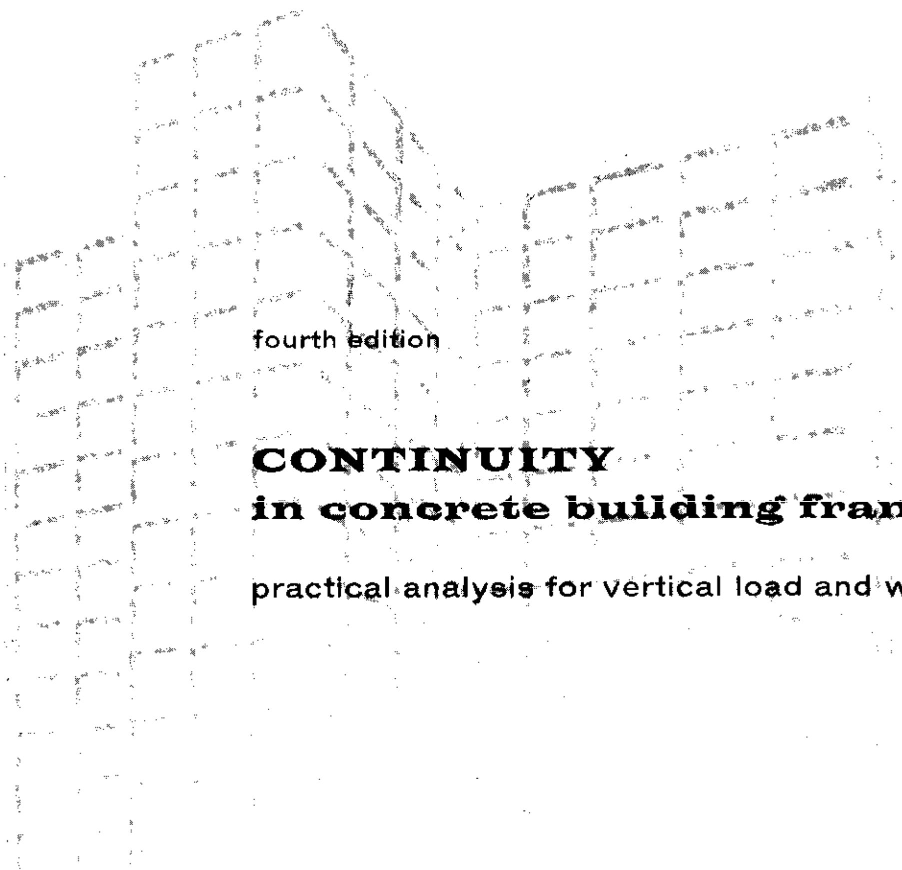
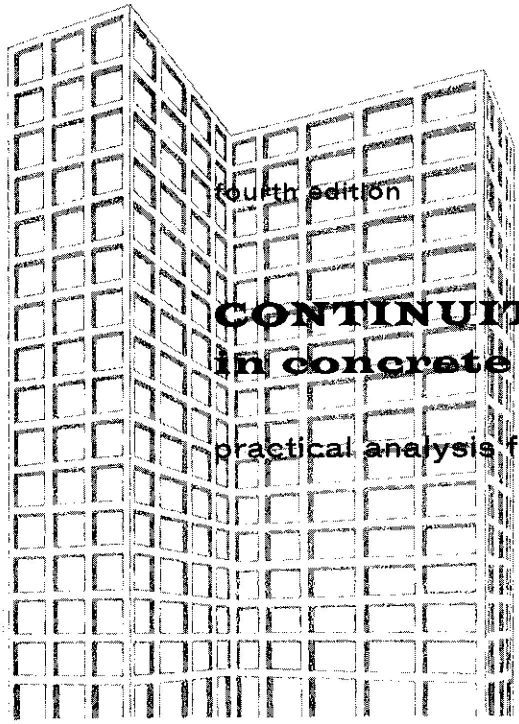


fourth edition

CONTINUITY in concrete building frames

practical analysis for vertical load and wind pressure

PORTLAND CEMENT ASSOCIATION



fourth edition

CONTINUITY in concrete building frames

practical analysis for vertical load and wind pressure

PORTLAND CEMENT ASSOCIATION

An organization of cement manufacturers to improve and extend the uses of portland cement and concrete through scientific research, engineering field work, and market development.

5420 Old Orchard Road, Skokie, Illinois 60077

preface

Continuous-frame analysis is a very important design subject for the structural engineer. In this field, he is confronted with the conflicting requirements of achieving sufficient accuracy and at the same time expending a minimum of effort and calculation. For this purpose, there are many analytical procedures available, such as the methods of elastic weights, virtual work, slope deflection and moment distribution. Each has certain advantages that make it specifically adaptable for particular conditions. In this text, moment distribution is treated in a manner suitable for office practice.

Interest in moment distribution had its origin in the presentation by Hardy Cross in 1929.* His method is applicable to even the most complicated frame problems. However, a condensed form was needed for ordinary building frame design in order to standardize certain features incidental to the analysis.

The moment-distribution procedure offered in this text is not a new method. However, it has been limited to two cycles for ordinary building frames. The two-cycle method of moment distribution has been tested over a period of years in the analysis of numerous building frames and in other work. The results have shown that the method speed and accuracy are of great assistance to designers. Some may choose to acquire a working knowledge of the mechanical details, which are readily learned and remembered. Others will consider it sufficient to use arbitrary coefficients. They will benefit by giving consideration to the tables included in this text for fixed-end moments, stiffness, points of inflection, and design of columns. These tables are also advantageous for those who continue to use individual types of analysis.

Section 22, "Design of Column Sections Subject to Combined Bending and Axial Load," has been revised for this edition. If designers adopt the procedure proposed, design of column sections subject to bending should be reduced from a time-consuming problem to one of simple routine.

Designers who do not wish to study the preliminary explanation and derivation may turn immediately to Section 10. However, a working knowledge of Tables 1 through

*See reference 3.

4 is needed. The special arrangement for two-cycle moment distribution is described in Sections 10 and 11. Subsequent sections treat supplementary problems.

The second part of this book, which is concerned with wind-stress analysis, is the same as in the previous edition.

The chronological list of references, pages 55-56, has been revised and brought up to date.

Miscellaneous changes in wording and references have been made in the text to incorporate code and handbook revisions and to include experience accumulated since the third edition was published.

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vertical load

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appendix

tables, vertical load

notations

a = fraction less than 1.00
A = area
b = width of compressive zone
b' = width of web in T-beam
C = a coefficient
d = depth of a section
D = distribution factor, or a ratio
e = eccentricity
E = modulus of elasticity
f = stress
F = a multiplier
h = height of column
I = moment of inertia
K = stiffness
L = span length
M = moment
M_{AB} = end moment at joint A of member AB
M^F = fixed-end moment
n = ratio of $\frac{E_s}{E_c}$
N = actual axial load on column section
p = percentage of reinforcement
P = equivalent axial load on column section
r = a ratio
R = radius of gyration, or angle of joint translation
t = depth of flange, or overall dimension of column section
U = unbalanced moment
v = relative shear in columns
V = total shear
w = load per linear foot
W = load on a span, or wind pressure
θ = angle of rotation

vertical load

1 / the concept of fixed-end moment

If a load, W, is placed on a simply supported beam AB with span L as in Fig. 1(a), moments in the beam may be computed as the product of a coefficient and WL. The coefficients are independent of adjacent beams.

When the load is applied on AB, the beam will deflect and the tangents at the ends of it will rotate through angles denoted as θ_A and θ_B. The designer need not be concerned with these angles if the beam ends are free to rotate.

Assume that AB is restrained at A in such a manner that the angle change at A is smaller than θ_A. The restraint may be represented by a moment M_{AB}, as illustrated in Fig. 1(b). Various degrees of restraint may be considered but the most important of these is the one illustrated in Fig. 1(c) where the angle changes are zero at both supports. In this case AB is said to have fixed ends, and the restraining moments are called fixed-end moments, M^F_{AB} and M^F_{BA}.

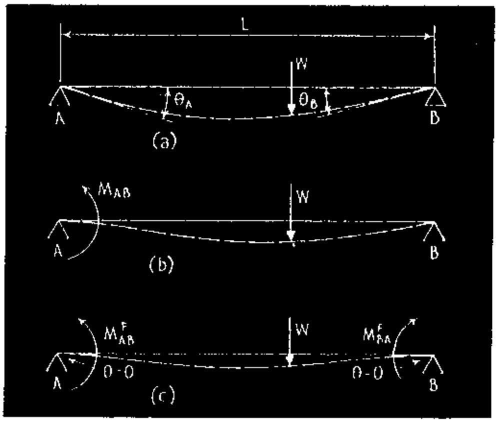


Fig. 1 – Beam with various degrees of restraint.

The beam with fixed ends has characteristics resembling those of simply supported beams. The following statements apply to both types of beams: *Moments may be computed as the product of a coefficient and* WL. *The coefficients are independent of adjacent beams.*

The fixed-end moment is particularly useful in beam design since it is independent of other members in the frame and also is a major part of the actual end moment in the beam. One objective in frame analysis is to determine the minor correction to the fixed-end moment to give the actual moment. When the correction is relatively small, as is often the case, it may be determined either by quick approximate procedures or by judgment.

2 / determination of fixed-end moments

The procedure to be illustrated is typical for all types of loading. Assume the problem is to determine the moments required to "fix" the ends *A* and *B* of a beam with span *L* supporting a load, *W*, placed a distance of *aL* from *A*.

To solve this problem, first place *W* on a beam *AB* considered simply supported as in Fig. 1(a). The angle changes in this beam are denoted as θ_A and θ_B. Then, as shown in Fig. 1(c), apply two end moments, M^F_{AB} and M^F_{BA}, of such direction and magnitude that the angle changes θ_A and θ_B are eliminated.

Angle changes and deflections may be determined by application of the two moment-area principles. Their use will be illustrated, but for a complete explanation refer to standard textbooks on structural theory.*

The procedure in this problem is as follows: For the load *W* acting alone, determine the moment curve in Fig. 2(a), assuming the beam to be simply supported. Let *E* denote modulus of elasticity and *I* denote moment of inertia. Divide all *M*-ordinates in Fig. 2(a) by the product of *EI* which gives the so-called "$\frac{M}{EI}$-diagram." Similarly, as in Fig. 2(b), draw an $\frac{M}{EI}$-diagram for M^F_{AB} and M^F_{BA}, which are the unknown quantities. Note that *M* denotes moments at any point in the beam considered simply supported.

The first moment-area principle states that *the angle between the tangents at any two points on a beam is equal to the area of the* $\frac{M}{EI}$*-diagram between the two points.* Since the tangents at *A* and *B* in the beams with fixed ends are assumed not to rotate, the angle between them equals zero. Both *E* and *I* are considered constant in this problem; therefore the product of *EI* cancels out, and we may write

$$-\tfrac{1}{2}M^F_{AB}L - \tfrac{1}{2}M^F_{BA}L + \tfrac{1}{2}Wa(1-a)L^2 = 0,^{**}$$

from which

$$M^F_{AB} + M^F_{BA} = a(1-a)WL. \tag{1}$$

The second moment-area principle states that *the deflection of any point on a beam measured from the tangent at any other point equals the moment about the first point of the* $\frac{M}{EI}$*-diagram between the two points.* The deflection of *A* measured from the tangent at *B* equals zero; therefore, canceling the constant product of *EI* and taking moments about *A*, we have

$$-\tfrac{1}{2}M^F_{AB}L \times \tfrac{1}{3}L - \tfrac{1}{2}M^F_{BA}L \times \tfrac{2}{3}L + \tfrac{1}{2}Wa(1-a)L^2 \times \tfrac{1}{3}(1+a)L = 0,$$

from which

$$M^F_{AB} + 2M^F_{BA} = a(1-a)\,(1+a)WL. \tag{2}$$

Subtracting equation (1) from equation (2) gives

$$M^F_{BA} = a^2(1-a)WL.$$

Similarly,

$$M^F_{AB} = a(1-a)^2WL. \tag{3}$$

*For instance, see reference 11.

**Moments M^F are here considered numerical values. Fixed-end moments due to gravity loads create tension in top fibers of beams and will subsequently be defined as negative quantities.

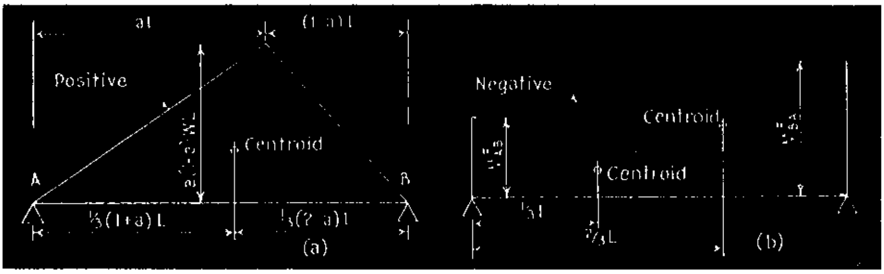
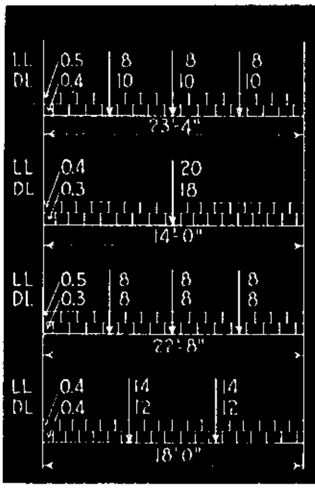


Fig. 2 — Moment curves for vertical load and restraint moments.

It is seen that M^F_{AB} and M^F_{BA} equal the product of WL and a coefficient that is a function of the type and position of the loading on span AB. Table 1 contains such coefficients for 15 types of loading on beams with fixed ends and constant moment of inertia. Coefficients are given so that moments may be computed also at intermediate points of the beams. For beams with variable I, similar data are available in *Handbook of Frame Constants* and *Continuous Concrete Bridges.**

3 / examples of fixed-end moments

The four beams in Fig. 3 are assumed to have fixed ends and a constant section throughout each beam. Moments at ends and at midspan are determined by using coefficients in Table 1. Time may be saved by selecting numerical values from Table 2,** which gives results without the use of a slide rule.



	End	LL	DL	Midspan	LL	DL
Conc.	$\frac{5}{48}\times54\times23.33$	131	73	0.6 × 131	79	44
Unif.	$\frac{1}{12}\times0.9\times23.33^2$	41	18	0.5 × 41	20	9
Total		172	91		99	53
Conc.	$\frac{1}{8}\times38\times14.00$	67	32	1.0 × 67	67	32
Unif.	$\frac{1}{12}\times0.7\times14.00^2$	11	5	0.5 × 11	6	2
Total		78	37		73	34
Conc.	$\frac{5}{48}\times48\times22.67$	113	57	0.6 × 113	68	34
Unif.	$\frac{1}{12}\times0.8\times22.67^2$	34	13	0.5 × 34	17	6
Total		147	70		85	40
Conc.	$\frac{2}{9}\times52\times18.00$	104	48	0.5 × 104	57	24
Unif.	$\frac{1}{12}\times0.8\times18.00^2$	22	11	0.5 × 22	11	5
Total		126	59		63	29

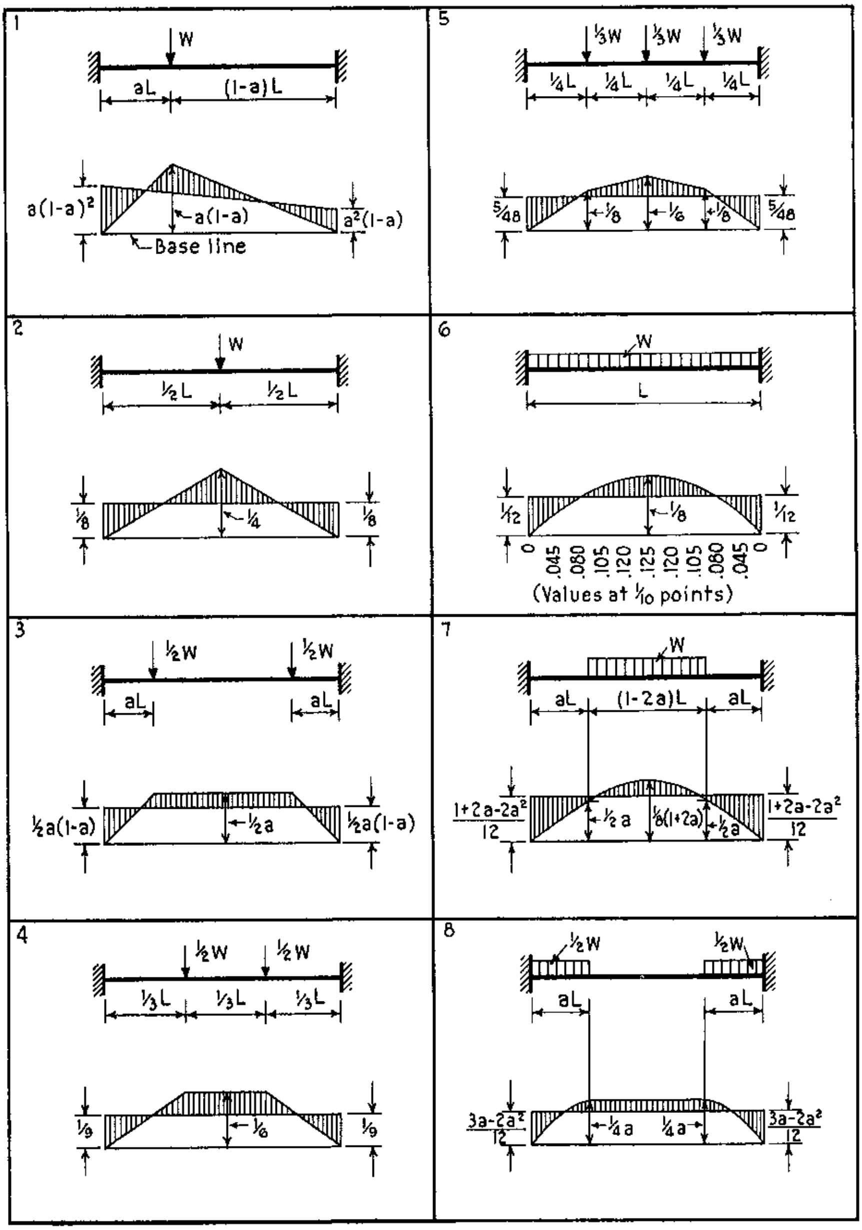
Fig. 3 — Moments in four beams with fixed ends.

*Both publications are available only in the United States and Canada from the Portland Cement Association.

