs the average bond stress has to be of the order of four times the tensile strength of the concrete, and this seems improbable. In fact the experimental crack spacings over the bars were much greater than those predicted and corresponded to more believable bond values.

Dr. Beeby: I stated in the paper that I had doubts about the direct applicability of the formula to wall or slab type specimens subjected to pure tension, and we are investigating this further at the moment.

The statement that the theory is purely geometrical is not entirely true. As is indicated in the paper, the coefficient k_2 depends on the bond characteristics of the bars. In the case of the German tension tests cited by Dr. Regan, the cover is relatively small and, to all intents and purposes, the equations in the paper will indicate that the cracks should vary more or less in proportion to ϕ/ρ. ϕ/ρ is equal to a constant times the concrete area divided by the total bar perimeter. Since the concrete area was constant, the formulae in the paper also suggest that the cracking over the bars will vary more or less proportionately to the sum of the perimeters. It has already been noted that a lesser increase in width is to be expected than the formulae predict.

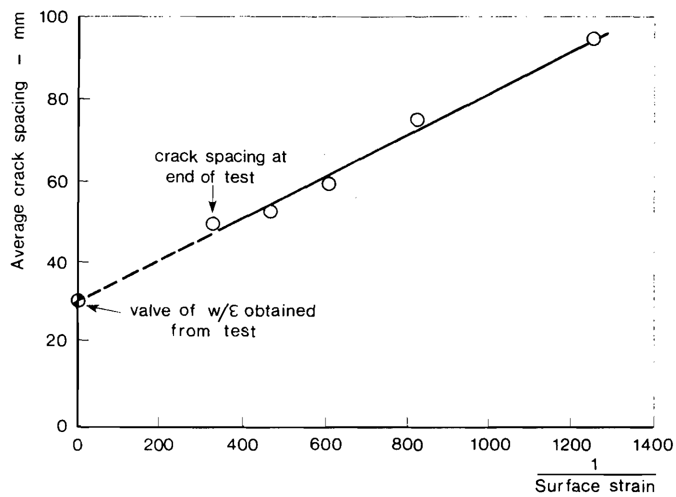

Fig 9. Relationship between crack spacing and surface strain (specimen SI)

Mr S. B. Desai (M): In marine environments, or for concrete in contact with soil, it is often recommended that the cover to the reinforcement should be 75 mm. This requires detailed crack control calculations, and it becomes problematic to restrict the crack width, simply because the steel is more remote from the external face. Would it not be more logical to check the crack width for a standard 'cover' distance (say, 25 mm or the diameter of bar) away from the tensile reinforcement? The enlarged crack width beyond this level may be considered not critical, and the 'extra' cover will stay as an extra precautionary measure.

Dr. Beeby: The point made by Mr Desai is a good one; I am quite convinced that, as far as durability is concerned, there can be no reason why the type of approach he suggests should not be adequate. Indeed, for very severe exposure, CP110 specifies a crack width limit of 0·004 times the cover. This is equivalent to checking for a crack width of 0·1 mm at a standard cover of 25 mm.

Dr. Clark: We have been hearing about the connection between crack widths and crack spacing, and I think that one should really think of this in terms of the final crack spacing and not an intermediate crack spacing. I believe I am right in thinking that a lot of problems have arisen in the past when researchers have related crack widths to spacing and they have done it for some intermediate spacing and not the final crack spacing that one would get theoretically at infinite strain. Perhaps Dr. Beeby would like to comment.

Dr. Beeby: What Dr. Clark has said is undoubtedly true—many workers have measured the spacings at the end of the test and presumed these to be the final spacing employed in the theory. In fact, the final spacing will not commonly be obtainable from tests. Fig 9 shows how crack spacing varies with 1/strain for a typical specimen from reference 5 in the paper. The final crack spacing is that which occurs when the strain approaches infinity ($1/\varepsilon = 0$). It will be seen that the value obtained at the end of the test was considerably above this. Equation (1) in the paper can be rearranged to give:

$$\omega_m/\varepsilon_m = S_m$$

A value for ω_m/ε_m can be obtained from the crack measurements, and this is plotted on the figure on the vertical axis. It will be seen that it is the value of S_m as ε_m ∞ that is required. Final crack spacings can be obtained from tests only by plotting the crack spacings in the way indicated in the figure and finding the intercept on the axis.