**Reproduced from *Reinforced Concrete Design Handbook*, published by the American Concrete Institute, Detroit, Mich.

table 1. coefficients for moments in beams with fixed ends

moments in beams of constant section and with fixed ends



$M = m \times W \times L$

m = coefficient taken from diagram
W = total load on beam
L = length of beam
a = fraction less than 1.00

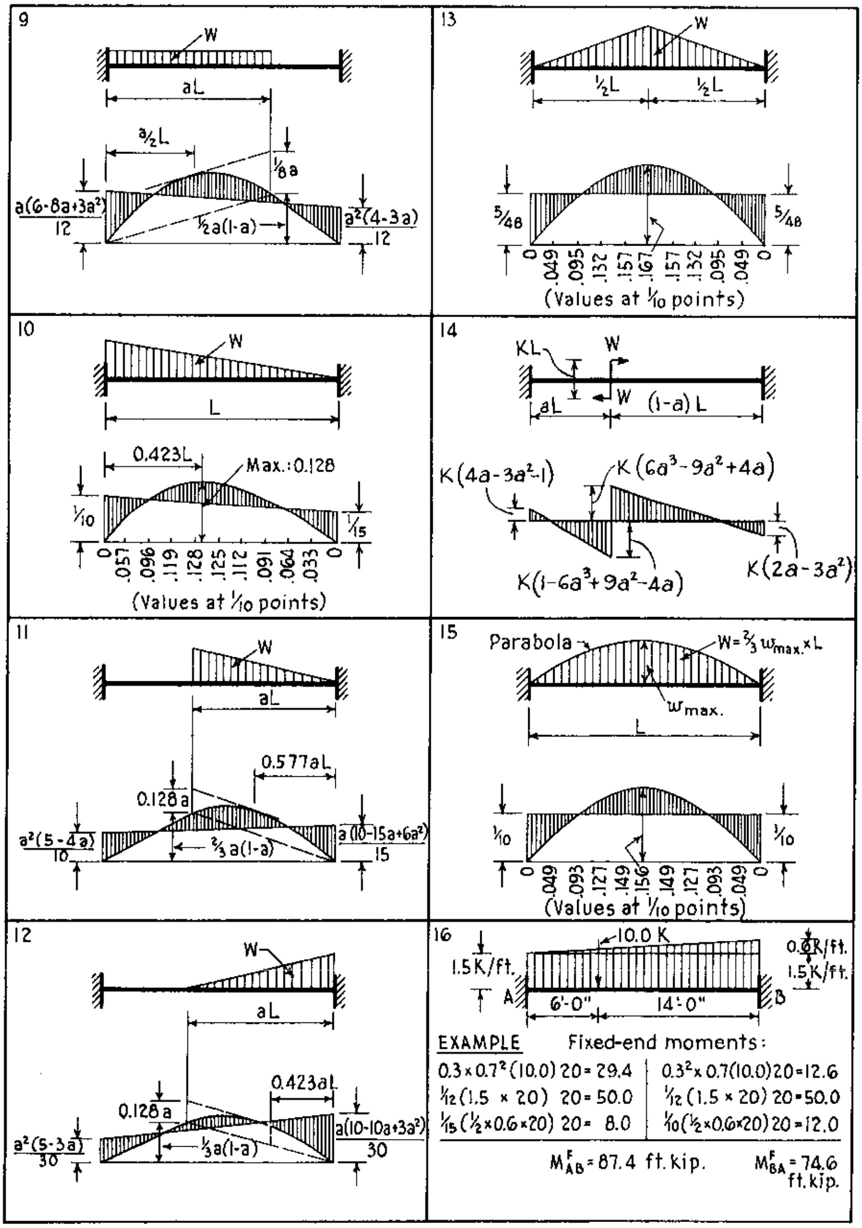
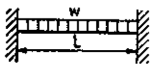


table 2. moments in beams with fixed ends

1. uniform load

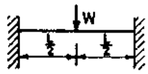


Fixed end moments: Table values $M = \frac{1}{12}wL^2$

Midspan moments: Table values × 0.5 $M = \frac{1}{24}wL^2$

Span L	Uniform Load w in kips per ft.									
	For following loads use table values directly									
	1	2	3	4	5	6	7	8	9	10
	For following loads use table values divided by 10									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5′-0″	2.08	4.17	6.25	8.33	10.4	12.5	14.6	16.7	18.7	20.8
5′-3″	2.30	4.59	6.89	9.19	11.5	13.8	16.1	18.4	20.7	23.0
5′-6″	2.52	5.04	7.56	10.1	12.6	15.1	17.6	20.2	22.7	25.2
5′-9″	2.76	5.51	8.27	11.0	13.8	16.5	19.3	22.0	24.8	27.6
6′-0″	3.00	6.00	9.00	12.0	15.0	18.0	21.0	24.0	27.0	30.0
6′-3″	3.26	6.51	9.77	13.0	16.3	19.5	22.8	26.0	29.3	32.6
6′-6″	3.52	7.04	10.6	14.1	17.6	21.1	24.6	28.2	31.7	35.2
6′-9″	3.80	7.59	11.4	15.2	19.0	22.8	26.6	30.4	34.2	38.0
7′-0″	4.08	8.17	12.3	16.3	20.4	24.5	28.6	32.7	36.8	40.8
7′-3″	4.38	8.76	13.1	17.5	21.9	26.3	30.7	35.0	39.4	43.8
7′-6″	4.69	9.38	14.1	18.8	23.4	28.1	32.8	37.5	42.2	46.9
7′-9″	5.01	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.1
8′-0″	5.33	10.7	16.0	21.3	26.7	32.0	37.3	42.7	48.0	53.3
8′-3″	5.67	11.3	17.0	22.7	28.4	34.0	39.7	45.4	51.1	56.7
8′-6″	6.02	12.0	18.1	24.1	30.1	36.1	42.2	48.2	54.2	60.2
8′-9″	6.38	12.8	19.1	25.5	31.9	38.3	44.7	51.0	57.4	63.8
9′-0″	6.75	13.5	20.3	27.0	33.7	40.5	47.3	54.0	60.8	67.5
9′-3″	7.13	14.3	21.4	28.5	35.6	42.8	49.9	57.0	64.2	71.3
9′-6″	7.52	15.0	22.6	30.1	37.6	45.1	52.6	60.2	67.7	75.2
9′-9″	7.92	15.8	23.8	31.7	39.6	47.5	55.5	63.4	71.3	79.2
10′-0″	8.33	16.7	25.0	33.3	41.7	50.0	58.3	66.7	75.0	83.3
10′-6″	9.19	18.4	27.6	36.8	45.9	55.1	64.3	73.5	82.7	91.9
11′-0″	10.1	20.2	30.3	40.3	50.4	60.5	70.6	80.7	90.8	101
11′-6″	11.0	22.0	33.1	44.1	55.1	66.1	77.2	88.2	99.2	110
12′-0″	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108	120
12′-6″	13.0	26.0	39.1	52.1	65.1	78.1	91.2	104	117	130
13′-0″	14.1	28.2	42.2	56.3	70.4	84.5	98.6	113	127	141
13′-6″	15.2	30.4	45.6	60.8	75.9	91.1	106	122	137	152
14′-0″	16.3	32.7	49.0	65.3	81.7	98.0	114	131	147	163
14′-6″	17.5	35.0	52.6	70.1	87.6	105	123	140	158	175
15′-0″	18.8	37.5	56.3	75.0	93.8	113	131	150	169	188
15′-6″	20.0	40.0	60.1	80.1	100	120	140	160	180	200
16′-0″	21.3	42.7	64.0	85.3	107	128	149	171	192	213
16′-6″	22.7	45.4	68.1	90.8	113	136	159	182	204	227
17′-0″	24.1	48.2	72.2	96.3	120	145	169	193	217	241
17′-6″	25.5	51.0	76.6	102	128	153	179	204	230	255
18′-0″	27.0	54.0	81.0	108	135	162	189	216	243	270
18′-6″	28.5	57.0	85.6	114	143	171	200	228	257	285
19′-0″	30.1	60.2	90.3	120	150	181	211	241	271	301
19′-6″	31.7	63.4	95.1	127	158	190	222	254	285	317
20′-0″	33.3	66.7	100	133	167	200	233	267	300	333
20′-6″	35.0	70.0	105	140	175	210	245	280	315	350
21′-0″	36.8	73.5	110	147	184	221	257	294	331	368
21′-6″	38.5	77.0	116	154	193	231	270	308	347	385
22′-0″	40.3	80.7	121	161	202	242	282	323	363	403
22′-6″	42.2	84.4	127	169	211	253	295	338	380	422
23′-0″	44.1	88.2	132	176	220	264	309	353	397	441
23′-6″	46.0	92.0	138	184	230	276	322	368	414	460
24′-0″	48.0	96.0	144	192	240	288	336	384	432	480
24′-6″	50.0	100	150	200	250	300	350	400	450	500
25′-0″	52.1	104	156	208	260	312	365	417	469	521
25′-6″	54.2	108	163	217	271	325	379	434	488	542
26′-0″	56.3	113	169	225	282	338	394	451	507	563
26′-6″	58.5	117	176	234	293	351	410	468	527	585
27′-0″	60.8	122	182	243	304	365	425	486	547	608
27′-6″	63.0	126	189	252	315	378	441	504	567	630
28′-0″	65.3	131	196	261	327	392	457	523	588	653
28′-6″	67.7	135	203	271	338	406	474	542	609	677
29′-0″	70.1	140	210	280	350	420	491	561	631	701
29′-6″	72.5	145	218	290	363	435	508	580	653	725
30′-0″	75.0	150	225	300	375	450	525	600	675	750

2. concentrated load at midspan

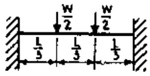


Fixed end moments: Table values $M = \frac{1}{8}WL$

Midspan moments: Table values $M = \frac{1}{8}WL$

Span L	Concentrated Load W in kips									
	For following loads use table values directly									
	10	20	30	40	50	60	70	80	90	100
	For following loads use table values divided by 10									
	1	2	3	4	5	6	7	8	9	10
5′-0″	6.3	12.5	18.8	25.0	31.3	37.5	43.8	50.0	56.3	62.5
5′-3″	6.6	13.1	19.7	26.3	32.8	39.4	45.9	52.5	59.1	65.6
5′-6″	6.9	13.8	20.6	27.5	34.4	41.3	48.1	55.0	61.9	68.8
5′-9″	7.2	14.4	21.6	28.8	35.9	43.1	50.3	57.5	64.7	71.9
6′-0″	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5	75.0
6′-3″	7.8	15.6	23.4	31.3	39.1	46.9	54.7	62.5	70.3	78.1
6′-6″	8.1	16.3	24.4	32.5	40.6	48.8	56.9	65.0	73.1	81.3
6′-9″	8.4	16.9	25.3	33.8	42.2	50.6	59.1	67.5	75.9	84.4
7′-0″	8.8	17.5	26.3	35.0	43.8	52.5	61.3	70.0	78.8	87.5
7′-3″	9.1	18.1	27.2	36.3	45.3	54.4	63.4	72.5	81.6	90.6
7′-6″	9.4	18.8	28.1	37.5	46.9	56.3	65.6	75.0	84.4	93.8
7′-9″	9.7	19.4	29.1	38.8	48.4	58.1	67.8	77.5	87.2	96.9
8′-0″	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100
8′-3″	10.3	20.6	30.9	41.3	51.6	61.9	72.2	82.5	92.8	103
8′-6″	10.6	21.3	31.9	42.5	53.1	63.8	74.4	85.0	95.6	106
8′-9″	10.9	21.9	32.8	43.8	54.7	65.6	76.6	87.5	98.4	109
9′-0″	11.3	22.5	33.8	45.0	56.3	67.5	78.8	90.0	101	113
9′-3″	11.6	23.1	34.7	46.3	57.8	69.4	80.9	92.5	104	116
9′-6″	11.9	23.8	35.6	47.5	59.4	71.3	83.1	95.0	107	119
9′-9″	12.2	24.4	36.6	48.8	60.9	73.1	85.3	97.5	110	122
10′-0″	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100	113	125
10′-6″	13.1	26.3	39.4	52.5	65.6	78.8	91.9	105	118	131
11′-0″	13.8	27.5	41.3	55.0	68.8	82.5	96.3	110	124	138
11′-6″	14.4	28.8	43.1	57.5	71.9	86.3	101	115	129	144
12′-0″	15.0	30.0	45.0	60.0	75.0	90.0	105	120	135	150
12′-6″	15.6	31.3	46.9	62.5	78.1	93.8	109	125	141	156
13′-0″	16.3	32.5	48.8	65.0	81.3	97.5	114	130	146	163
13′-6″	16.9	33.8	50.6	67.5	84.4	101	118	135	152	169
14′-0″	17.5	35.0	52.5	70.0	87.5	105	122	140	157	175
14′-6″	18.1	36.3	54.4	72.5	90.6	109	127	145	163	181
15′-0″	18.8	37.5	56.3	75.0	93.8	112	131	150	169	188
15′-6″	19.4	38.8	58.1	77.5	96.9	116	136	155	174	194
16′-0″	20.0	40.0	60.0	80.0	100	120	140	160	180	200
16′-6″	20.6	41.3	61.9	82.5	103	124	144	165	186	206
17′-0″	21.3	42.5	63.8	85.0	106	128	149	170	191	213
17′-6″	21.9	43.8	65.6	87.5	109	131	153	175	197	219
18′-0″	22.5	45.0	67.5	90.0	112	135	158	180	202	225
18′-6″	23.1	46.3	69.4	92.5	116	139	162	185	208	231
19′-0″	23.8	47.5	71.3	95.0	119	142	166	190	214	238
19′-6″	24.4	48.8	73.1	97.5	122	146	171	195	219	244
20′-0″	25.0	50.0	75.0	100	125	150	175	200	225	250
20′-6″	25.6	51.3	76.9	103	128	154	179	205	231	256
21′-0″	26.3	52.5	78.8	105	131	158	184	210	236	263
21′-6″	26.9	53.8	80.6	108	134	161	188	215	242	269
22′-0″	27.5	55.0	82.5	110	138	165	192	220	248	275
22′-6″	28.1	56.3	84.4	113	141	169	197	225	253	281
23′-0″	28.8	57.5	86.3	115	144	172	201	230	259	288
23′-6″	29.4	58.8	88.1	118	147	176	206	235	264	294
24′-0″	30.0	60.0	90.0	120	150	180	210	240	270	300
24′-6″	30.6	61.3	91.9	123	153	184	214	245	276	306
25′-0″	31.3	62.5	93.8	125	156	188	219	250	281	313
25′-6″	31.9	63.8	45.6	128	159	191	223	255	287	319
26′-0″	32.5	65.0	97.5	130	162	195	228	260	292	325
26′-6″	33.1	66.3	99.4	133	166	199	232	265	298	331
27′-0″	33.8	67.5	101	135	169	202	236	270	304	338
27′-6″	34.4	68.8	103	138	172	206	241	275	309	344
28′-0″	35.0	70.0	105	140	175	210	245	280	315	350
28′-6″	35.6	71.3	107	143	178	214	249	285	321	356
29′-0″	36.3	72.5	109	145	181	218	254	290	326	363
29′-6″	36.9	73.8	111	148	184	221	258	295	332	369
30′-0″	37.5	75.0	113	150	188	225	262	300	338	375

3. concentrated loads at third points

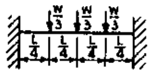


Fixed end moments: Table values $M = \frac{1}{9}WL$

Midspan moments: Table values × 0.5 $M = \frac{1}{18}WL$

Span L	Concentrated Load W in kips — For following loads use table values directly: 10 / For following loads use table values divided by 10: 1	20 / 2	30 / 3	40 / 4	50 / 5	60 / 6	70 / 7	80 / 8	90 / 9	100 / 10
5′-0″	5.6	11.1	16.7	22.2	27.8	33.3	38.9	44.4	50.0	55.6
5′-3″	5.8	11.7	17.5	23.3	29.2	35.0	40.8	46.7	52.5	58.3
5′-6″	6.1	12.2	18.3	24.4	30.6	36.7	42.8	48.9	55.0	61.1
5′-9″	6.4	12.8	19.2	25.6	31.9	38.3	44.7	51.1	57.5	63.9
6′-0″	6.7	13.3	20.0	26.7	33.3	40.0	46.7	53.3	60.0	66.7
6′-3″	6.9	13.9	20.8	27.8	34.7	41.7	48.6	55.6	62.5	69.4
6′-6″	7.2	14.4	21.7	28.9	36.1	43.3	50.6	57.8	65.0	72.2
6′-9″	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5	75.0
7′-0″	7.8	15.6	23.3	31.1	38.9	46.7	54.4	62.2	70.0	77.8
7′-3″	8.1	16.1	24.2	32.2	40.3	48.3	56.4	64.4	72.5	80.6
7′-6″	8.3	16.7	25.0	33.3	41.7	50.0	58.3	66.7	75.0	83.3
7′-9″	8.6	17.2	25.8	34.4	43.1	51.7	60.3	68.9	77.5	86.1
8′-0″	8.9	17.8	26.7	35.6	44.4	53.3	62.2	71.1	80.0	88.9
8′-3″	9.2	18.3	27.5	36.7	45.8	55.0	64.2	73.3	82.5	91.7
8′-6″	9.4	18.9	28.3	37.8	47.2	56.7	66.1	75.6	85.0	94.4
8′-9″	9.7	19.4	29.2	38.9	48.6	58.3	68.1	77.8	87.5	97.2
9′-0″	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100
9′-3″	10.3	20.6	30.8	41.1	51.4	61.7	71.9	82.2	92.5	103
9′-6″	10.6	21.1	31.7	42.2	52.8	63.3	73.9	84.4	95.0	106
9′-9″	10.8	21.7	32.5	43.3	54.2	65.0	75.8	86.7	97.5	108
10′-0″	11.1	22.2	33.3	44.4	55.6	66.7	77.8	88.9	100	111
10′-6″	11.7	23.3	35.0	46.7	58.3	70.0	81.7	93.3	105	117
11′-0″	12.2	24.4	36.7	48.9	61.1	73.3	85.6	97.8	110	122
11′-6″	12.8	25.6	38.3	51.1	63.9	76.7	89.4	102	115	128
12′-0″	13.3	26.7	40.0	53.3	66.7	80.0	93.3	107	120	133
12′-6″	13.9	27.8	41.7	55.6	69.4	83.3	97.2	111	125	139
13′-0″	14.4	28.9	43.3	57.8	72.2	86.7	101	116	130	144
13′-6″	15.0	30.0	45.0	60.0	75.0	90.0	105	120	135	150
14′-0″	15.6	31.1	46.7	62.2	77.8	93.3	109	124	140	156
14′-6″	16.1	32.2	48.3	64.4	80.6	96.7	113	129	145	161
15′-0″	16.7	33.3	50.0	66.7	83.3	100	117	133	150	167
15′-6″	17.2	34.4	51.7	68.9	86.1	103	121	138	155	172
16′-0″	17.8	35.6	53.3	71.1	88.9	107	124	142	160	178
16′-6″	18.3	36.7	55.0	73.3	91.7	110	128	147	165	183
17′-0″	18.9	37.8	56.7	75.6	94.4	113	132	151	170	189
17′-6″	19.4	38.9	58.3	77.8	97.2	117	136	156	175	194
18′-0″	20.0	40.0	60.0	80.0	100	120	140	160	180	200
18′-6″	20.6	41.1	61.7	82.2	103	123	144	164	185	206
19′-0″	21.1	42.2	63.3	84.4	106	127	148	169	190	211
19′-6″	21.7	43.3	65.0	86.7	108	130	152	173	195	217
20′-0″	22.2	44.4	66.6	88.8	111	133	156	178	200	222
20′-6″	22.8	45.6	68.3	91.1	114	137	159	182	205	228
21′-0″	23.3	46.7	70.0	93.3	117	140	163	187	210	233
21′-6″	23.9	47.8	71.7	95.6	119	143	167	191	215	239
22′-0″	24.4	48.9	73.3	97.8	122	147	171	196	220	244
22′-6″	25.0	50.0	75.0	100	125	150	175	200	225	250
23′-0″	25.6	51.1	76.7	102	128	153	179	204	230	256
23′-6″	26.1	52.2	78.3	104	131	157	183	209	235	261
24′-0″	26.7	53.3	80.0	107	133	160	187	213	240	267
24′-6″	27.2	54.4	81.7	109	136	163	191	218	245	272
25′-0″	27.8	55.6	83.3	111	139	167	194	222	250	278
25′-6″	28.3	56.7	85.0	113	142	170	198	227	255	283
26′-0″	28.9	57.8	86.7	116	144	173	202	231	260	289
26′-6″	29.4	58.9	88.3	118	147	177	206	236	265	294
27′-0″	30.0	60.0	90.0	120	150	180	210	240	270	300
27′-6″	30.6	61.1	91.7	122	153	183	214	244	275	306
28′-0″	31.1	62.2	93.3	124	156	187	218	249	280	311
28′-6″	31.7	63.3	95.0	127	158	190	222	253	285	317
29′-0″	32.2	64.4	96.7	129	161	193	226	258	290	322
29′-6″	32.8	65.6	98.3	131	164	197	229	262	295	328
30′-0″	33.3	66.7	100	133	167	200	233	267	300	333