Dr. Clark: I should like to give a word of warning to people carrying out model tests and to give a possible reason for some of the results that we have been hearing about tonight. Strictly speaking, one should consider the strain minus the strain at which cracking occurs. In most full-sized tests the cracking strain is quite small compared with the total strain, but the cracking strain increases as the size of the model decreases and, to get a strain line relationship between crack spacing and the reciprocal of strain, it is necessary to plot the reciprocal of the strain minus the cracking strain. This could explain some of the disagreement that we have heard about tonight, since some of the tests in question were carried out on models.

Dr. G. Somerville (M): (Cement & Concrete Association): I wonder whether Dr. Beeby could elaborate on scale effect a little more. Dr. Clark referred to scale effect in terms of the size of the members, but perhaps there is rather more to it than that. We must remember that prototype members may come in a variety of sizes, and may contain different diameters of reinforcing bar, or indeed different sizes of aggregate in the concrete mix. How does this affect the whole picture? I think scale (and size) effect is very important in drawing general conclusions from models about a wide range of prototype structural members.

Dr. Beeby: In some respects, I am not the best person to answer this and maybe it is a buck that could be passed to Dr. Clark! We did try testing some slab units where otherwise identical specimens were made with 20, 10 and 5 mm maximum sized aggregate. No noticeable differences were found between these.

More generally, Dr. Clark has done a great deal of work on the modelling of cracking, and I hope that he will say a few words.

Dr. Clark: One of the best things that I can do is to refer you to reference 10 of Dr. Beeby's paper. Dr. Somerville referred to scale effects, but I would prefer to think in terms of size effects because I am convinced that it is the physical size, rather than the scale, of the model that is important.

For large models, the type of concrete has little influence on the scaling of crack widths, but the reinforcement does because, if one compares, for example, a 6 mm and a 25 mm bar, it is found that the bar deformations do not scale.

For small models, the type and size of the aggregate definitely has an effect on crack similitude, and the type of model reinforcement is extremely important. Plain wire model reinforcement tends to simulate prototype unbonded reinforcement; threaded rod reinforcement simulates deformed prototype bars very well; and the deformed wire type of model reinforcement, which is generally produced by machines based on a machine developed at Cornell University in the United States, tends to simulate plain prototype bars.

Dr. Regan: We have been told in the paper that there is no difference between plain bars and deformed bars. Now we are told that the type of reinforcement is important. I must admit that I get a little worried! If it is important in the model scale, I assume it is also important in the prototype?

Dr. Clark: In equation (5) of Dr. Beeby's paper, there is a factor K_2 which Dr. Beeby did not have time to discuss in his presentation. K_2 is a function of bar type and predicts that crack widths between the bars are influenced only slightly by the type of bar, but that the crack widths over the bars are influenced to a greater extent by the type of bar. The influence of the type of bar is greater for model than for prototype bars.

R. W. Allison (written contribution): The crack width formulae proposed by Dr. Beeby have been obtained from the equation:

$$w_m = S_m \varepsilon_m \qquad \ldots (1)$$

where

w_m is the average crack width

S_m is the mean *final* crack spacing

ε_m is the average surface strain

Equation (1) is a particular case of the equation of geometrical compatibility:

$$w_m = S(\varepsilon_m - \varepsilon_{cm}) \qquad \ldots (2)$$

where S is the mean crack spacing, assumed in equation (1) to have reached its final value, and ε_{cm} is the average residual surface strain in the concrete between cracks, assumed in equation (1) to be zero.