4. concentrated loads at fourth points



Fixed end moments: Table values $M = \frac{5}{48}WL$

Midspan moments: Table values × 0.6 $M = \frac{3}{48}WL$

Span L	Concentrated Load W in kips — For following loads use table values directly: 10 / For following loads use table values divided by 10: 1	20 / 2	30 / 3	40 / 4	50 / 5	60 / 6	70 / 7	80 / 8	90 / 9	100 / 10
5′-0″	5.2	10.4	15.6	20.8	26.0	31.2	36.5	41.7	46.9	52.1
5′-3″	5.5	10.9	16.4	21.9	27.3	32.8	38.3	43.8	49.2	54.7
5′-6″	5.7	11.5	17.2	22.9	28.6	34.4	40.1	45.8	51.6	57.3
5′-9″	6.0	12.0	18.0	24.0	29.9	35.6	41.9	47.9	53.9	60.0
6′-0″	6.3	12.5	18.8	25.0	31.3	37.5	43.8	50.0	56.3	62.5
6′-3″	6.5	13.0	19.5	26.0	32.6	39.1	45.6	52.1	58.6	65.1
6′-6″	6.8	13.5	20.3	27.1	33.9	40.6	47.4	54.2	60.9	67.7
6′-9″	7.0	14.1	21.1	28.1	35.2	42.2	49.2	56.3	63.3	70.3
7′-0″	7.3	14.6	21.9	29.2	36.5	43.8	51.0	58.3	65.6	72.9
7′-3″	7.6	15.1	22.7	30.2	37.8	45.3	52.9	60.4	68.0	75.5
7′-6″	7.8	15.6	23.4	31.3	39.1	46.9	54.7	62.5	70.3	78.1
7′-9″	8.1	16.1	24.2	32.3	40.4	48.4	56.5	64.6	72.7	80.7
8′-0″	8.3	16.7	25.0	33.3	41.7	50.0	58.3	66.7	75.0	83.3
8′-3″	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.7	77.3	85.9
8′-6″	8.9	17.7	26.6	35.4	44.3	53.1	62.0	70.8	79.7	88.5
8′-9″	9.1	18.2	27.3	36.5	45.6	54.7	63.8	72.9	82.0	91.1
9′-0″	9.4	18.8	28.1	37.5	46.9	56.3	65.6	75.0	84.4	93.8
9′-3″	9.6	19.3	28.9	38.5	48.2	57.8	67.4	77.1	86.7	96.4
9′-6″	9.9	19.8	29.7	39.6	49.5	59.4	69.3	79.2	89.1	99.0
9′-9″	10.2	20.3	30.5	40.6	50.8	60.9	71.1	81.2	91.4	102
10′-0″	10.4	20.8	31.3	41.7	52.1	62.5	72.9	83.3	93.8	104
10′-6″	10.9	21.9	32.8	43.8	54.7	65.6	76.6	87.5	98.4	109
11′-0″	11.5	22.9	34.4	45.8	57.3	68.7	80.2	91.7	103	115
11′-6″	12.0	24.0	35.9	47.9	59.9	71.9	83.9	95.8	108	120
12′-0″	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100	113	125
12′-6″	13.0	26.0	39.1	52.1	65.1	78.1	91.1	104	117	130
13′-0″	13.5	27.1	40.6	54.2	67.7	81.3	94.8	108	122	135
13′-6″	14.1	28.1	42.2	56.3	70.3	84.4	98.4	113	127	141
14′-0″	14.6	29.2	43.7	58.3	72.9	87.5	102	117	131	146
14′-6″	15.1	30.2	45.3	60.4	75.5	90.6	106	121	136	151
15′-0″	15.6	31.3	46.9	62.5	78.1	93.8	109	125	141	156
15′-6″	16.1	32.3	48.4	64.6	80.7	96.9	113	129	145	161
16′-0″	16.7	33.3	50.0	66.7	83.3	100	117	133	150	167
16′-6″	17.2	34.4	51.6	68.8	85.9	103	120	138	155	172
17′-0″	17.7	35.4	53.1	70.8	88.5	106	124	142	159	177
17′-6″	18.2	36.5	54.7	72.9	91.1	109	128	146	164	182
18′-0″	18.8	37.5	56.3	75.0	93.8	113	131	150	169	188
18′-6″	19.3	38.5	57.8	77.1	96.4	116	135	154	173	193
19′-0″	19.8	39.6	59.4	79.2	99.0	119	139	158	178	198
19′-6″	20.3	40.6	60.9	81.3	102	122	142	163	183	203
20′-0″	20.8	41.7	62.5	83.3	104	125	146	167	187	208
20′-6″	21.4	42.7	64.1	85.4	107	128	149	171	192	214
21′-0″	21.9	43.8	65.6	87.5	109	131	153	175	197	219
21′-6″	22.4	44.8	67.2	89.6	112	134	157	179	202	224
22′-0″	22.9	45.8	68.8	91.7	115	138	160	183	206	229
22′-6″	23.4	46.9	70.3	93.8	117	141	164	188	211	234
23′-0″	24.0	47.9	71.9	95.8	120	144	168	192	216	240
23′-6″	24.5	49.0	73.4	97.9	122	147	171	196	220	245
24′-0″	25.0	50.0	75.0	100	125	150	175	200	225	250
24′-6″	25.5	51.0	76.6	102	128	153	179	204	230	255
25′-0″	26.0	52.1	78.1	104	130	156	182	208	234	260
25′-6″	26.6	53.1	79.7	106	133	159	186	212	239	266
26′-0″	27.1	54.2	81.2	108	135	162	190	217	244	271
26′-6″	27.6	55.2	82.8	110	138	166	193	221	248	276
27′-0″	28.1	56.3	84.4	112	141	169	197	225	253	281
27′-6″	28.6	57.3	85.9	115	143	172	200	229	258	286
28′-0″	29.2	58.3	87.5	117	146	175	204	233	262	292
28′-6″	29.7	59.4	89.1	119	148	178	208	238	267	297
29′-0″	30.2	60.4	90.6	121	151	181	211	242	272	302
29′-6″	30.7	61.5	92.2	123	154	184	215	246	277	307
30′-0″	31.3	62.5	93.8	125	156	188	219	250	281	313

Note that end moments for total load (*TL*) are computed in Fig. 3 as a coefficient multiplied by WL. All other values equal certain proportions of the moments in the first column.

4 / stiffness and carry-over factor

It has been shown in Section 2, "Determination of Fixed-End Moments," that moments at fixed ends may be determined by multiplying the product of load and span by a coefficient. Since ends of beams in buildings are not fixed, the fixed-end moments must be modified to suit whatever rotation takes place at the joints. The effect of rotating one end of a beam will now be discussed, including the concepts of stiffness and carry-over factor.

In the member AB in Fig. 4(a), joint A is fixed and there is no load on the beam between A and B. Applying a moment M_{BA} at B will cause a change of angle, θ_B, and induce a resisting moment M_{AB} at A. Consider the problem to determine the relationship between M_{BA} and θ_B, and between M_{AB} and M_{BA}.

The moment diagrams corresponding to M_{AB} and M_{BA} are shown in Fig. 4(b) and then divided into two constitutent $\frac{M}{EI}$-diagrams as in Fig. 4(c) and 4(d). Since the rotation of B creates tension on top of the beam at A, M_{AB} is negative, while M_{BA}, producing tension on the bottom of the beam, is positive. According to the first of the moment-area principles, the area of the $\frac{M}{EI}$-diagrams between A and B equals the angle θ_B:

$$-\frac{\frac{1}{2}M_{AB}}{EI} \times L + \frac{\frac{1}{2}M_{BA}}{EI} \times L = \theta_B.^{*}$$

According to the second principle, the moment of the $\frac{M}{EI}$-diagrams about A equals the deflection of A measured from the tangent at B:

$$-\frac{\frac{1}{2}M_{AB}L}{EI} \times \tfrac{1}{3}L + \frac{\frac{1}{2}M_{BA}L}{EI} \times \tfrac{2}{3}L = \theta_B L.$$

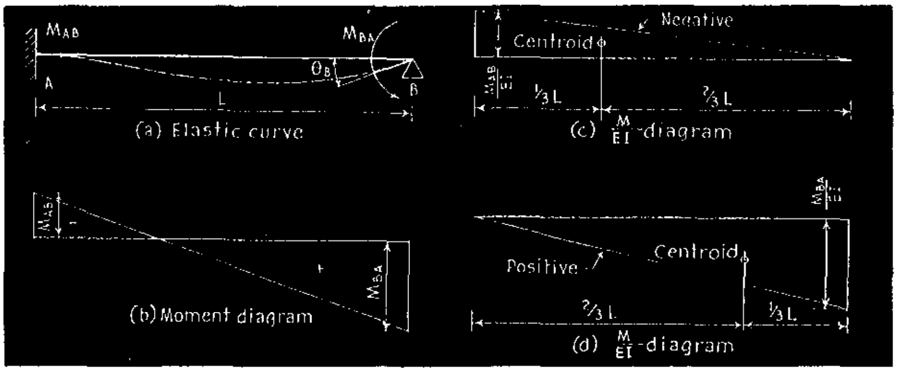


Fig. 4 — Moments in beam with one fixed end, other end being rotated.

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*M_{AB} and M_{BA} are considered numerical values.

Inserting $K = \frac{4EI}{L}$ and rearranging give

$$-2M_{AB} + 2M_{BA} = K\theta_B;$$
$$-2M_{AB} + 4M_{BA} = 3K\theta_B;$$

from which

$$M_{BA} = K\theta_B;$$
$$M_{AB} = \tfrac{1}{2}M_{BA}.$$

K is called the stiffness of the member. For members with constant section, K equals $\frac{4EI}{L}$, which is referred to as the absolute value. A relative value of $K = \frac{I}{L}$ is preferred when E is constant throughout a frame. It is seen by inspection of the two equations derived that

1. The stiffness K at B equals the moment at B required to give B a unit rotation when A is fixed.
2. The moment required to rotate B through a given angle is proportional to the stiffness K.
3. Applying a moment M_{BA} at B will induce at A a moment $M_{AB} = \tfrac{1}{2}M_{BA}$. The factor of ½ is called "the carry-over factor."*

The concepts of stiffness and carry-over factor together with the concept of fixed-end moment are used in the procedure of analysis known as moment distribution.

5 / tables of stiffness for beams and columns

The relative stiffness of all beams and columns must be established regardless of the analytical method used. Stiffnesses are functions of cross-sectional dimensions, but are not initially known and must be estimated. The selection of stiffness factors is simplified by use of Tables 3 and 4. The specific assumptions on which these tables are based are discussed in this section and also in Section 18, "Effect of Variation in Stiffness."

For beams, the question arises regarding the effect of flange on stiffness. The ACI Code specifies that in computing the value of I for relative stiffness of beams, the reinforcement may be neglected, but allowance shall be made for the effect of flange in T-shaped sections.

One procedure is to compute I for a T-beam as the product of $\tfrac{1}{12}b'd^3$ and a coefficient C, values of which may be selected from Fig. 5. The width of the web is denoted as b' and the total beam depth as d. Stiffness equals $C\left(\frac{\tfrac{1}{12}b'd^3}{L}\right)$ and the value of $I = \tfrac{1}{12}b'd^3$ may be selected from Table 3. It is often difficult to select the flange width, b, and the assumption that the entire flange width available is fully effective across the span may be questionable. Therefore, results obtained by using Fig. 5 are only as accurate as the assumptions made.

*The value of ½ applies to prismatic members only. For other types of members, values of carry-over factors may be selected from *Handbook of Frame Constants* and *Continuous Concrete Bridges*, available only in the United States and Canada from the Portland Cement Association. These publications also give stiffness factors.

table 3. stiffness of beams

values of *K* for T-beams $K = \frac{2I^*}{10L}$ $d =$ depth $b' =$ width of web $I = \frac{b'd^3}{12}$

d	b'	I	L : Length of beam (feet) 8	10	12	14	16	20	24	30
8	6	256	6	5	4	4	3	3	2	2
	8	341	9	7	6	5	4	3	3	2
	10	427	11	9	7	6	5	4	4	3
	11½	491	12	10	8	7	6	5	4	3
	13	555	14	11	9	8	7	6	5	4
	15	640	16	13	11	9	8	6	5	4
	17	725	18	15	12	10	9	7	6	5
	19	811	20	16	14	12	10	8	7	5
10	6	500	13	10	8	7	6	5	4	3
	8	667	17	13	11	10	8	7	6	4
	10	833	21	17	14	12	10	8	7	6
	11½	958	24	19	16	14	12	10	8	6
	13	1083	27	22	18	15	14	11	9	7
	15	1250	31	25	21	18	16	13	10	8
	17	1417	35	28	24	20	18	14	12	9
	19	1583	40	32	26	23	20	16	13	11
12	6	864	22	17	14	12	11	9	7	6
	8	1152	29	23	19	16	14	12	10	8
	10	1440	36	29	24	21	18	14	12	10
	11½	1656	41	33	28	24	21	17	14	11
	13	1872	47	37	31	27	23	19	16	12
	15	2160	54	43	36	31	27	22	18	14
	17	2448	61	49	41	35	31	25	20	16
	19	2736	68	55	46	39	34	27	23	18
14	6	1372	34	27	23	20	17	14	11	9
	8	1829	46	37	30	26	23	18	15	12
	10	2287	57	46	38	33	29	23	19	15
	11½	2630	66	53	44	38	33	26	22	18
	13	2973	74	59	50	42	37	30	25	20
	15	3430	86	69	57	49	43	34	29	23
	17	3887	97	78	65	56	49	39	32	26
	19	4345	109	87	72	62	54	43	36	29
16	6	2048	51	41	34	29	26	20	17	14
	8	2731	68	55	46	39	34	27	23	18
	10	3413	85	68	57	49	43	34	28	23
	11½	3925	98	79	65	56	49	39	33	26
	13	4437	111	89	74	63	55	44	37	30
	15	5120	128	102	85	73	64	51	43	34
	17	5803	145	116	97	83	73	58	48	39
	19	6485	162	130	108	93	81	65	54	43
18	6	2916	73	58	49	42	36	29	24	19
	8	3888	97	78	65	56	49	39	32	26
	10	4860	122	97	81	69	61	49	41	32
	11½	5589	140	112	93	80	70	56	47	37
	13	6318	158	126	105	90	79	63	53	42
	15	7290	182	146	122	104	91	73	61	49
	17	8262	207	165	138	118	103	83	69	55
	19	9234	231	185	154	132	115	92	77	62
20	6	4000	100	80	67	57	50	40	33	27
	8	5333	133	107	89	76	67	53	44	36
	10	6667	167	133	111	95	83	67	56	44
	11½	7667	192	153	128	110	96	77	64	51
	13	8667	217	173	144	124	108	87	72	58
	15	10000	250	200	167	143	125	100	83	67
	17	11333	283	227	189	162	142	113	94	76
	19	12667	317	253	211	181	158	127	106	84
22	6	5324	133	106	89	76	67	53	44	36
	8	7099	177	142	118	101	89	71	59	47
	10	8873	222	177	148	127	111	89	74	59
	11½	10204	255	204	170	146	128	102	85	68
	13	11535	288	231	192	165	144	115	96	77
	15	13310	333	266	222	190	166	133	111	89
	17	15085	377	302	251	215	189	151	126	101
	19	16859	421	337	281	241	211	169	141	112

d	b'	I	L : Length of beam (feet) 8	10	12	14	16	20	24	30
24	8	9216	230	185	155	130	115	90	75	60
	10	11520	290	230	190	165	145	115	95	75
	11½	13248	330	265	220	190	165	130	110	90
	13	14976	375	300	250	215	185	150	125	100
	15	17280	430	345	290	245	215	175	145	115
	17	19584	490	390	325	280	245	195	165	130
	19	21888	545	440	365	315	275	220	180	145
	21	24192	605	485	405	345	300	240	200	160
26	8	11717	295	235	195	165	145	115	100	80
	10	14647	365	295	245	210	185	145	120	100
	11½	16844	420	335	280	240	210	170	140	110
	13	19041	475	380	315	270	240	190	160	125
	15	21970	550	440	365	315	275	220	185	145
	17	24899	620	500	415	355	310	250	205	165
	19	27829	695	555	465	400	350	280	230	185
	21	30758	770	615	515	440	385	310	255	205
28	8	14635	365	295	245	210	185	145	120	100
	10	18293	455	365	305	260	230	185	150	120
	11½	21037	525	420	350	300	265	210	175	140
	13	23781	595	475	395	340	295	240	200	160
	15	27440	685	550	455	390	345	275	230	185
	17	31099	775	620	520	445	390	310	260	205
	19	34757	870	695	580	495	435	350	290	230
	21	38416	960	770	640	550	480	385	320	255
30	8	18000	450	360	300	255	225	180	150	120
	10	22500	565	450	375	320	280	225	190	150
	11½	25875	645	520	430	370	325	260	215	175
	13	29250	730	585	490	420	365	295	245	195
	15	33750	845	675	565	480	420	340	280	225
	17	38250	955	765	640	545	480	385	320	255
	19	42750	1070	855	715	610	535	430	355	285
	21	47250	1180	945	790	675	590	475	395	315
36	8	31104	780	620	520	445	390	310	260	205
	10	38880	970	780	650	555	485	390	325	260
	11½	44712	1120	895	745	640	560	445	375	300
	13	50544	1260	1010	840	720	630	505	420	335
	15	58320	1460	1170	970	835	730	585	485	390
	17	66096	1650	1320	1100	945	825	660	550	440
	19	73872	1850	1480	1230	1060	925	740	615	490
	21	81648	2040	1630	1360	1170	1020	815	680	545
42	8	49392	1230	990	825	705	615	495	410	330
	10	61740	1540	1230	1030	880	770	615	515	410
	11½	71001	1780	1420	1180	1010	890	710	590	475
	13	80262	2010	1610	1340	1150	1000	805	670	535
	15	92610	2320	1850	1540	1320	1160	925	770	615
	17	104958	2620	2100	1750	1500	1310	1050	875	700
	19	117306	2930	2350	1950	1680	1470	1170	975	780
	21	129654	3240	2590	2160	1850	1620	1300	1080	865
48	8	73728	1840	1470	1230	1050	920	735	615	490
	10	92160	2300	1840	1540	1320	1150	920	770	615
	11½	105984	2650	2120	1770	1510	1320	1060	885	705
	13	119808	3000	2400	2000	1710	1500	1200	1000	800
	15	138240	3460	2760	2300	1970	1730	1380	1150	920
	17	156672	3920	3130	2610	2240	1960	1570	1310	1040
	19	175104	4380	3500	2920	2500	2190	1750	1460	1170
	21	193536	4840	3870	3230	2760	2420	1940	1610	1290
54	8	104976	2620	2100	1750	1500	1310	1050	875	700
	10	131220	3280	2620	2190	1880	1640	1310	1090	875
	11½	150903	3770	3020	2510	2160	1890	1510	1260	1010
	13	170586	4260	3410	2840	2440	2130	1710	1420	1140
	15	196830	4920	3940	3280	2810	2460	1970	1640	1310
	17	223074	5580	4460	3720	3190	2790	2230	1860	1490
	19	249318	6230	4990	4160	3560	3120	2490	2080	1660
	21	275562	6890	5510	4590	3940	3440	2760	2300	1840

*** See page 20 for explanation of coefficient 2 in numerator. Coefficient 10 in denominator is introduced simply to reduce the magnitude of relative stiffness values.**

table 4. stiffness of columns

values of *K* for columns $K=\frac{I}{10h}$ $d=\text{depth}$ $b=\text{width}$ $I=\frac{bd^3}{12}$

d	b	I	h : Height of column (feet)								d	b	I	h : Height of column (feet)							
			8	9	10	11	12	14	16	20				8	9	10	11	12	14	16	20
8	10	427	5	5	4	4	4	3	3	2	24	12	13824	175	155	140	125	115	100	85	70
	12	512	6	6	5	5	4	4	3	3		14	16128	200	180	160	145	135	115	100	80
	14	597	7	7	6	5	5	4	4	3		18	20738	260	230	205	190	175	150	130	105
	18	768	10	9	8	7	6	5	5	4		22	25344	315	280	255	230	210	180	160	125
	22	939	12	10	9	9	8	7	6	5		26	29952	375	335	300	270	250	215	185	150
	26	1109	14	12	11	10	9	8	7	6		30	34560	430	385	345	315	290	245	215	175
	30	1280	16	14	13	12	11	9	8	6		36	41472	520	460	415	375	345	295	260	205
	36	1536	19	17	15	14	13	11	10	8		42	48384	605	540	485	440	405	345	300	240
10	10	833	10	9	8	8	7	6	5	4	26	12	17576	220	195	175	160	145	125	110	90
	12	1000	13	11	10	9	8	7	6	5		14	20505	255	230	205	185	170	145	130	105
	14	1167	15	13	12	11	10	8	7	6		18	26364	330	295	265	240	220	190	165	130
	18	1500	19	17	15	14	13	11	9	8		22	32223	405	360	320	295	270	230	200	160
	22	1833	23	20	18	17	15	13	11	9		26	38081	475	425	380	345	315	270	240	190
	26	2167	27	24	22	20	18	16	14	11		30	43940	550	490	440	400	365	315	275	220
	30	2500	31	28	25	23	21	18	16	13		36	52728	660	585	525	480	440	375	330	265
	36	3000	38	33	30	27	25	21	19	15		42	61516	770	685	615	560	515	440	385	310
12	10	1440	18	16	14	13	12	10	9	7	28	12	21952	275	245	220	200	185	155	135	110
	12	1728	22	19	17	16	14	12	11	9		14	25611	320	285	255	235	215	185	160	130
	14	2016	25	22	20	18	17	14	13	10		18	32928	410	365	330	300	275	235	205	165
	18	2592	32	29	26	24	22	19	16	13		22	40245	505	445	400	365	335	285	250	200
	22	3168	40	35	32	29	26	23	20	16		26	47563	595	530	475	430	395	340	295	240
	26	3744	47	42	37	34	31	27	23	19		30	54880	685	610	550	500	455	390	345	275
	30	4320	54	48	43	39	36	31	27	22		36	65856	825	730	660	600	550	470	410	330
	36	5184	65	58	52	47	43	37	32	26		42	76832	960	855	770	700	640	550	480	385
14	10	2287	29	25	23	21	19	16	14	11	30	12	27000	340	300	270	245	225	195	170	135
	12	2744	34	30	27	25	23	20	17	14		14	31500	395	350	315	285	265	225	195	160
	14	3201	40	36	32	29	27	23	20	16		18	40500	505	450	405	370	340	290	255	205
	18	4116	51	46	41	37	34	29	26	21		22	49500	620	550	495	450	415	355	310	250
	22	5031	63	56	50	46	42	36	31	25		26	58500	730	650	585	530	490	420	365	295
	26	5945	74	66	59	54	50	42	37	30		30	67500	845	750	675	615	565	480	420	340
	30	6860	86	76	69	62	57	49	43	34		36	81000	1010	900	810	735	675	580	505	405
	36	8232	103	91	82	75	69	59	51	41		42	94500	1180	1050	945	860	790	675	590	475
16	10	3413	43	38	34	31	28	24	21	17	32	12	32768	410	365	330	300	275	235	205	165
	12	4096	51	46	41	37	34	29	26	20		14	38229	480	425	380	350	320	275	240	190
	14	4779	60	53	48	43	40	34	30	24		18	49152	615	545	490	445	410	350	305	245
	18	6144	77	68	61	56	51	44	38	31		22	60075	750	670	600	545	500	430	375	300
	22	7509	94	83	75	68	63	54	47	38		26	70997	885	790	710	645	590	505	445	355
	26	8875	111	99	89	81	74	63	55	44		30	81920	1020	910	820	745	685	585	510	410
	30	10240	128	114	102	93	85	73	64	51		36	98304	1230	1090	985	895	820	700	615	490
	36	12288	154	137	123	112	102	88	77	61		42	114688	1430	1270	1150	1040	955	820	715	575
18	10	4860	61	54	49	44	41	35	30	24	34	12	39304	490	435	395	355	330	280	245	195
	12	5832	73	65	58	53	49	42	36	29		14	45855	575	510	460	415	380	330	285	230
	14	6804	85	76	68	62	57	49	43	34		18	58956	735	655	590	535	490	420	370	295
	18	8748	109	97	87	80	73	62	55	44		22	72057	900	800	720	655	600	515	450	360
	22	10692	134	119	107	97	89	76	67	53		26	85159	1060	945	850	775	710	610	530	425
	26	12636	158	140	126	115	105	90	79	63		30	98260	1230	1090	985	895	820	700	615	490
	30	14580	182	162	146	133	122	104	91	73		36	117912	1470	1310	1180	1070	980	840	735	590
	36	17496	219	194	175	159	146	125	109	87		42	137564	1720	1530	1380	1250	1150	985	860	690
20	10	6667	83	74	67	61	56	48	42	33	36	12	46656	585	520	465	425	390	335	290	235
	12	8000	100	89	80	73	67	57	50	40		14	54432	680	605	545	495	455	390	340	270
	14	9333	117	104	93	85	78	67	58	47		18	69984	875	780	700	635	585	500	435	350
	18	12000	150	133	120	109	100	86	75	60		22	85536	1070	950	855	780	715	610	535	430
	22	14667	183	163	147	133	122	105	92	73		26	101088	1260	1120	1010	920	840	720	630	505
	26	17333	217	193	173	158	144	124	108	87		30	116640	1460	1300	1170	1060	970	835	730	585
	30	20000	250	222	200	182	167	143	125	100		36	139968	1750	1560	1400	1270	1170	1000	875	700
	36	24000	300	267	240	218	200	171	150	120		42	163296	2040	1810	1630	1480	1360	1170	1020	815
22	10	8873	111	99	89	81	74	63	55	44	38	12	54872	685	610	550	500	460	390	345	275
	12	10648	133	118	106	97	89	76	67	53		14	64017	800	710	640	580	535	455	400	320
	14	12422	155	138	124	113	104	89	78	62		18	82308	1030	915	825	750	685	590	515	410
	18	15972	200	177	160	145	133	114	100	80		22	100599	1260	1120	1010	915	840	720	630	505
	22	19521	244	217	195	177	163	139	122	98		26	118889	1490	1320	1190	1080	990	850	745	595
	26	23071	288	256	231	210	192	165	144	115		30	137180	1710	1520	1370	1250	1140	980	855	685
	30	26620	333	296	266	242	222	190	166	133		36	164616	2060	1830	1650	1500	1370	1180	1030	825
	36	31944	399	355	319	290	266	228	200	160		42	192052	2400	2130	1920	1750	1600	1370	1200	960

*See footnote to Table 3.

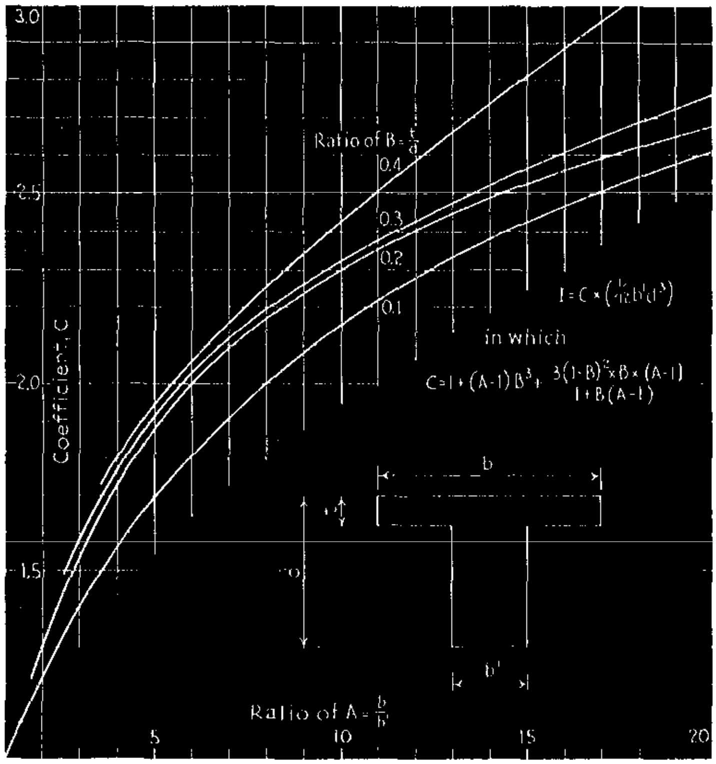


Fig. 5 — Coefficients for moment of inertia of T-beams.

A quicker and usually acceptable procedure in building design is to select K for T-beams from Table 3. Allowance has been made for effect of flange by doubling the moment of inertia of the gross web section. Fig. 5 indicates that for values of $\frac{t}{d}$ between 0.2 and 0.4, a multiplier of 2 corresponds closely to a flange width equal to six times the web width. This will be considered a reasonable allowance for most T-beams. As seen from Fig. 5, variations in depth ratio, $\frac{t}{d}$, have relatively little effect on I. For rectangular beams the factor of 2 in Table 3 should be omitted.

Table 4 contains relative stiffnesses for columns computed on basis of gross concrete section, neglecting reinforcement as is done for beams. This is in accordance with Section 702 of the 1956 edition of the ACI Code. Other building codes, such as the 1936 edition of the ACI Code, required that allowance be made for reinforcement in columns. If this is to be done, the best procedure is probably to add a percentage to the I and K values taken from Table 4. An increase of 10 per cent is considered reasonable for usual column sections.

6/signs

Two sign conventions are in general use. One must be chosen and used throughout the operation of moment distribution. Fixed-end moments for gravity loads may be recorded either as (1) negative on both sides of a joint, or (2) negative on one side of the joint and positive on the other side. Both have advantages. The choice between them depends on the type of problem. Convention (1) is usually applied to problems involving distribution within a single level. It is identical to the usual design concept that considers moments to be negative when they produce tension in the top of beams. However, (2) is preferred when moments are distributed from floor to floor.* Convention (1) has been adopted here.

One simple, sure way to determine signs is to visualize curvature of beams and rotation of joints. In accordance with the sign convention chosen, moments are negative in "humps" (tension in top) and positive in "sags" (tension in bottom).

For illustration, a fixed-ended beam when loaded conforms to the shape indicated in Fig. 6(a). The central portion sags (plus) and the outer portions hump (minus). Therefore, moments at fixed ends are negative in horizontal beams with gravity loading.

Examples of clockwise and counterclockwise rotation about a central support, *B*, of a continuous, fixed-ended beam is illustrated in Fig. 6(b) and

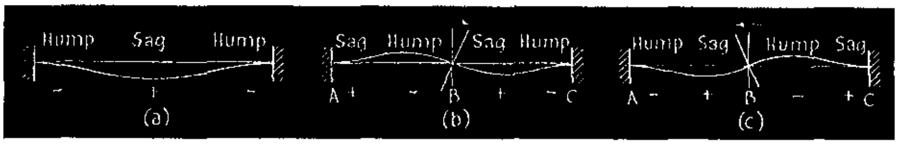


Fig. 6 — Signs illustrated by means of curvature and deflection of beams.

6(c). The beam sags on one side and humps on the other side of the support. It can readily be seen that the sag adjacent to *B* would be on the span that had the greater fixed-end moment at *B*. When the beam sags at one end of a member because of joint rotation, it will hump at the opposite end.

The fundamental sign concepts illustrated in Fig. 6 are sufficient for the type of analysis in this text and will be the sign convention used in the following sections.

7/moment distribution at one joint

Consider the frame in Fig. 7(a), which consists of four members fixed at their far ends. Apply at their common end, joint *B*, an external moment *U*. This moment will rotate joint *B* until the sum of the resisting moments induced in the four members is equal to *U*. Since all members are rigidly con-

*As illustrated in *Moment Distribution Applied to Continuous Concrete Structures* and *Concrete Building Frames Analyzed by Moment Distribution*, available only in the United States and Canada from the Portland Cement Association.

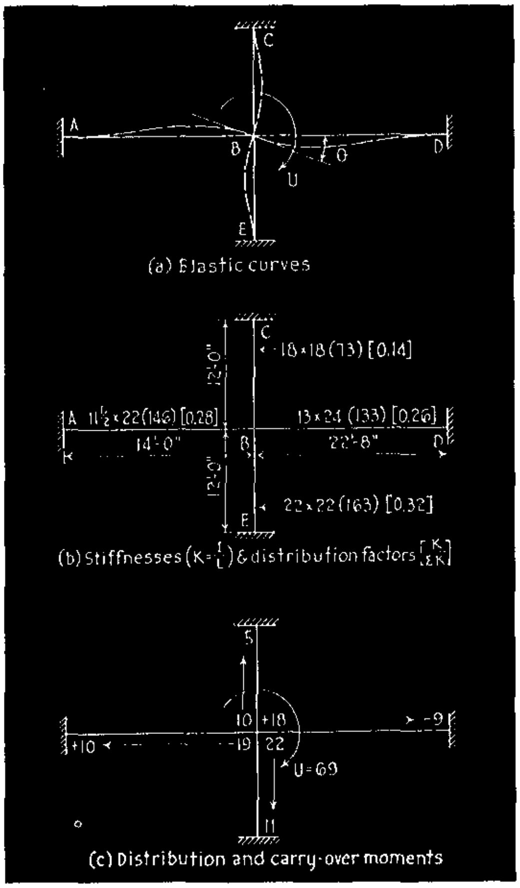


Fig. 7 – Frame consisting of four members with far ends fixed.

nected at B, each member will rotate through the same angle at this joint. The problem is to determine the moments induced at both ends of each of the four members.

First compute the relative stiffnesses $K = \frac{I}{L}$ for all members; then their sum, ΣK; and finally the four ratios of K divided by ΣK. These ratios are called " distribution factors" and will be denoted at D_{BA}, D_{BC}, D_{BD} and D_{BE}. It will be shown that the moments induced in the beams at B, called "distributed moments," equal

$$M_{BA} = D_{BA} \times U;$$
$$M_{BC} = D_{BC} \times U;$$
$$M_{BD} = D_{BD} \times U;$$
$$M_{BE} = D_{BE} \times U.$$

Summation: $\Sigma M_{BX} = U \Sigma D_{BX} = U.$

It has been stated that the sum of the distributed moments at B must equal the external moment U, or that $\Sigma M_{BX} = U$. This requirement is satisfied since the sum of the four distribution factors ΣD_{BX} equals unity. It has been shown in Section 4, "Stiffness and Carry-over Factor," that moments required to produce a given angle change are proportional to the stiffness K. This requirement is also satisfied since the D-factors are proportional to the K-factors. Therefore, *the distributed moment* M_{BX} *equals* U *multiplied by the distribution factor* D_{BX}.

According to one of the equations derived in Section 4 for a prismatic member, half of the distributed moment is "carried over" to the opposite fixed end.

8/ example of moment distribution at one joint

The frame in Fig. 7(b) is the same as that in Fig. 7(a), but numerical values have been inserted. Sizes and lengths of beams and columns are given for which stiffnesses may be selected from Tables 3 and 4. Joint B is being rotated clockwise by an external moment, $U = 69$ ft.kips. The problem is to determine the distributed moments and the carried-over moments.

Initially, calculate the sum of the four stiffnesses, $\Sigma K = 146 + 73 + 133 + 163 = 515$, and the distribution factors, $D = \frac{K}{\Sigma K}$. These are recorded in Fig. 7(b) and, it should be noted, add up to unity around a joint. The distributed moments induced at B in Fig. 7(c) equal UD_{BX}, which gives 19 and 18 in the beams, and 10 and 22 in the columns. The four distributed moments must add up to 69. The rotation of joint B also produces moments at the opposite fixed ends of all the members. These carry-over moments are half of the distributed moment.

The sketch of the distorted frame in Fig. 7(a) indicates that the clockwise rotation of joint B creates a hump to the left, but a sag to the right. Therefore, 19 is negative, but 18 is positive. There is also a sag at A and a hump at D; therefore the carried-over moments are $+10$ at A and -9 at D. No signs are given for the column moments.

In moment distribution, U is called "unbalanced moment" and is computed as the numerical difference between adjacent fixed-end moments. For illustration, let beams AB and BD in Fig. 7 be loaded as shown in the second and third beam in Fig. 3. The fixed-end moments for total load are $M^F_{BA} = 78$, and $M^F_{BD} = 147$. The numerical difference is $U = 69$ ft.kips.

9/ limitations in two-cycle moment distribution

The procedure described in Sections 7 and 8 in regard to moment distribution at *one* joint is an elemental part of the general procedure, in which *many* joints are involved. The entire frame may be divided into "unit frames," each of which is treated as in Fig. 7. Each joint may be rotated and relocked one or more times. One operation of rotating and relocking corresponds to what is known as a "cycle." The main problem in these operations is the recording of calculations. For the general case involving distribution of moments between various levels, a type of recording is discussed and illustrated in *Concrete Building Frames Analyzed by Moment Distribution.**

The scope of this text is limited to that type of building frame in which

*Available only in the United States and Canada from the Portland Cement Association.

the following assumption is permissible, as stated in part in Section 702 of the ACI Code under the heading "Conditions of Design": ". . . the far ends of the columns may be assumed as fixed." This assumption is accepted generally and simplifies the moment analysis to a great extent. As a result, beams in one floor may be designed without regard to those above and below. Also, analytical work is simplified. All building frame analyses for vertical load discussed in this text are based on this assumption.

10/special arrangement of moment distribution for building frames

Fig. 8 contains five groups of calculations for moments at ends of four beams. The loads on the beams are shown in Fig. 3, in which moments have been computed for beams with fixed ends. Since stiffnesses are not known beforehand, it will be assumed that they are all equal. In this case, the stiffness ratio or distribution factor for each member at any joint equals 1 *divided by the number of all adjacent members,** recorded as ⅓ or ¼ in Fig. 8. The problem is to determine maximum end moments in the beams.

To determine maximum end moment at A, place total load on AB and dead load on BC as shown in (A). Since B is considered fixed, the end moments at B are 172 to the left and 37 to the right. The difference is $U = 135$. When B is released, the moment distributed to the left is $UD = 135 \times 1/4$; and the moment carried to A while it remains fixed is $UD \times 1/2 = 135 \times 1/4 \times 1/2 = 17$. Refer to Fig. 6(c) for a deflection curve illustrating this case. The counterclockwise rotation of joint B creates a hump in the beam at A that results in a negative value for the carry-over moment. This value is written in Fig. 8(A), but neither the external moment U nor the distributed moment UD is recorded. Joint B is then relocked in its new position.

The next step is to examine A, which so far has been considered locked. The original fixed-end moment is -172, but the release and rotation of B transfers an additional moment to A. At this stage, the modified total fixed-end moment is $-172 - 17 = -189$. Since there is no fixed-end moment to the left of A, U at A equals 189. Releasing A and permitting it to rotate induces a distributed moment at A equal to $UD = 189 \times 1/3 = 63$. When joint A rotates clockwise, it tends to create a sag in the beam at A, which results in a positive moment of 63 and a final maximum moment at A of $-189 + 63 = -126$ ft.kips.

The procedure explained in the last two paragraphs takes much longer to describe than to perform, and the explanation is superfluous for designers who are familiar with moment distribution. In Fig. 8, the only new feature is the manner of recording and the arrangement of the calculations. The full advantage of the modification proposed will be discussed later, but first a brief description will be given in connection with group (B) in Fig. 8.

To determine moments at B, place loading as illustrated in Fig. 8(B),

*The general expression is

$$\text{distribution factor} = \frac{\text{stiffness of member}}{\text{sum of stiffnesses of all members at joint}}.$$

For further discussion, see Sections 5 and 18.

and release joints A and C. The figure clearly presents the computation of the two moments, 29 and 1, carried over to B. When A and C are released, they rotate so as to create a hump on both sides of the fixed joint B. Therefore, both 29 and 1 are negative. While B is still considered fixed, the modified total fixed-end moments at B are -201 to the left and -79 to the right. The unbalanced moment at B is numerically equal to $201 - 79 = 122$. It is multiplied by the distribution factor of $\frac{1}{4}$ at either side when joint B is released. In regard to signs, refer to Fig. 6(c) for the counterclockwise rotation of joint B. Distributed moments at columns C, D and E are determined by the same procedure.

The operations illustrated in Fig. 8 cover *two* complete cycles of distribution, which in the ordinary type of recording means that moments are distributed *twice*. However, in Fig. 8 only one distribution is in evidence, because the usual two distributions have been combined in one operation. Moments are carried over first and are included with fixed-end moments *before* the distribution is made.

One advantage of the proposed arrangement is that it automatically limits the analytical work to the degree required for reasonable accuracy. Two cycles of distribution are all that are needed when columns are assumed fixed at ends above and below the floor considered. Designers who fail to

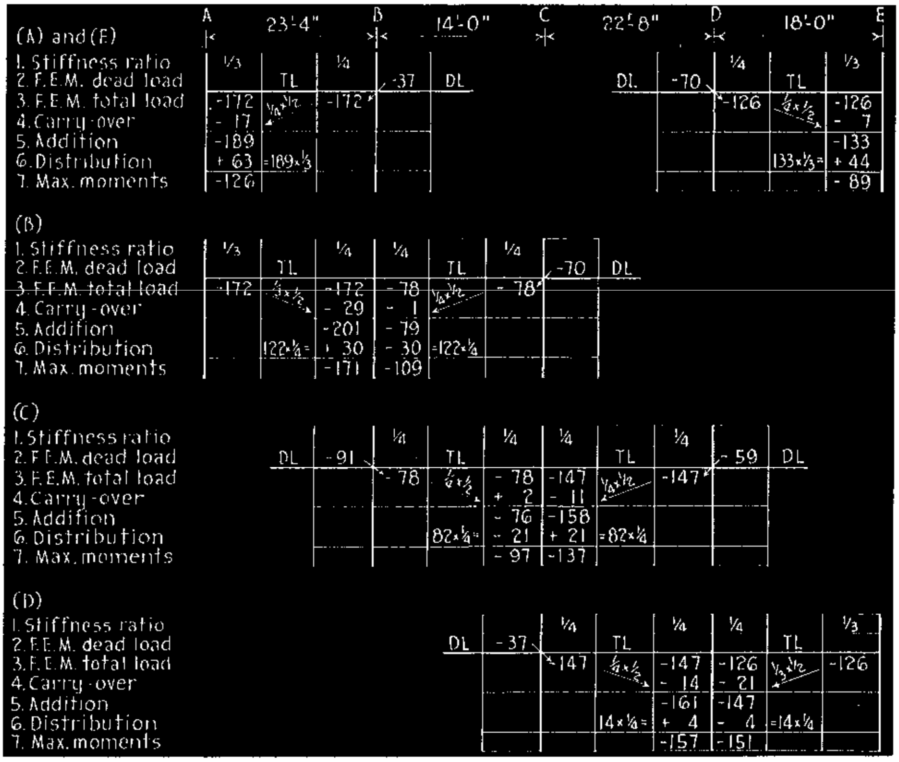


Fig. 8 — Moment distribution illustrated in its various elements.

realize this often include three or even four cycles of distribution at considerable waste of time.