Hence, theoretically, the crack width formulae given in the paper are applicable only where the final crack spacing has been reached, i.e. where the stable crack pattern has formed. Reference to published results[6,7] shows that this does not occur until surface strains in the order of 2000 $\mu\varepsilon$ to 3000 $\mu\varepsilon$ are attained; before that, the average crack spacing may be considerably greater than S_m, and decreases as the strain increases. It is not obvious, therefore, that formulae derived from equation (1) can be applied to reinforced concrete structures at the serviceability limit state, where surface strains will seldom exceed 2000 $\mu\varepsilon$. Nor is it certain that equation (1) can be applied in all cases when the final crack pattern has formed. This may be demonstrated by the following argument.

It was shown in an earlier report[6] that an inverse relationship existed between mean crack spacing and average strain. Some test results from that report are shown here in Fig 10. As noted, the mean final crack spacing is reached when the strain is above 2000 $\mu\varepsilon$, so presumably the line AB in Fig 10 is the one that test results would lie on if it were possible to increase the strain further without causing yield in the reinforcement. The final crack spacing S_m is therefore greater than the spacing S_{oo}, obtained by projecting the line of best fit, CB, back to the vertical axis in Fig 10. However, in his reply to a contribution to the oral discussion, the author confirmed an earlier conclusion[6], i.e. that the value of (w_m/ε_m) was found in tests to be equal to S_{oo}, and therefore less than S_m. In the example illustrated here, the difference between S_{oo} and S_m is quite large, indicating that the error in applying equation (1) would be quite large also.

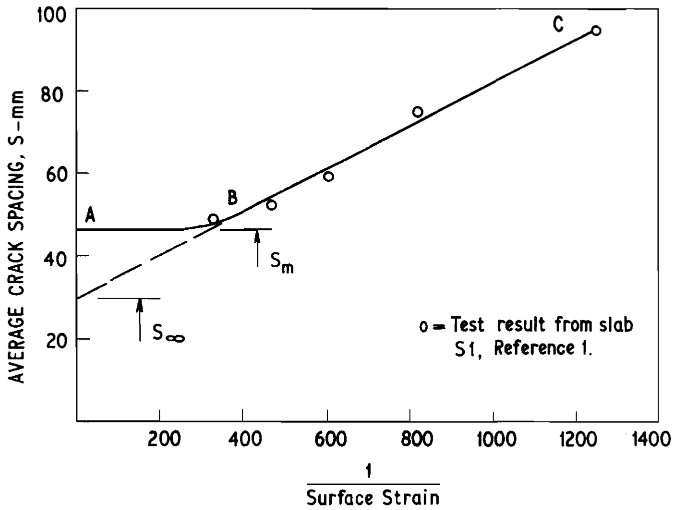

Fig 10. Plot of crack spacing against surface strain

Existing crack width formulae predict that crack widths are proportional to average surface strain, and this is, indeed, the behaviour observed during tests on reinforced concrete elements, irrespective of whether a stable crack pattern has been formed or not. Nonetheless, the relationship is an empirical one—although geometrical compatibility (equation (2)) does not preclude the possibility that crack widths will be proportional to strain, it can be used to predict this relationship only when the crack spacing has reached a constant value, S_m, and when ε_{cm} is either zero or a constant proportion of ε_m.

Dr. Beeby: Mr. Allison misses an important point. As discussed earlier, the significance of measured crack spacings has been a frequent cause of confusion, and some further clarification may help. If one inspects Fig 2 in the paper, it will be seen that the stress in the concrete is influenced by a crack only over a relatively short distance on either side of the crack designated s_o. The stress conditions in all other parts of the member are independent of the crack. The corollary of this is that the crack itself is influenced only by events that occur within this distance of $\pm S_o$ from itself. The crack width is thus a direct function of S_o and strain. The final crack spacing, S_{oo} is also related directly to S_o. The spacing in other situations is not, and it is quite irrelevant to the arguments what the actual spacing may be while it is greater than S_{oo}.

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