In Fig. 8, the five groups of calculations have five different arrangements of load. The total load is carried on spans adjacent to the particular joint at which maximum moments are to be computed, but dead load only is carried on the next adjacent spans. The calculations are so arranged that all five groups in Fig. 8 can be consolidated into one single group, as has been done in Fig. 9.

Note that all the moments in line 7 of Fig. 9 are *maximum* values and that it requires five types of loading to produce them. Computing moments as in Fig. 9, therefore, will save considerable time. In addition, some of the blank spaces in Fig. 9 are available for a quick, convenient determination of maximum moments at midspan. Such midspan moments, which ordinarily are determined only after a rather tedious set of calculations, may be recorded directly in Fig. 9. This operation is illustrated in Fig. 10 and described in Section 11, "Maximum Moments at Midspan."

The arrangement suggested accommodates any type of loading, whether uniform or concentrated, symmetrical or unsymmetrical. It is effective for any combination of stiffnesses of the various beams and columns, and can be used also for haunched beams and flared columns. For highly irregular cases in which it is necessary to discard the assumption of columns' being fixed above and below, the fundamental calculations remain unchanged. The proposed method needs merely to be extended, not to be discarded.

It may also be considered an advantage to start with the fixed-end moments, which generally make up the bulk of the final moments. In many instances, corrections may not need to be added to the fixed-end moments, or they may be estimated. If the corrections must be computed, calculations without the use of a slide rule will often be sufficient. The calculations that follow the recording of fixed-end moments are relatively unimportant and may be made with great speed at little risk of serious error.

Yet another advantage results from the use of fixed-end moments. When the analysis begins, cross-sectional dimensions must be estimated. If there is any doubt about sizes of beams, the fixed-end moments in line 3 of Fig. 9 should be computed first and used for preliminary design. Stiffnesses may then be selected from Tables 3 and 4 and stiffness ratios recorded in line 1 of Fig. 9. If this is done, it will seldom be necessary to revise the distribution of moments. Another convenient use of fixed-end moments is discussed in Section 17, "Point of Inflection."

	A	23'-4"	B		14'-0"	C		22'-8"	D		18'-0"	E
1. Stiffness ratio	1/3		1/4	1/4		1/4	1/4		1/4	1/4		1/3
2. F.E.M. dead load	—		-91	-37		-37	-70		-70	-59		—
3. F.E.M. total load	-172		-172	-78		-78	-147		-147	-126		-126
4. Carry-over	-17		-29	-1		+2	-11		-14	-21		-7
5. Addition	-189		-201	-79		-76	-158		-161	-147		-133
6. Distribution	+63		+30	-30		-21	+21		+4	-4		+44
7. Max. moments	-126		-171	-109		-97	-137		-157	-151		-89

Fig. 9 — Special arrangement for building frames.

	A	23'-4"	B		14'-0"	C		22'-8"	D		18'-0"	E
Column moments	63		–			–			–			44
Stiffness ratio	1/3		1/4	1/4		1/4	1/4		1/4	1/4		1/3
F.E.M. dead load	–		- 91	- 37		- 37	- 70		- 70	- 59		–
F.E.M. total load	-172	+ 99	-172	- 78	+ 73	- 78	-147	+85	-147	-126	+ 63	-126
	- 17	+ 11	- 29	- 1	+ 1	+ 2	- 11	+ 7	- 14	- 21	+ 13	- 7
	-189	+ 18	-201	- 79	- 1	- 76	-158	+ 9	-161	-147	+ 5	-133
	+ 63	–	+ 30	- 30	–	- 21	+ 21	–	+ 4	- 4	–	+ 44
Max. beam moments	-126	+128	-171	-109	+ 73	- 97	-137	+101	-157	-151	+ 81	- 89
Column moments	63		–			–			–			45

Example: $\frac{17}{2}(1+\frac{1}{3}) = 11$, $\frac{29}{2}(1+\frac{1}{4}) = 18$

Fig. 10 — Complete schedule including maximum moments at midspan.

11/maximum moments at midspan*

The calculations recorded in Fig. 9 are repeated in Fig. 10 and others are added for the determination of maximum moments at midspan.

The usual procedure for calculating midspan moments is to consider two loading conditions, in each of which alternate spans have live loads. Since the object is to determine end moments for each of these loadings, this step involves calculations occupying approximately twice the space given in Fig. 9. The average value of moments at opposite ends of each beam is finally computed and deducted from the midspan moment in beams considered simply supported.

It is much faster to determine maximum moments at midspan, as in Fig. 10. The positive midspan moments shown as 99, 73, 85 and 63 are taken from the data in Fig. 3 for beams with fixed ends. Certain corrections are to be added to these moments in order to obtain the final maximum moments at midspan.

The procedure will be illustrated for span AB. Multiply -17 at A by $-\frac{1}{2}(1 + \frac{1}{3})$, in which $\frac{1}{3}$ is the distribution factor at A, and record the result, $+11$. Multiply -29 at B by $-\frac{1}{2}(1 + \frac{1}{4})$, in which $\frac{1}{4}$ is the distribution factor at B, and record the result, $+18$. The sum, $+99 + 11 + 18 = +128$, is the maximum moment at midspan. All the other corrections are determined in the same manner. An additional example is given in Section 19 for haunched beams, to which reference is made for explanation and derivation. The corrections for prismatic beams in Fig. 10 are simply a special case of those discussed in Section 19 for haunched beams.

The accuracy of the two-cycle procedure in Fig. 10 is illustrated in Fig. 11. All moments in Fig. 11 are based on the fixed-ended beams taken from Fig. 3, the stiffness ratios taken from Fig. 10, and on the assumption that columns are fixed at ends above and below the floor considered. The results of both the two-cycle and the four-cycle method of moment distribution are

	A		B			C			D			E
Mom. from two cycles	-126	+128	-171	-109	+73	-97	-137	+101	-157	-151	+81	-89
Mom. from four cycles	-130	+127	-172	-109	+72	-95	-137	+100	-159	-151	+79	-91

Fig. 11 — Accuracy of two-cycle procedure.

*In certain irregular cases, it may be necessary to determine maximum positive moment at points other than at midspan.

in close agreement for this example. However, the determination of maximum moment at midspan assumes that rotation in adjacent joints is relatively small, with negligible effect on midspan moment. When adjacent joints have large unbalanced moments and are very flexible, consideration should be given to the carried-over moment.

12 / minimum moments at midspan

In the frame analyzed in Fig. 10, the second span from the left, span *BC*, is only 14 ft. long and is flanked by much longer spans. It is possible that negative moments may extend across the short intermediate span. This possibility will now be investigated.

The loading in Fig. 12 has dead load only on span *BC* and total load on the adjacent spans. The end moments of —172, —37 and —147, together with the midspan moment +34, are taken from Fig. 3. The same fixed-end moments as those in Fig. 10 are used, but in a different arrangement.

The procedure is the same as that described in previous sections. For further explanation of Fig. 12, consider *B* fixed while *C* is permitted to rotate. The unbalanced moment at *C*, $147 - 37 = 110$, is to be multiplied by $\frac{1}{4} \times \frac{1}{2}$. The result, 14, is the moment carried to *B*. Since the individual rotations of *B* and *C* create sag at the respective opposite joints, the signs of the carry-over moments are positive. Multiply +14 by $-\frac{1}{2}(1 + \frac{1}{4})$ and +17 by $-\frac{1}{2}(1 + \frac{1}{4})$. Record the results and add them to +34; this gives a *minimum* moment of +14 at midspan. Similarly, the minimum moment at midspan of *DE* is +28.

These moments are much smaller than those recorded in Fig. 10 but they are still positive. With certain framing proportions, however, the minimum moments are negative. The matter is discussed further in Section 17, "Point of Inflection," and Fig. 12 is referred to again in Section 21, "Determination of Column Moments."

The same consideration should be given to carried-over moments from very flexible joints as that mentioned at the end of Section 11.

	A	23'-4"	B		14'-0"	C		22'-0"	D		18'-0"	E
Column moments	–	TL	45		DL	35		TL	23		DL	–
Stiffness ratio	1/3		1/4	1/4		1/4	1/4		1/4	1/4		1/3
F.E.M. TL or DL	-172		-172	-37	+34	-37	-147		-147	-59	+29	-59
			-29	+14	-9	+17	-11		14	-10	+6	+11
			-201	-23	-11	-20	-158		-161	69	-7	
				-45								
Min. beam moments				-68	+14						+28	
Column moments	–		45			35			23			–

Fig. 12 — Minimum moments at midspan.

13 / clear span and center-to-center span

In analysis of frames, members are usually represented by their centerlines. The ACI Code specifies that "in analysis of continuous frames, center-to-

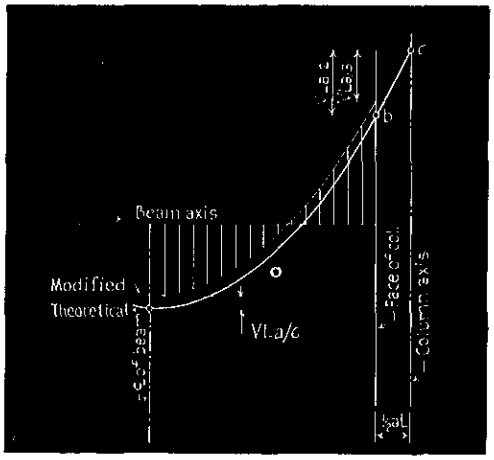


Fig. 13—Clear span versus center-to-center span.

center distances may be used in the determination of moments. Moments of faces of supports may be used for design of beams and girders."

These simplifications in design imply that reactions are concentrated at the column axes and that the moments of inertia at the ends of the beams and girders are unaffected by the stiffening effect of the adjoining supports. For average design conditions, the error introduced by neglecting these factors is small. However, it should be pointed out that while these assumptions yield a conservative value for moment at the centerline, they underestimate the critical moments at the face of the support. For this reason, corrections should be applied to the moment curve determined on the basis of center-to-center distances, especially when the width of the support is large.

Other than a rigorous, two-dimensional analysis, no exact, easily applied method is available for computing the correction. Such accuracy, however, is unnecessary. In all cases, the magnitude of the correction can be established on the basis of limiting assumptions.

With respect to the distribution of the reaction over the column, the centroid of the reaction must occur between the face of the column and its axis. If it is assumed that the reaction is concentrated at the face of the support, but that the span of the beam is still measured from center to center of columns, the correction applied at *b* to the theoretical moment curve shown in Fig. 13 is $\frac{1}{4}\,VLa^2$. For usual values of *a*, this correction is insignificant and will be ignored.

On the other hand, the effect of the restraint imparted by the column is more pronounced. The use of center-to-center span distance assumes that the beam is free to deflect at *b*. This movement is restricted by the column. The effect of such restriction can be approximated by assuming that the moment of inertia of the beam over the column varies. A reasonable assumption is that the moment of inertia is infinite in this area. On this basis the moment at *b* computed by means of Table 56 in *Handbook of Frame Constants* is $\frac{1}{6}\,VLa$ greater than that indicated by the theoretical curve in Fig. 13. This correction applies along the entire length of the beam and therefore the modified moment curve is ⅙ VLa higher than the theoretical curve. This corresponds to a reduction of the moment at the column face of ⅓ VLa.

For columns, it appears reasonable to take the length equal to the story height. Theoretical column moments obtained in this manner are larger than those existing at the top and bottom of the beams. This will be considered in the discussion of column moments given in Section 20, "Bending in Columns."

14 / shear in continuous beams

Shear at the end of a beam that is part of a frame is determined as the sum of the shear in the beam considered simply supported and a correction due to the difference between end moments produced by the frame action. The correction is usually small compared with the simple beam shear, especially in interior spans.

In end spans the correction may be obtained from the moment calculations in Fig. 10. As an illustration: In span AB, the end moments are 171 and 126. The difference between them is 45, and the shear correction is 45 divided by the span length ($L = 23$ ft. 4 in.), which equals 1.9 kips. The end shear at B in the beam AB considered simply supported is 37.5 kips taken from loads in Fig. 3. Therefore, the total shear at B is $37.5 + 1.9 = 39.4$ kips; at A it is $37.5 - 1.9 = 35.6$ kips. Similarly, the shear at D in DE is $33.2 + \frac{151 - 89}{18} = 33.2 + 3.4 = 36.6$ kips.

For interior beams the loading conditions for maximum moments are not quite as favorable for determination of maximum shears. For illustration, consider the problem to determine maximum shear at D in CD. The shear in the simply supported beam is 33.1 kips. In Fig. 10, 157 ft.kips is the maximum moment at end D, but 137 ft.kips at C is not the moment due to the loading that will result in maximum shear at D. The moment at C is too large. Therefore, computing the corrections as $\frac{157 - 137}{22.67} = 0.9$ kips is not on the safe side. The correction is small in comparison with the figure it modifies. As a result, it is often sufficient to use some rough approximations such as twice its value. In this case, the shear would be $33.1 + (2 \times 0.9) = 34.9$ kips.

It may be necessary under special circumstances to determine the shear correction accurately. The end moment M_{CD} to be substituted for 137 ft.kips

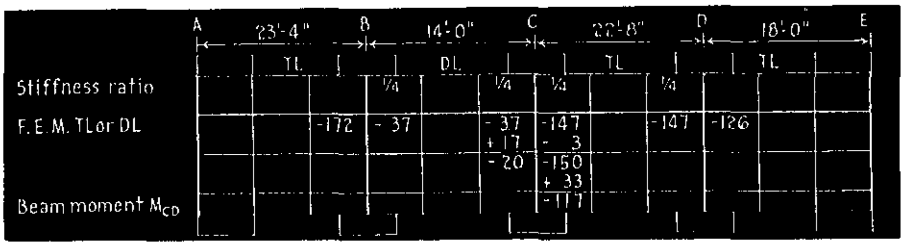


Fig. 14 – End moment for shear determination.

in the example above may be easily computed as shown in Fig. 14. The fixed-end moments in Fig. 14 are available from Fig. 10 and the distribution shown is the procedure explained in connection with Fig. 8. The shear correction equals $\frac{157 - 117}{22.67} = 1.8$ kips. This represents only 5 per cent of the total shear, $33.1 + 1.8 = 34.9$ kips.

15 / example of reduction in theoretical moments

As discussed in Section 13, "Clear Span and Center-to-Center Span," moments determined on basis of centerline distances should be reduced at the face of columns before being used for proportioning of the members. It was recommended that the reduction be $\frac{1}{3}\,VLa$ for end moments and $\frac{1}{6}\,VLa$ for positive moments. V is the end shear and may for this purpose be taken as the shear in simply supported beams. The width of support, aL, in this example will be taken as 20 in. for all five columns.

	A	23'-4"	B		14'-0"	C		22'-8"	D		18'-0"	E
Theoretical max. mom.	-126	+128	-171	-109	+73	-97	-137	+101	-157	-151	+81	-89
V: Max. end shear	37		37	23		23	32		32	33		33
Deduct ⅓ VLa or ⅙ VLa	21	10	21	13	6	13	18	9	18	18	9	18
Design moments	-105	+118	-150	-96	+67	-84	-119	+92	-139	-133	+72	-71
Tensile steel required	3.5	3.9	4.9	3.2	2.2	2.8	3.9	3.0	4.6	4.4	2.4	2.3
Top at support	2-#7 + 2-#10		2-#10 + 2-#10			2-#10 + 2-#9			2-#9 + 2-#10			2-#10 + 2-#6
Trussed bars		2-#10			2-#10			2-#9			2-#10	
Straight bars, bott.		2-#8			2-#6			2-#7			2-#6	
Tensile steel provided	3.74	4.12	5.08	5.08	3.42	4.54	4.54	3.20	4.54	4.54	3.42	3.42

Fig. 15 — Deductions in theoretical moments and proportioning of reinforcement.

The theoretical moments taken from Fig. 10 are recorded in Fig. 15, with end shears determined from the loads and spans (minus 20 in.) taken from Fig. 3. Values for the ends and midspans are computed and deducted from the theoretical moments.

16 / proportioning of reinforcement in beams

To continue the example in Section 15, consider the problem to proportion all tensile reinforcement for $f_s = 20{,}000$ psi and $d = 21$ in. by the accepted straightline theory of flexure. The first four lines in Fig. 15 were discussed in Section 15. The areas and arrangement of tensile reinforcement are recorded in the next four lines. Negative reinforcement is given first and consists of trussed bars with the exception of the first and last items, which are short, straight top bars. Positive reinforcement is given in the next two lines for trussed bars and straight bottom bars, respectively.

Comparing areas required with areas provided, it is seen that the latter is often much larger than the former. The most conspicuous fact is the deviation from the customary rule-of-thumb of "bending up one-half of the bars." Actually, a far greater proportion of positive reinforcement is bent.

17/point of inflection

The designer should specify where to bend up bars and how far negative reinforcement shall extend into adjacent spans. The generally adopted rule is that reinforcing bars shall be extended at least 12 diameters beyond the point of inflection or beyond the point at which they are no longer needed to resist stress. In the discussion that follows, special attention is given to negative reinforcement.

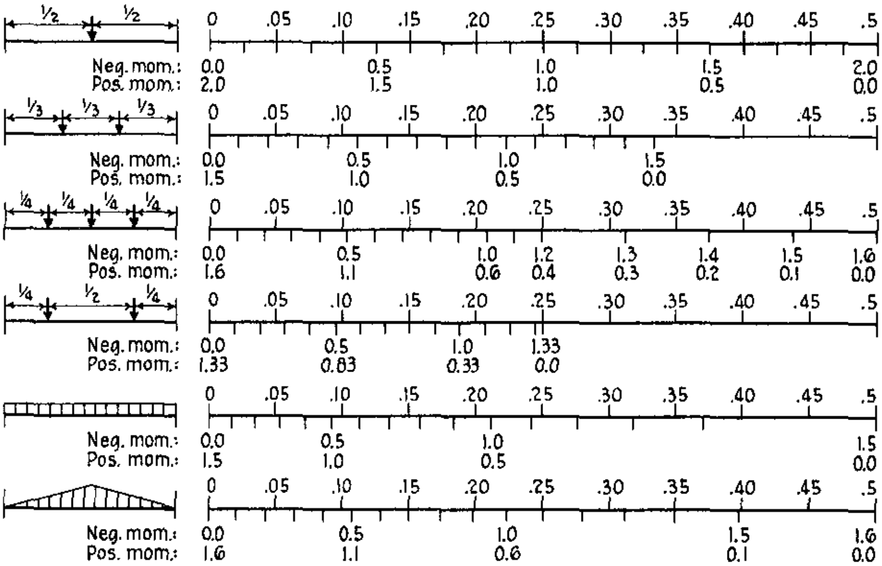
The problem is to determine the point of inflection for negative moments near B in beam BC. Refer to Fig. 10.

The final maximum moment M_{BC} is 109 and with the original fixed-end moment M^F_{BC} of 78 has a ratio of $109 \div 78 = 1.4$. The greater portion of the loading on BC is concentrated load at midspan. Locate this type of loading in Table 5 and proceed in the line marked "Neg. mom." to the right until the ratio of 1.4 is reached. Just above that point on the adjacent scale, the value of 0.35 appears. This signifies that the point of inflection is a distance of $0.35L$ from the support, L being the span length.

Span BC is particularly short in comparison with the adjacent spans. Under such circumstances, it is possible that a greater distance to the point of inflection may be obtained with minimum loading on BC. This loading case is treated in Fig. 12, from which the ratio of final moment to fixed-end moment may be computed as $68 \div 37 = 1.8$. The value in Table 5 for this ratio is $0.45L$ and is farther from the support than the point based on maximum loading. Therefore, negative reinforcement must extend at least 12 diameters beyond the 0.45-point of the span.

The construction of the scales in Table 5 merits a brief explanation. Fig. 16 illustrates the method of construction for a concentrated load at midspan.

***table 5.* points of inflection**



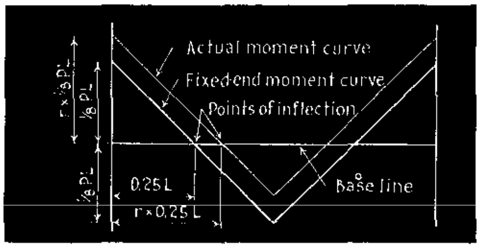


Fig. 16 – Point of inflection.

The heavy white line is the moment curve in a beam with fixed ends, and the point of inflection for this curve is at the quarter-point. If M^F is the fixed-end moment and rM^F is the final moment in the beam, the distance to the point of inflection must be $0.25rL$. This determines the relationship between the scales in Table 5.

Distances to the point of inflection for positive moments are determined in a similar way. Data for several types of loading are given in Table 5. In all instances, actual moments whether at end or at midspan are to be divided by fixed-end moments. The data in Table 5 are correct only for cases in which the moment curves are symmetrical. However, it is usually satisfactory to use Table 5 for cases of dissymmetry. It is applicable for members of constant or variable moment of inertia and may also be used to determine where a certain percentage of the total reinforcement is no longer needed.

Returning to the example in this section, assume that two negative bars extend from B to midspan of BC and can carry a moment of 60 ft.kips. Compute the ratio of $\frac{109 - 60}{78} = 0.63$, which corresponds to $0.16L$ in Table 5. This is the point at which the two bars can carry the tensile stress without help from other trussed bars. The latter cannot be bent down closer to the support than $0.16L$ plus 12 diameters.

18/effect of variation in stiffness

It was stated in Section 10 (page 24) that "since stiffnesses are not known beforehand, it will be assumed that they are all equal. In this case, the stiffness ratio or distribution factor for each member at any joint equals 1 divided by the number of all adjacent members." It is of interest to examine the effect a change in stiffness may have on the results of an analysis.

Inspection of Table 4 indicates that column stiffness is approximately doubled if the dimension of a square column is increased from 12 to 14 in. or from 22 to 26 in. This shows that column stiffness is quite sensitive to change in column size. It is not unusual for a designer to increase the column

Ratio: $\frac{K_{col.}}{K_{beam}}$	A–B 23'-4"			B–C 14'-0"			C–D 22'-8"			D–E 18'-0"		
0.5	-97	+144	-170	-124	+73	-105	-133	+108	-163	-160	+91	-68
1.0	-126	+128	-171	-109	+73	-97	-137	+101	-157	-151	+81	-89
2.0	-147	+117	-171	-97	+73	-90	-141	+95	-153	-142	+73	-105
4.0	-159	+108	-171	-88	+73	-85	-144	+91	-151	-135	+68	-115

Fig. 17 – Variation in stiffness affecting moments in beams.

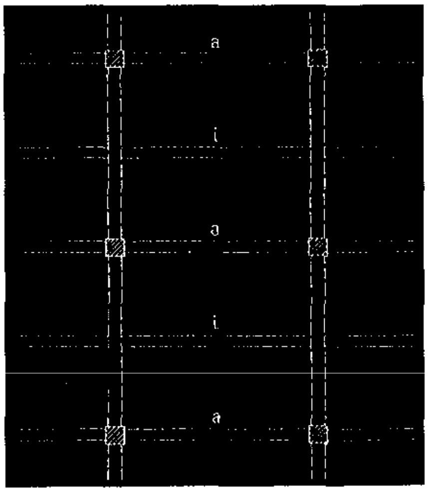


Fig. 18—Stiffness of floor system with two beams per column.

sizes estimated by 2 or even 4 in. when making allowance for bending moment in columns. As a result, the stiffnesses and the analysis may have to be back-checked and perhaps revised.

The effect of variations in stiffness is illustrated in Fig. 17 for ratios of columns to beam stiffness of 0.5, 1.0, 2.0 and 4.0. The tabulated values indicate that some moments, especially those in exterior spans, are sensitive to changes in column stiffness, whereas others are not. It is advisable to be sure that appropriate stiffness values are used in the analysis.

Some question may arise as to what moment of inertia should be adopted for a floor system such as that in Fig. 18. Some designers compute *I* only for the beams marked *a*; others use the sum of *I*-values for beams marked *a* and *i*. The former procedure gives an *I* that is too small and the latter gives an *I* that is too large. The intermediate beams contribute to the actual *I* for the floor construction, the amount depending on the torsional stiffness of the girder.

The beam marked *i* is a part of the frame and its stiffness (or part of it) must be included in the *I*-value for the floor construction. It is probably best to make all the beams identical. Select the *K*-value for one beam from Table 3 and use twice this value for stiffness of one panel of the floor in Fig. 18.

19/ haunched beams

Moments in continuous beams are usually much greater at ends than at midspan. It is unfortunate that only the web is available to take compression at the ends where the moments are greater. As a result, there is a tendency

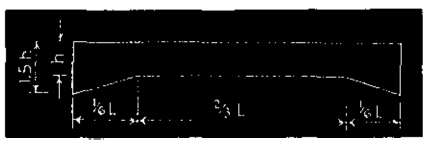


Fig. 19 — Haunched beam.

to deepen the web at the supports and to use haunched beams. The ACI Code specifies that if this is done, "the effect of haunches shall be considered both in determining bending moments and in computing stresses."

Haunching beams at their ends changes fixed-end moments, stiffness, and carry-over factor. For illustration, compare the haunched beam in Fig. 19 with a straight beam. The following values obtain:

	Straight	*Haunched*
Fixed-end moment coefficient for uniform load	0.083	0.093
Stiffness	1.00	1.50
Carry-over factor	0.50	0.59

The changes due to the haunches are so great that they cannot be ignored. Coefficients for haunched members may be selected from *Handbook of Frame Constants.* Many examples involving haunched members are given in *One-Story Concrete Frames Analyzed by Moment Distribution.**

An example of analysis for haunched beams will now be given. The beam loading and span lengths in this example are the same as in Fig. 3. Assume that all beams are symmetrically haunched, that the ratio of maximum depth to minimum depth of beam is 1.5, and that the length of haunch divided by length of span is 0.17 in all beams. Under these circumstances, it can be shown that all the fixed-end moments are approximately 12 per cent greater in the haunched beams than in the prismatic beams. The 12 per cent increase will be used in this example. Moment coefficients for more accurate work may be selected from the references given in the preceding paragraph.** The stiffness of 1.5 and the carry-over factor of 0.6 were selected from the same data.†

In this example all beam stiffnesses are increased 50 per cent because of the haunches. The stiffness ratios or distribution factors equal

$$\frac{1.5}{1.5+1.0+1.0} = 0.4 \text{ for exterior end of exterior beams;}$$

$$\frac{1.5}{1.5+1.5+1.0+1.0} = 0.3 \text{ for all other ends of beams.}$$

The moments in Fig. 20, when distributed and carried over from exterior joints, are multiplied by $0.4 \times 0.6 = 0.24$. In all other cases multiply by $0.3 \times 0.6 = 0.18$. It is seen that the procedure is exactly the same as for prismatic members. The two corrections for maximum midspan moment and the derivation of the corrections +15 and +22 may be computed as illustrated in Fig. 20. For example, the correction originating from −27 at *A* equals $\frac{27}{2}\left(\frac{1}{0.6} + 0.4 - 1.0\right)$. The values of 0.4 and 0.3 are distribution factors, and 0.6 is the carry-over factor.

*Available only in the United States and Canada from the Portland Cement Association.

**These coefficients were obtained by plotting the values given in Tables 42, 43 and 44 in the *Handbook of Frame Constants*, page 19, and interpolating. The use of these tables is discussed in the handbook.

†Note that stiffness for prismatic members is given as 4 in Table 52a of the *Handbook of Frame Constants*, page 22, but it is, of course, only the relative value with which we are concerned.

	A	23'-4"	B		14'-0"	C		22'-8"	D		18'-0"	E
Column moments	66		–			–			–			45
Stiffness ratio	0.4	←0.6→	0.3	0.3	←0.6→	0.3	0.3	←0.6→	0.3	0.3	←0.6→	0.4
F. E. M. dead load			102	41		41	78		78	66		
F. E. M. total load	-193	+78	-193	-87	+64	-87	-165	+67	-165	-141	+48	-141
	-27	+15*	-46	-2	+1	+3	-18	+9	-22	-34	+17	-11
	-220	+22†	-239	-89	-1	-84	-183	+11	-187	-175	+6	-152
	+88		+45	-45		-30	+30		+4	-4		+61
Max. beam moments	-132	+115	-194	-134	+64	-114	-153	+87	-183	-179	+71	-91
Column moments	66		–			–			–			46

Carry-over: 0.6 ∴ * $\frac{27}{2}\left(\frac{1}{0.6} + 0.4 - 1.0\right) = 14.4$, say, 15. † $\frac{46}{2}\left(\frac{1}{0.6} + 0.3 - 1.0\right) = 22.3$, say, 22. Others similar.

Fig. 20 – Haunched beam, distribution of moments.

	A		B	
Stiffness ratio	0.4	←0.6→	0.3	
F.E.M. TL or DL	-193.0	+78.0	-193.0	-41.0
Distribution	+77.2		+45.6	
Carry-over	-27.4		-46.3	
Distribution	+11.0		+13.9	
Addition	-132.2		-179.8	

Positive moment at midspan:
Fixed-end beam = +78.0
Simply supported beam = +78.0 + 193.0 = 271.0
Actual conditions = 271.0 − ½ (132.2 + 179.8) = 115.0

Fig. 21 – Maximum midspan moment by ordinary method.

The ordinary method is shown in Fig. 21. It is to determine the end moments and deduct their average value from the midspan moment in *AB* considered simply supported. The fixed-end moments are based on a loading pattern that produces maximum positive moment at midspan of *AB*. The result, +115, is the actual maximum midspan moment.

A more convenient procedure is to add two corrections to the midspan moment of +78. From Fig. 21, it is seen that the two corrections equal

$$\frac{+77.2 - 27.4 + 11.0}{2} + \frac{+45.6 - 46.3 + 13.9}{2}$$

$$= \frac{+77.2 - 46.3 + 13.9}{2} + \frac{+45.6 - 27.4 + 11.0}{2}$$

$$= \frac{+77.2 - 77.2 \times 0.6 + 77.2 \times 0.6 \times 0.3}{2}$$

$$+ \frac{+45.6 - 45.6 \times 0.6 + 45.6 \times 0.6 \times 0.4}{2}$$

$$= \frac{+77.2 \times 0.6}{2}\left(\frac{1}{0.6} - 1 + 0.3\right) + \frac{+45.6 \times 0.6}{2}\left(\frac{1}{0.6} - 1 + 0.4\right)$$

$$= \frac{+46.3}{2}\left(\frac{1}{0.6} + 0.3 - 1\right) + \frac{+27.4}{2}\left(\frac{1}{0.6} + 0.4 - 1\right)$$

$$= 14.4 + 22.3, \text{ say, } 15 + 22.$$

Note that 46.3 and 27.4 have been calculated and are recorded as −46 and −27 in Fig. 20. These values must be multiplied by the quantities as shown. The result is the two corrections calculated above, which added to

+78 give the final moment, +115. Since the carry-over factor is ½ in prismatic beams, the quantity within the parentheses becomes, for prismatic beams, 1 plus the distribution factor.

20/bending in columns

The two subjects discussed in this section are (1) determination of moments in columns, and (2) proportioning of column sections subject to combined bending and axial load.

Section 1108 of the 1956 ACI Code states: "In computing moments in columns, the far ends may be considered fixed. Columns shall be designed to resist the axial forces from loads on all floors plus the maximum bending due to loads on a single adjacent span of the floor under consideration.

"Resistance to bending moments at any floor level shall be provided by distributing the moment between the columns immediately above and below the given floor in proportion to their relative stiffnesses and conditions of restraint."

The simplest procedure is to use the moments obtained from the regular beam analysis illustrated in Fig. 10. Greater moments may be produced in the exterior columns, but it is doubtful whether the effort required to calculate these is justifiable.

It is generally conceded that moments cannot be determined in columns with the same degree of accuracy as in beams. A beam moment is obtained as the sum of fixed-end moment and an additional term or a correction derived by analysis. But a column moment equals the corrections obtained by analysis and is far more sensitive to changes in assumptions and much more susceptible to faulty analysis.

In addition, columns appear to have a marked ability to "select" the amount of moment they are capable of supporting. Consider for illustration a column supporting an axial load and assume that one end of it is also being subjected to a gradually increasing rotation. At a certain stage of the rotation, the column section may be overstressed, and it may crack or yield. When this occurs, there is a sudden drop in the moment required to produce the rotation.

These two arguments are representative of a group from which the following conclusion may be drawn: The elastic theory is not at present close enough in accordance with facts to justify an elaborate procedure for determination of moments in columns. For multistory buildings, it is considered satisfactory to compute column moments under the same assumption used for beam moments. As previously stated, far ends of columns are fixed above and below the floor at which moments are to be determined. The procedure is illustrated in Section 21, "Determination of Column Moments."

In regard to proportioning of column sections, the 1956 ACI Code permits the use of the assumption that gross concrete section may be considered effective even if some of it is in tension because of a relatively large bending moment. The Code does not allow this assumption to be used for eccentricities greater than two-thirds the dimension of the column section.

Proportioning may be made simple if concrete is considered "uncracked," or effective in both compression and tension. When the design is based on the assumption of a "cracked section," proportioning of column sections is always cumbersome and difficult, especially in corner columns where there is bending in two directions. The former assumption is by far the more desirable one from the viewpoint of the professional engineer. This in itself is significant.

It may be argued that analysis and proportioning should both be made under the same assumption of either cracked or uncracked section. The common procedure is to use gross section for stiffnesses in the analysis. It would be difficult to determine the stiffness under any other assumption. The 1956 ACI Code allows "any reasonable assumption for computing the relative stiffnesses of columns and floor systems," provided that it is consistent throughout the analysis.

21/determination of column moments

From the considerations in Section 20, column moments will be determined on the basis of the assumption underlying the calculations made for beams in Fig. 10. Moments in exterior columns may then be taken directly from this figure. For illustration, the moment at the exterior end of beam *AB* is 126. This moment must equal the sum of the moments in the columns at *A* and should be distributed to them in proportion to their stiffness ratios or distribution factors.

The moments in interior columns are not recorded in Fig. 10 because the end moments are based on live load on both sides of each individual joint. Most codes specify that column moments be computed for unbalanced floor loading, that is, live load on one side only.

Fig. 12 serves the additional purpose of obtaining moments in interior columns produced by unbalanced floor loading. Live load is placed on the alternate long spans in Fig. 12. The fixed-end moments are the same as in Fig. 10, but arranged differently.

Irregularities in spans or loading may be great enough to necessitate an analysis for beams more extensive than that shown in Fig. 10. The general form of moment distribution may be used and should be employed for both beams and columns. For a detailed description of a loading pattern arranged to give maximum moments in columns, refer to *Concrete Building Frames Analyzed by Moment Distribution,* page 8.

22/design of column sections subject to combined bending and axial load

For uncracked sections, Section 1109 of the 1956 ACI Code gives a new form of the formula for proportioning columns.

The 1951 ACI Code formula (28) was: $P = N\left(1 + \frac{CDe}{t}\right)$.

The 1956 ACI Code formula (18) is: $\frac{f_a}{F_a} + \frac{f_b}{F_b} \leqq 1.00$.

The 1956 ACI Code limits the ratio of eccentricity, $\frac{e}{t}$, to ⅔; its former limit was 1.0.

Formula (20) of the 1956 Code is: $P = N\left(1 + \frac{Be}{t}\right)$.

The old values CD are combined in the single symbol B. This formula can be used in both preliminary selection and final design of the column. The 1951 Code formula (28) is more convenient for column design, but the 1956 ACI Code formula (18) is more advantageous for investigation of stresses.

A derivation of the 1951 ACI method is presented in the ACI *Reinforced Concrete Design Handbook* (Second Edition, 1955) on page 98, with further information on page 31.

To illustrate that the 1951 and 1956 formulas give the same results, the following derivation is presented:

Concrete: f_a = actual axial stress;
f_b = actual bending stress;
F_a = allowable axial stress when no bending stress exists;
F_b = allowable bending stress when no axial stress exists;
f_p = allowable stress for combination of axial compression and flexure;
f_c' = ultimate compressive strength.
Steel: f_s = allowable stress in vertical column reinforcement.

Supplementary notation is given on page 7.

In the 1951 formula (28), the allowable equivalent axial load, combining the effects of axial load and moment, is:

$$P = N\left(1 + \frac{CDe}{t}\right). \qquad (28)$$

For an axially loaded column:

$$P = F_a A\,[1 + (n-1)p]. \qquad (1)$$

Equating formulas (28) and (1):

$$N\left(1 + \frac{CDe}{t}\right) = F_a A\,[1 + (n-1)p]. \qquad (2)$$

This can be written as:

$$\frac{N}{A}\left[\frac{1 + \frac{De}{t}}{1 + (n-1)p}\right] = F_a\left(\frac{1 + \frac{De}{t}}{1 + \frac{CDe}{t}}\right). \qquad (3)$$

When the entire concrete area, A, is considered effective in a section subject to an eccentric force N at a distance e from the centerline, the total extreme fiber stress is expressed as:

$$f_c = f_a + f_b = \frac{N}{A\,[1 + (n-1)p]} + \frac{Net}{2I}. \qquad (4)$$

The moment of inertia equals:

$$I = R^2 A\,[1 + (n-1)p], \qquad (5)$$

and $\frac{t^2}{2R^2}$ is denoted as D. (6)

Inserting (5) and (6) into (4) gives:

$$f_c = f_a + f_b = \frac{N}{A}\left[\frac{1+\dfrac{De}{t}}{1+(n-1)p}\right]. \tag{7}$$

The objective of design is to make the actual and allowable stresses equal, that is, $f_c \leqq f_p$. Then, from formulas (3) and (7):

$$f_p = F_a\left(\frac{1+\dfrac{De}{t}}{1+\dfrac{CDe}{t}}\right). \tag{8}$$

This is formula (29) of the 1951 ACI Code except that the term F_a has been used instead of f_a to avoid conflict of terminology.

By definition, $C = \dfrac{F_a}{F_b}$. (9)

Therefore,

$$f_p = \frac{1+\dfrac{De}{t}}{\dfrac{1}{F_a}+\dfrac{De}{F_b t}}. \tag{10}$$

Multiply numerator and denominator by $\dfrac{N}{A\,[1+(n-1)p]}$:

$$f_p = \frac{\dfrac{N}{A}\left[\dfrac{1+\dfrac{De}{t}}{1+(n-1)p}\right]}{\dfrac{N}{F_a A[1+(n-1)p]}+\dfrac{NDe}{A[1+(n-1)p]F_b t}}. \tag{11}$$

Substituting (4), (5), (6) and (7) into (11) gives:

$$f_p = \frac{f_a+f_b}{\dfrac{f_a}{F_a}+\dfrac{f_b}{F_b}}. \tag{12}$$

Equation (12) can be transposed as follows to show the ratio of actual to allowable stress:

$$\frac{f_a}{F_a}+\frac{f_b}{F_b} = \frac{f_a+f_b}{f_p}. \tag{13}$$

Now the sum of the actual stresses, f_a and f_b, should be less than the allowable stress, f_p; therefore the column should be proportioned so that:

$$\frac{f_a}{F_a}+\frac{f_b}{F_b} \leqq 1.00. \tag{14}$$

This is the same as formula (18) of the 1956 ACI Code, which was to be demonstrated.

23/proportioning of a column section

Consider the problem to design a 20-in. square section with a 17-in. spiral core subject to an axial load, $N = 200$ kips, combined with a moment $M = 70$ ft.kips. Use intermediate-grade bars, $f'_c = 3{,}000$ psi, hot rolled spiral, and select column section from Tables 20, 21 and 22 for spiral columns in the *Reinforced Concrete Design Handbook*, pages 61–63. These tables are based on the 1951 ACI Code.

Compute $e = \dfrac{70 \times 12}{200} = 4.2$ in.

Then $\dfrac{e}{t} = \dfrac{4.2}{20} = 0.21 =$ less than 0.67.

From Table 7, for $g = 0.75$ and in the group headed "Square Sections with Spirals," it is seen that $D = 6.2$ is a good average covering a wide range of values of $(n - 1)p$.

table 6. coefficients f_a and C for design of columns

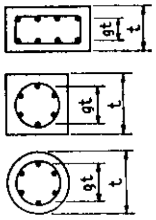
values of $f_a = \dfrac{0.225f'_c + f_s p}{1 + (n-1)p}$ for spiral columns; 0.8 times this value for tied columns

f'_c	n	Tied Columns						Spiral Columns									
		Values of p															
		0.010	0.015	0.020	0.025	0.030	0.040	0.010	0.015	0.020	0.025	0.030	0.040	0.050	0.060	0.070	0.080
		$f_s = 16{,}000$															
2000	15	428	456	481	504	524	559	535	570	602	630	655	699	735	766	793	816
2500	12	521	551	579	604	627	668	651	689	723	755	784	835	879	917	950	980
3000	10	613	645	675	702	728	774	766	806	843	878	909	967	1017	1062	1101	1137
3750	8	750	785	817	847	875	927	938	981	1021	1059	1094	1159	1218	1270	1318	1361
5000	6	979	1016	1051	1084	1117	1177	1224	1270	1314	1356	1396	1471	1540	1604	1663	1718
		$f_s = 20{,}000$															
2000	15	456	496	531	563	592	641	570	620	664	704	739	801	853	897	934	967
2500	12	550	592	631	667	699	757	687	740	789	833	874	946	1008	1062	1109	1150
3000	10	642	687	729	767	803	868	803	859	911	959	1004	1085	1155	1218	1273	1323
3750	8	780	828	873	915	955	1027	975	1035	1091	1144	1193	1284	1366	1439	1506	1567
5000	6	1010	1060	1109	1156	1200	1283	1262	1326	1386	1444	1500	1604	1700	1788	1870	1946

values of $C = \dfrac{f_a}{0.45f'_c}$

f'_c	n	Tied Columns						Spiral Columns									
		Values of p															
		0.010	0.015	0.020	0.025	0.030	0.040	0.010	0.015	0.020	0.025	0.030	0.040	0.050	0.060	0.070	0 080
		$f_s = 16{,}000$															
2000	15	0.48	0.51	0.53	0.56	0.58	0.62	0.59	0.63	0.67	0.70	0.73	0.78	0.82	0.85	0.88	0.91
2500	12	0.46	0.49	0.51	0.54	0.56	0.59	0.58	0.61	0.64	0.67	0.70	0.74	0.78	0.82	0.84	0.87
3000	10	0.45	0.48	0.50	0.52	0.54	0.57	0.57	0.60	0.62	0.65	0.67	0.72	0.75	0.79	0.82	0.84
3750	8	0.44	0.46	0.48	0.50	0.52	0.55	0.56	0.58	0.61	0.63	0.65	0.69	0.72	0.75	0.78	0.81
5000	6	0.44	0.45	0.47	0.48	0.50	0.52	0.54	0.56	0.58	0.60	0.62	0.65	0.68	0.71	0.74	0.76
		$f_s = 20{,}000$															
2000	15	0.51	0.55	0.59	0.63	0.66	0.71	0.63	0.69	0.74	0.78	0.82	0.89	0.95	1.00	1.04	1.07
2500	12	0.49	0.53	0.56	0.59	0.62	0.67	0.61	0.66	0.70	0.74	0.78	0.84	0.90	0.94	0.99	1.02
3000	10	0.48	0.51	0.54	0.57	0.59	0.64	0.59	0.64	0.67	0.71	0.74	0.80	0.86	0.90	0.94	0.98
3750	8	0.46	0.49	0.52	0.54	0.57	0.61	0.58	0.61	0.65	0.68	0.71	0.76	0.81	0.85	0.89	0.93
5000	6	0.45	0.47	0.49	0.51	0.53	0.57	0.56	0.59	0.62	0.64	0.67	0.71	0.76	0.79	0.83	0.86

table 7. coefficients *D* for design of columns



$$D = \frac{1 + (n-1)p}{\frac{1}{6} + \frac{1}{2}(n-1)pg^2}$$

$$D = \frac{1 + (n-1)p}{\frac{1}{6} + \frac{1}{4}(n-1)pg^2}$$

$$D = \frac{1 + (n-1)p}{\frac{1}{8} + \frac{1}{4}(n-1)pg^2}$$

values of $D = \frac{t^2}{2R^2}$

in which R = radius of gyration

$p = \frac{A_s}{A_g}$, in which A_g = gross area of concrete section

All reinforcement is arranged symmetrically

g	$(n-1)p$																
	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80
	Rectangular Sections with Ties																
1.00	6.0	5.5	5.1	4.8	4.5	4.3	4.1	4.0	3.8	3.7	3.6	3.5	3.4	3.4	3.3	3.2	3.2
0.95	6.0	5.5	5.2	4.9	4.7	4.5	4.3	4.2	4.0	3.9	3.8	3.7	3.7	3.6	3.5	3.5	3.4
0.90	6.0	5.6	5.3	5.1	4.8	4.7	4.5	4.4	4.3	4.2	4.1	4.0	3.9	3.8	3.8	3.7	3.7
0.85	6.0	5.7	5.4	5.2	5.0	4.9	4.7	4.6	4.5	4.4	4.3	4.2	4.2	4.1	4.1	4.0	4.0
0.80	6.0	5.7	5.5	5.4	5.2	5.1	4.9	4.8	4.7	4.7	4.6	4.5	4.5	4.4	4.3	4.3	4.3
0.75	6.0	5.8	5.6	5.5	5.4	5.3	5.2	5.1	5.0	4.9	4.9	4.8	4.8	4.7	4.7	4.6	4.6
0.70	6.0	5.9	5.7	5.6	5.6	5.5	5.4	5.3	5.3	5.2	5.2	5.1	5.1	5.1	5.0	5.0	5.0
0.65	6.0	5.9	5.9	5.8	5.7	5.7	5.6	5.6	5.6	5.5	5.5	5.5	5.4	5.4	5.4	5.4	5.4
0.60	6.0	6.0	6.0	5.9	5.9	5.9	5.9	5.9	5.9	5.9	5.8	5.8	5.8	5.8	5.8	5.8	5.8
	Square Sections with Spirals																
1.00	6.0	5.9	5.7	5.6	5.5	5.5	5.4	5.3	5.3	5.2	5.1	5.1	5.1	5.0	5.0	4.9	4.9
0.95	6.0	5.9	5.8	5.7	5.7	5.6	5.5	5.5	5.5	5.4	5.4	5.3	5.3	5.3	5.2	5.2	5.2
0.90	6.0	5.9	5.9	5.8	5.8	5.7	5.7	5.7	5.7	5.6	5.6	5.6	5.5	5.5	5.5	5.5	5.5
0.85	6.0	6.0	5.9	5.9	5.9	5.9	5.9	5.9	5.9	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8
0.80	6.0	6.0	6.0	6.0	6.0	6.0	6.1	6.1	6.1	6.1	6.1	6.1	6.1	6.1	6.1	6.1	6.1
0.75	6.0	6.0	6.1	6.1	6.2	6.2	6.2	6.3	6.3	6.3	6.3	6.4	6.4	6.4	6.4	6.4	6.4
0.70	6.0	6.1	6.1	6.2	6.3	6.3	6.4	6.4	6.5	6.5	6.6	6.6	6.7	6.7	6.7	6.8	6.8
0.65	6.0	6.1	6.2	6.3	6.4	6.5	6.5	6.6	6.7	6.8	6.8	6.9	7.0	7.0	7.1	7.1	7.2
0.60	6.0	6.1	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0	7.1	7.2	7.2	7.3	7.4	7.5	7.5
	Round Sections with Spirals																
1.00	8.0	7.6	7.3	7.1	6.9	6.7	6.5	6.4	6.2	6.1	6.0	5.9	5.8	5.7	5.7	5.6	5.5
0.95	8.0	7.7	7.5	7.2	7.1	6.9	6.7	6.6	6.5	6.4	6.3	6.2	6.1	6.1	6.0	6.0	5.9
0.90	8.0	7.8	7.6	7.4	7.3	7.1	7.0	6.9	6.8	6.7	6.6	6.6	6.5	6.4	6.4	6.3	6.3
0.85	8.0	7.8	7.7	7.6	7.4	7.3	7.3	7.2	7.1	7.0	7.0	6.9	6.9	6.8	6.8	6.7	6.7
0.80	8.0	7.9	7.8	7.7	7.6	7.6	7.5	7.5	7.4	7.4	7.3	7.3	7.2	7.2	7.2	7.1	7.1
0.75	8.0	7.9	7.9	7.9	7.8	7.8	7.8	7.7	7.7	7.7	7.7	7.7	7.6	7.6	7.6	7.6	7.6
0.70	8.0	8.0	8.0	8.0	8.0	8.0	8.0	8.0	8.0	8.0	8.1	8.1	8.1	8.1	8.1	8.1	8.1
0.65	8.0	8.1	8.1	8.2	8.2	8.3	8.3	8.3	8.4	8.4	8.4	8.5	8.5	8.5	8.5	8.6	8.6
0.60	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.6	8.7	8.8	8.8	8.9	8.9	9.0	9.0	9.1	9.1

Refer to Table 6 in the group headed "Values of C" for spiral columns, $f_s = 16{,}000$ and $f'_c = 3{,}000$. Select $C = 0.65$ (estimating $p = 0.025$).

Compute: $CD\frac{M}{t} = 0.65 \times 6.2\left(\frac{70 \times 12}{20}\right) = 169$ kips

Add: $N = 200$ kips

Design section for total load: $P = 369$ kips

From Table 20 (*Handbook*), load on concrete $= 270$ kips

Balance to be carried by longitudinal bars $= 99$ kips

From Table 21 (*Handbook*), select eight No. 8 bars: 101 kips. Select spiral from Table 22 (*Handbook*): ⅝-in. round rod at 2¾-in. pitch.

Since p actually equals 0.016, the value of C taken from Table 6 should be reduced from 0.65 to 0.60. This reduces the term $CD\frac{M}{t}$ by 13 kips. The load to be carried by the bars becomes 86 kips, and the number of No. 8 bars may be reduced from eight to seven.

It is customary in office work to "run down" column loads in a column schedule. This arrangement may still be retained when bending is included. Space should be allowed for recording of the bending term, $CD\frac{M}{t}$; the axial load, N; and the summation of these terms, P. The value of M is taken from Fig. 10 or 12; of C from Table 6; and of D from Table 7. In the case of bending in two directions, there will be two terms of the type $CD\frac{M}{t}$, one for each direction, and P will be the sum of three items. This type of proportioning of columns is quick and simple.

24/moments in one-way slabs and joists

For design of ordinary one-way slabs, it is not customary to use a regular moment analysis. Moments in slabs are usually determined by means of arbitrary coefficients. Such coefficients may also be useful for beams of approximately equal spans with uniformly distributed loads.

Boase and Howell have presented extensive tables of moment coefficients.* One of their tables, reproduced as Table 8, is based on the following assumptions:

Spans are all of the same length.

Horizontal members have the same stiffness.

Vertical members have the same stiffness.

Vertical members are fixed at ends above and below the floor considered.

Load is uniformly distributed.

Ratio of live to dead load is the same in all beams.

Coefficients are tabulated separately for frames with two spans, three spans, and four or more spans. Five ratios of live to dead load and seven ratios of column to beam stiffness are included. The coefficients are to be multiplied by the product of unit load, w, and the square of span length, L. In accordance with the ACI Code specifications for the application of prescribed moment coefficients, it is recommended that for positive moments, L be taken as clear span; and that for negative moments, L be taken as the average for two adjacent clear spans. The ratio of the longer to the shorter of two adjacent spans shall not exceed 1.20.

The use of Table 8 enables the designer to ascertain at a glance how a change in stiffness affects the results. For slabs and joists, he may then select stiffness ratios in such a manner that his design is reasonably conservative.

The procedure outlined for one-way slabs and joists is also useful for a number of other cases involving beams with uniform load and approximately equal spans. Further refinements and additional tables have been introduced, including three types of concentrated loading. For detailed description and illustrative examples, refer to the appendix of *Reinforced Concrete Design Handbook.*

*"Design Coefficients for Building Frames," American Concrete Institute *Journal*, September 1939. The tables are republished, enlarged and elaborated in the appendix to the ACI *Reinforced Concrete Design Handbook*, pages 103–120.

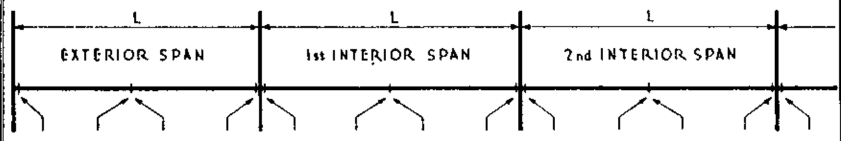
table 8. moment coefficients for slabs and joists

maximum moment coefficients, C_1

$\frac{w_{live}}{w_{dead}}$	$\frac{\Sigma K_{col.}}{K_{beam}}$	TWO-SPAN FRAMES				THREE-SPAN FRAMES						
		L EXTERIOR SPAN			℄	L EXTERIOR SPAN				L INTERIOR SPAN	℄	
		Max. −	Max. +	Min. +	Max. −	Max. −	Max. +	Min. +	Max. −	Max. −	Max. +	Min. +
0	0	0	+.063	+.063	−.125	0	+.075	+.075	−.100	−.100	+.025	+.025
	0.5	−.028	+.056	+.056	−.111	−.030	+.061	+.061	−.098	−.091	+.034	+.034
	1	−.042	+.052	+.052	−.104	−.044	+.055	+.055	−.096	−.088	+.037	+.037
	2	−.055	+.049	+.049	−.097	−.058	+.049	+.049	−.093	−.085	+.040	+.040
	4	−.066	+.046	+.046	−.092	−.067	+.046	+.046	−.090	−.084	+.041	+.041
	8	−.074	+.044	+.044	−.088	−.074	+.044	+.044	−.087	−.084	+.041	+.041
	Infinity	−.083	+.042	+.042	−.083	−.083	+.042	+.042	−.083	−.083	+.042	+.042
0.5	0	0	+.073	+.031	−.125	0	+.083	+.042	−.106	−.106	+.042	0
	0.5	−.031	+.061	+.031	−.111	−.033	+.066	+.035	−.101	−.096	+.044	+.012
	1	−.045	+.056	+.031	−.104	−.047	+.058	+.033	−.098	−.092	+.044	+.017
	2	−.058	+.051	+.030	−.097	−.060	+.052	+.031	−.094	−.089	+.044	+.022
	4	−.069	+.047	+.029	−.092	−.069	+.048	+.029	−.091	−.086	+.044	+.025
	8	−.075	+.045	+.028	−.088	−.075	+.045	+.029	−.088	−.085	+.043	+.026
	Infinity	−.083	+.042	+.028	−.083	−.083	+.042	+.028	−.083	−.083	+.042	+.028
1	0	0	+.078	+.016	−.125	0	+.088	+.025	−.108	−.108	+.050	−.013
	0.5	−.032	+.064	+.019	−.111	−.034	+.069	+.023	−.103	−.098	+.049	+.002
	1	−.046	+.058	+.020	−.104	−.048	+.060	+.022	−.099	−.094	+.048	+.008
	2	−.060	+.052	+.021	−.097	−.061	+.053	+.021	−.095	−.091	+.047	+.013
	4	−.070	+.048	+.021	−.092	−.070	+.048	+.021	−.091	−.088	+.045	+.016
	8	−.076	+.045	+.021	−.088	−.076	+.045	+.021	−.088	−.085	+.044	+.018
	Infinity	−.083	+.042	+.021	−.083	−.083	+.042	+.021	−.083	−.083	+.042	+.021
2	0	0	+.083	0	−.125	0	+.092	+.008	−.111	−.111	+.058	−.025
	0.5	−.033	+.067	+.007	−.111	−.035	+.071	+.010	−.104	−.100	+.054	−.009
	1	−.048	+.060	+.009	−.104	−.050	+.062	+.011	−.100	−.096	+.052	−.002
	2	−.061	+.053	+.011	−.097	−.062	+.054	+.012	−.095	−.093	+.049	+.003
	4	−.071	+.049	+.012	−.092	−.071	+.049	+.013	−.091	−.089	+.047	+.008
	8	−.077	+.046	+.013	−.088	−.077	+.046	+.013	−.088	−.086	+.045	+.011
	Infinity	−.083	+.042	+.014	−.083	−.083	+.042	+.014	−.083	−.083	+.042	+.014
3	0	0	+.086	−.008	−.125	0	+.094	0	−.113	−.113	+.063	−.031
	0.5	−.034	+.068	+.001	−.111	−.036	+.073	+.004	−.105	−.102	+.057	−.014
	1	−.049	+.061	+.004	−.104	−.050	+.063	+.005	−.101	−.097	+.054	−.007
	2	−.062	+.054	+.007	−.097	−.063	+.055	+.007	−.096	−.094	+.050	−.001
	4	−.071	+.049	+.008	−.092	−.071	+.049	+.008	−.091	−.090	+.047	+.004
	8	−.077	+.046	+.009	−.088	−.077	+.046	+.009	−.088	−.087	+.045	+.007
	Infinity	−.083	+.042	+.010	−.083	−.083	+.042	+.010	−.083	−.083	+.042	+.010

$$M = C_1(w_{dead} + w_{live})L^2 \text{ where: } \begin{cases} M = \text{Moment in ft.kips} \\ w_{live} = \text{Uniform live load in kips per ft.} \\ w_{dead} = \text{Uniform dead load in kips per ft.} \\ L = \text{Span in ft.} \end{cases}$$

FOUR OR MORE SPAN FRAMES



$\frac{w_{live}}{w_{dead}}$	$\frac{\Sigma K_{col.}}{K_{beam}}$	Exterior span Max. −	Exterior span Max. +	Exterior span Min. +	Exterior span Max. −	1st interior span Max. −	1st interior span Max. +	1st interior span Min. +	1st interior span Max. −	2nd interior span Max. −	2nd interior span Max. +	2nd interior span Min. +	2nd interior span Max. −	Max. −
0	0	0	+.072	+.072	−.106	−.106	+.034	+.034	−.077	−.077	+.044	+.044	−.085	−.085
	0.5	−.030	+.060	+.060	−.101	−.095	+.038	+.038	−.080	−.081	+.043	+.043	−.084	−.084
	1	−.044	+.054	+.054	−.098	−.090	+.039	+.039	−.081	−.082	+.042	+.042	−.084	−.084
	2	−.057	+.050	+.050	−.094	−.087	+.041	+.041	−.082	−.083	+.042	+.042	−.083	−.083
	4	−.067	+.046	+.046	−.090	−.085	+.042	+.042	−.083	−.083	+.042	+.042	−.083	−.083
	8	−.074	+.044	+.044	−.087	−.084	+.042	+.042	−.083	−.083	+.042	+.042	−.083	−.083
	Infinity	−.083	+.042	+.042	−.083	−.083	+.042	+.042	−.083	−.083	+.042	+.042	−.083	−.083
0.5	0	0	+.081	+.039	−.110	−.110	+.049	+.007	−.088	−.088	+.057	+.016	−.094	−.094
	0.5	−.033	+.065	+.035	−.104	−.099	+.048	+.016	−.087	−.088	+.052	+.019	−.091	−.091
	1	−.046	+.058	+.033	−.100	−.094	+.047	+.019	−.087	−.088	+.049	+.021	−.089	−.089
	2	−.060	+.052	+.031	−.095	−.090	+.045	+.022	−.086	−.087	+.047	+.023	−.087	−.087
	4	−.069	+.048	+.029	−.091	−.087	+.044	+.025	−.085	−.086	+.044	+.025	−.086	−.086
	8	−.076	+.045	+.029	−.088	−.085	+.043	+.026	−.085	−.085	+.043	+.026	−.085	−.085
	Infinity	−.083	+.042	+.028	−.083	−.083	+.042	+.028	−.083	−.083	+.042	+.028	−.083	−.083
1	0	0	+.085	+.023	−.113	−.113	+.056	−.006	−.094	−.094	+.063	+.002	−.099	−.099
	0.5	−.034	+.068	+.022	−.105	−.101	+.052	+.004	−.091	−.092	+.057	+.008	−.095	−.095
	1	−.048	+.060	+.022	−.100	−.096	+.050	+.009	−.090	−.091	+.052	+.011	−.092	−.092
	2	−.061	+.053	+.021	−.095	−.091	+.048	+.013	−.088	−.089	+.049	+.014	−.089	−.089
	4	−.070	+.048	+.021	−.091	−.088	+.046	+.016	−.087	−.087	+.046	+.017	−.087	−.087
	8	−.076	+.045	+.021	−.088	−.086	+.044	+.018	−.085	−.085	+.044	+.018	−.085	−.085
	Infinity	−.083	+.042	+.021	−.083	−.083	+.042	+.021	−.083	−.083	+.042	+.021	−.083	−.083
2	0	0	+.090	+.007	−.115	−.115	+.064	−.019	−.099	−.099	+.070	−.011	−.104	−.104
	0.5	−.035	+.070	+.009	−.106	−.103	+.057	−.007	−.095	−.096	+.061	−.004	−.098	−.098
	1	−.050	+.062	+.011	−.101	−.098	+.054	−.001	−.093	−.094	+.056	0	−.095	−.095
	2	−.063	+.054	+.013	−.096	−.093	+.050	+.004	−.090	−.091	+.051	+.005	−.091	−.091
	4	−.071	+.049	+.013	−.091	−.089	+.047	+.008	−.088	−.088	+.047	+.008	−.088	−.088
	8	−.077	+.046	+.013	−.088	−.086	+.045	+.011	−.086	−.086	+.045	+.011	−.086	−.086
	Infinity	−.083	+.042	+.014	−.083	−.083	+.042	+.014	−.083	−.083	+.042	+.014	−.083	−.083
3	0	0	+.092	−.002	−.116	−.116	+.068	−.026	−.102	−.102	+.073	−.018	−.106	−.106
	0.5	−.036	+.071	+.003	−.107	−.104	+.060	−.012	−.097	−.098	+.063	−.010	−.100	−.100
	1	−.050	+.063	+.005	−.101	−.099	+.055	−.006	−.094	−.095	+.057	−.005	−.096	−.096
	2	−.064	+.055	+.007	−.096	−.094	+.051	0	−.091	−.092	+.052	0	−.092	−.092
	4	−.072	+.049	+.008	−.091	−.090	+.048	+.004	−.089	−.089	+.048	+.004	−.089	−.089
	8	−.077	+.046	+.009	−.088	−.087	+.045	+.007	−.086	−.087	+.045	+.007	−.087	−.087
	Infinity	−.083	+.042	+.010	−.083	−.083	+.042	+.010	−.083	−.083	+.042	+.010	−.083	−.083

wind pressure

25 / introduction

Some theoretical treatises on wind pressure are confined to the simple case in which a single bent in a building is subject to a known wind pressure. However, the amount of pressure acting on each bent is generally not known beforehand.

In a wind-pressure problem, it is essential first to ascertain the pressure on each individual bent. This is particularly important in reinforced concrete construction because all concrete members are integrally and rigidly connected with adjacent members. Also, all bents extending in a given direction cooperate in resisting the wind pressure acting in that direction.

The share of wind pressure resisted by each bent in a building is a function of the pressure necessary to give the bent a unit deflection. The relationship between pressure and deflection may make it difficult to solve the problem in its general form. A special, simplified way to solve the problem is presented in this text.*

Consider a floor in which all joints are part of bents that cooperate in resisting a given total wind pressure, W, acting above that floor. Each joint

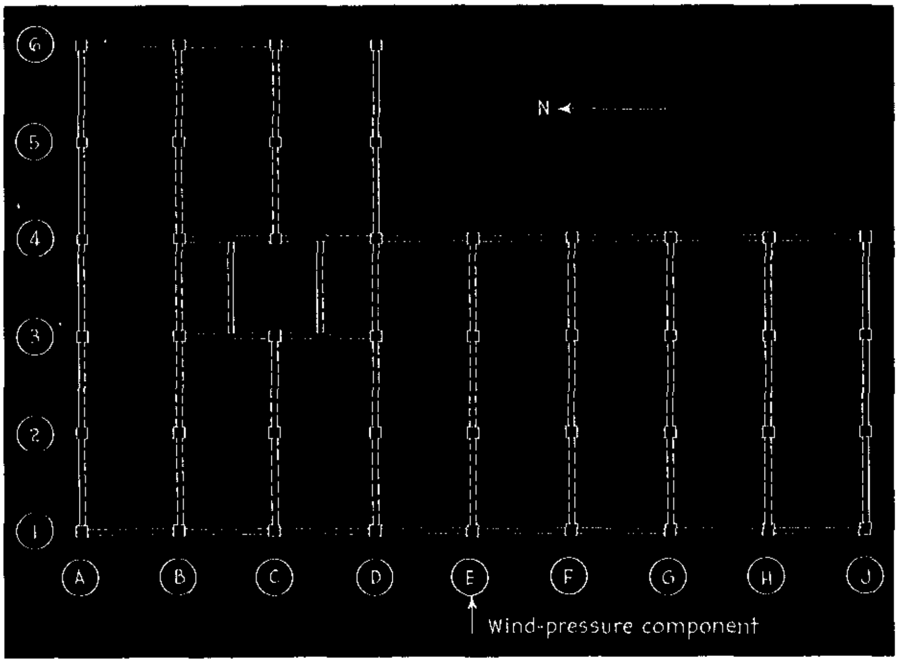


Fig. 22 — Framing plan of floor.

*See reference 29; also reference 28.

Joint	Joint coefficient	Columns: Shear, kips	Columns: Moment, ft. kips	Beams: Moment, ft. kips	Beams: Shear, kips
A1	$4\times\frac{0+1}{0+1+4+4}=.44$	6.5	32.5	65.0	6.2
A2	$4\times\frac{1+1}{1+1+4+4}=.80$	11.8	59.0	59.0	5.9
A3	A2 .80	same as joint		A2	5.9
A4	A2 .80	same as joint		A2	5.9
A5	A2 .80	same as joint		A2	6.2
A6	A1 .44	same as joint		A1	
	4.08	4.08 × 8 × 20 = 653			
B1	$4\times\frac{0+1.5}{0+1.5+4+4}=.63$	9.3	46.5	93.0	9.3
B2	$8\times\frac{1.5+1.5}{1.5+1.5+8+8}=1.26$	18.5	92.5	92.5	9.3
B3	B2 1.26	same as joint		B2	9.3
B4	B2 1.26	same as joint		B2	9.3
B5	B2 1.26	same as joint		B2	9.3
B6	B1 .63	same as joint		B1	
	6.30	6.30 × 7 × 20 = 882			
C1	B1 .63	same as joint		B1	9.3
C2	B2 1.26	same as joint		B2	9.7
C3	$8\times\frac{1.5+0}{1.5+0+8+8}=.69$	10.1	50.5	101.0 / 0	0
C4	C3 .69	10.1	50.5	0 / 101.0	9.7
C5	B2 1.26	same as joint		B2	9.3
C6	B1 .63	same as joint		B1	
	5.16	5.16 × 6 × 20 = 620			
D1	B1 .63	same as joint		B1	9.3
D2	B2 1.26	same as joint		B2	9.3
D3	B2 1.26	same as joint		B2	9.4
D4	$8\times\frac{1.5+1}{1.5+1+8+8}=1.08$	15.9	79.5	95.0 / 64.0	6.2
D5	A2 .80	same as joint		A2	6.2
D6	A1 .44	same as joint		A1	
	5.47	5.47 × 5 × 20 = 547			

Joint	Joint coefficient	Columns: Shear, kips	Columns: Moment, ft. kips	Beams: Moment, ft. kips	Beams: Shear, kips
E1	B1 .63	same as joint		B1	9.3
E2	B2 1.26	same as joint		B2	9.3
E3	B2 1.26	same as joint		B2	9.3
E4	B1 .63	same as joint		B1	
	3.78	3.78 × 4 × 20 = 302			
F1	B1 .63	same as joint		B1	9.3
F2	B2 1.26	same as joint		B2	9.3
F3	B2 1.26	same as joint		B2	9.3
F4	B1 .63	same as joint		B1	
	3.78	3.78 × 3 × 20 = 227			
G1	B1 .63	same as joint		B1	9.3
G2	B2 1.26	same as joint		B2	9.3
G3	B2 1.26	same as joint		B2	9.3
G4	B1 .63	same as joint		B1	
	3.78	3.78 × 2 × 20 = 151			
H1	B1 .63	same as joint		B1	9.3
H2	B2 1.26	same as joint		B2	9.3
H3	B2 1.26	same as joint		B2	9.3
H4	B1 .63	same as joint		B1	
	3.78	3.78 × 1 × 20 = 76			
J1	(A1:0.44) 1.33	19.5	97.5	195.0	18.6
J2	(A2:0.80) 2.40	35.3	176.5	176.5	17.7
J3	(A2:0.80) 2.40	same as joint		J2	18.6
J4	(A1:0.44) 1.33	same as joint		J1	
	(2.48) 7.46	7.46 × 0 × 20 = 00			

Sum of all joint coefficients: 43.59
Moment of joint coefficients with respect to bent J: 3458
Eccentricity of joint coefficients: $80.0-\frac{3458}{43.59}=0.6$ ft.
Factor for joint coefficients: $\frac{640.0}{43.59}=14.7$ kips

Fig. 23 — Tabulation of wind-pressure calculations.

in the floor is the intersection of one or two columns with one or two beams, or its equivalent portion of floor construction. The concept of "joint" will in this connection include physical properties such as stiffnesses of the adjacent members in the direction of the wind pressure.

A joint taken in this enlarged sense is illustrated in Fig. 26 with certain theoretical derivations. On the basis of certain assumptions, it can be demonstrated that the resistance of a joint against deformation or deflection may be expressed as a function of the $\frac{I}{L}$ values of the members at the joint. The particular function of the stiffnesses will be called the "joint coefficient." If the coefficient for any joint in the floor is denoted as v_x and the sum of all coefficients in the floor considered is Σv_x, the share of the wind pressure carried by each joint is $\left(\frac{v_x}{\Sigma v_x}\right) W$. An illustration for a complete floor level is given in Figs. 22 and 23.

Total shear in a story, caused by wind pressure, may be distributed to each joint in the floor below by means of a particularly simple set of calculations. However, the centroid of wind shear and that of all joint coefficients must coincide. This may generally be accomplished by altering certain beam or column sizes. If joint coefficients cannot be adjusted sufficiently, a correction for the eccentricity may be introduced as illustrated in Figs. 24 and 25.

The treatment of wind pressure given in this text is sufficient and adequate for design of wind pressure on all reinforced concrete buildings except tall, towerlike structures. For these, refer to publications listed in the bibliography; for example, see reference 31, which uses an exhaustive analysis based on the elastic theory, the conventional theory for reinforced concrete design.

The procedures presented for wind pressure are also useful for investigation of earthquake stresses, provided the design can be based on the assumption of "static loading," in which the effect of an earthquake shock is assumed to be equivalent to a static horizontal load similar to wind pressure. For earthquake design based on the "dynamic-loading" assumption, refer to publications in the bibliography; for example, reference 35.

Bent	$v_x \times x^2 = I_x$	Bent	$v_y \times y^2 = I_y$
A	$4.08 \times 70.5^2 = 20{,}300$	1	$6.48 \times 41.3^2 = 11{,}100$
B	$6.30 \times 50.5^2 = 16{,}100$	3	$2.64 \times 1.3^2 = 00$
C	$5.16 \times 30.5^2 = 4{,}800$	4	$6.67 \times 18.7^2 = 2{,}300$
D	$5.47 \times 10.5^2 = 600$	6	$2.48 \times 58.7^2 = 3{,}500$
E	$3.78 \times 9.5^2 = 300$		
F	$3.78 \times 29.5^2 = 3{,}300$		
G	$3.78 \times 49.5^2 = 9{,}300$		$I_y =$ 21,900
H	$3.78 \times 69.5^2 = 18{,}200$		$I_x =$ 92,800
J	$2.48 \times 89.5^2 = 19{,}900$		
	$I_x =$ 92,800		$I_x + I_y =$ 114,700

Fig. 24 — Moments of inertia of columns resisting eccentric wind pressure.

Bent	$\frac{Wex}{I_x + I_y}$	$F = 16.6 + \frac{Wex}{I_x + I_y}$	$\frac{F}{14.7}$
A	$.053 \times (-70.5) = -3.7$	12.9	0.88
B	$.053 \times (-50.5) = -2.7$	13.9	0.95
C	$.053 \times (-30.5) = -1.6$	15.0	1.02
D	$.053 \times (-10.5) = -.6$	16.0	1.09
E	$.053 \times (+9.5) = +.5$	17.1	1.16
F	$.053 \times (+29.5) = +1.6$	18.2	1.24
G	$.053 \times (+49.5) = +2.6$	19.2	1.31
H	$.053 \times (+69.5) = +3.7$	20.3	1.38
J	$.053 \times (+89.5) = +4.7$	21.3	—

Fig. 25 — Determination of shear due to eccentric wind pressure.

26 / concentric wind pressure on a building

Fig. 22 is a framing plan for a floor 20 stories below the roof of a building in which each story is 10 ft. high. The direction of the wind is east-west and its intensity is 20 psf. All bays are 20 ft. long. The relative stiffnesses, $K = \frac{I}{L}$, of the members of the floor in the east-west direction are:

Type of member	*Relative stiffness*
Spandrel beams	$20 \div 20 = 1.0$
Interior beams	$30 \div 20 = 1.5$
Wall columns	$40 \div 10 = 4.0$
Interior columns	$80 \div 10 = 8.0$

The distribution of wind pressure to columns above each joint in the floor considered will be determined.

The total shear due to wind pressure above the floor is $W = (8 \times 20) \times (20 \times 10) \times 20 = 640$ kips, and its centroid lies midway between bents A and J, that is, 80 ft. from J.

The nine bents from A to J in the east-west direction resist wind pressure. Each column in Fig. 22 will carry a certain portion of the 640 kips. Resistance of each joint or the shear induced in each column above is proportional to a joint coefficient. The following expression is derived in Section 29:

$$v_x = K \text{ for column} \left(\frac{\text{sum of } K\text{-values for adjacent beams}}{\text{sum of } K\text{-values for adjacent members}} \right).$$

As mentioned in Section 25, the portion of W that is resisted by each column is $\left(\frac{v_x}{\Sigma v_x} \right) W$. Calculations may conveniently be arranged as shown in Fig. 23. The nine bents, A to J, are tabulated separately, and each group is subdivided to provide space for individual joints in that bent. Joint coefficients are computed in the second column with a summation for each bent.

The relative resistance of each bent against horizontal displacement is proportional to the summation of joint coefficients for that bent. If the center of gravity of these nine resistances coincides with the centroid of the shear due to wind pressure, the wind pressure will give the floor a parallel displacement. If it does not coincide, a parallel displacement must be combined with a rotation of the floor as a whole about some vertical axis.

The joint coefficients in bent J based on the original K-values are in parentheses and their sum is 2.48. This value together with the other eight summations gives a centroid of resistance that is 89.5 ft.* from bent J. Since the wind-pressure component lies 80 ft. from J, the object is to eliminate the eccentricity of 9.5 ft. This may be done by adjusting sizes of certain beams and columns. The adjustment will be made in the J-bent because it is farthest from the centroid, which gives the change in J relatively greater weight. It is assumed that structural changes in bent J are not objectionable from an architectural viewpoint.

*Computed as $\frac{\Sigma \text{ (joint coefficients times distance from } J)}{\Sigma \text{ (joint coefficients)}}$.

In Fig. 23, the joint coefficients in the *J*-bent have been trebled; their new summation is 7.46. This value in conjunction with the other eight summations, which remain unchanged, gives a centroid of resistance that is 79.4 ft. from *J*. The eccentricity of 0.6 ft. is considered negligible. Calculations are needed to ascertain what changes in dimensions will be necessary to produce the new *K*-values recorded for the *J*-bent. This is settled by a procedure of trial and error and does not involve wind-pressure theory.

After the adjustment is made in the *J*-bent and the eccentricity is made negligible, the sum of all joint coefficients in Fig. 23 is 43.59. Each unit of bent resistance must withstand a wind pressure equal to $640 \div 43.59 = 14.7$ kips. Multiplying each individual joint coefficient in Fig. 23 by 14.7 gives the portion of wind pressure withstood by each joint or the wind shear resisted by each column above.

Column moments are taken as column shear multiplied by one-half the column height. At each joint, the sum of column moments equals the sum of beam moments and is distributed to the beams in proportion to their *K*-values. Beam shears are taken as the sum of the two end moments in the beam divided by the length of the beam.

At columns *C*3 and *C*4, it is assumed that there is not enough torsional stiffness in the lateral girders at the opening to transmit bending to the east-west beams. As a result, credit is given only for beams to one side of the column. The beam moments at *D*4 vary according to the stiffness of the spandrel and interior beams.

A brief discussion must be added in regard to the adjustment in bent *J*. The stiffening of this bent may cause the beam shears to increase greatly. The increased uplift on the windward side of such a bent may approach the point at which there is insufficient dead load available to counteract the uplift. This may be remedied by removing some of the stiffness from such bents to adjacent bents.

An interesting point may be demonstrated by making a similar analysis with smaller *K*-values for the columns at another typical floor several stories above the one considered. It will show that the percentage of wind pressure carried by each bent remains surprisingly uniform even when all *K*-values are one-fourth of their original value. This uniformity in distribution greatly reduces the analytical work required for a group of typical floors.

27 / eccentric wind pressure on a building

Consider the example in Section 26, but assume that the joint coefficients for the *J*-bent remain unchanged. Their sum equals 2.48 (see Fig. 23) and the sum of all joint coefficients equals 38.61. The centroid of resistance is $3{,}458 \div 38.61 = 89.5$ ft. from *J*, and the wind-pressure eccentricity is $e = 9.5$ ft. Under these assumptions, determine the shear induced in all the columns by a wind pressure of $W = 640$ kips.

If the wind pressure had been concentric, all joint coefficients would have been multiplied by the same factor, $\frac{W}{\Sigma v_x} = \frac{640}{38.61} = 16.6$. All joints would then be given the same translation. In the case of eccentric pressure, the floor will get both a translation and a rotation about some vertical axis. It is proposed to account for the combined effect by a method that amounts to using a multiplier equal to

$$F = \frac{W}{\Sigma v_x} + \frac{Wex}{I_x + I_y},$$

in which

$\Sigma v_x =$ sum of all joint coefficients in the x-bents (east-west);
$x =$ distance from any x-bent to centroid of joint coefficients;
$I_x =$ moment of inertia of joint coefficients about their centroid;
$I_y =$ the same as I_x but for bents in the perpendicular direction.

Values of I_x and I_y are computed in Fig. 24, in which joint coefficients, v_x are taken from Fig. 23. The calculations leading to v_y and y for bents 1, 3, 4 and 6 (running north-south) are not shown, but may be derived in the same manner from the data in Section 26. *K*-values for the floor slab are low and are ignored since its stiffness is small in comparison with the stiffness of the beams. Therefore, bents 2 and 5 do not appear in Fig. 24.

Inserting numerical values in the above formula for F gives:

$$F = \frac{640}{38.61} + \frac{640 \times 9.5 \times x}{114{,}700} = 16.6 + 0.053x.$$

Values of F are computed in Fig. 25. The next step is to determine column shears by multiplying joint coefficients in Fig. 23 by corresponding values of F in Fig. 25. These calculations will not be illustrated here. It is of more interest to compare results obtained by eccentric and concentric analysis.

In the example in which the *J*-bent is stiffened, all joint coefficients are multiplied by 14.7. But if the low *K*-values are maintained in *J*, all joint coefficients are to be multiplied by F taken from Fig. 25. The ratio of $F \div 14.7$ compares the column shears in the two examples. It is seen that changing from concentric to eccentric wind pressure reduces the shear by 12 per cent in bent *A* and increases it by 38 per cent in bent *H*. These changes have been brought about merely by varying the sizes of members in the *J*-bent.

28 / warping of floors

Bents subject to wind pressure have deflection due to shear and moment. Shear deflection signifies that floors are translated but not tilted, and originates in bending deformation of columns. Moment deflection, signifying that floors are tilted, is caused by change in column length. The latter type of deflection cannot be disregarded in tall, towerlike structures but has been ignored in the procedure employed in Sections 26 and 27.

One point in regard to moment deflection and its effect on reinforced concrete bents deserves brief attention. Refer for illustration to the calculations for bent *B* in Fig. 23. The shear is 9.3 kips in all beams, both interior and exterior. Since shears have opposite directions in beam ends adjacent to interior columns, the wind pressure creates no additional axial load in the interior columns. However, in exterior columns, an axial load of 9.3 kips is added to the gravity load in the column on the leeward side and deducted on the windward side. The result is a nonuniform change in length of column; the floor warps and a secondary distribution of moments and shears takes place.

Ordinarily the effect of warping is not of any consequence, but it may sometimes be desirable to approach the ideal condition in which there is no warping of floors. To do this, it is necessary to adjust dimensions in the bents so that interior beams carry much more shear than exterior beams. This can be accomplished by making the coefficients at interior joints large in comparison with those for exterior joints. Suitable dimensions are established by trial. The purpose is to make the additional column load due to wind pressure proportional to the distance of the columns from the midpoint of the bent.

Such refinements as those described in this section are considered justifiable only in relatively tall buildings, especially if the outer spans are comparatively short and their stiffnesses great.

29 / derivation of formula for joint coefficient

A, *B*, *C*, *D* and *F* in Fig. 26 are joints in a bent that is deformed by bending due to wind pressure. During the investigation of the conditions around joint A, the following assumptions were made and incorporated in Fig. 26:

1. Joints *F*, *A* and *C* lie on a straight line.
2. Joints *B*, *A* and *D* lie on a straight line.
3. The angle change, θ_A, is the same at F, A and C, θ_A being measured from a horizontal line.

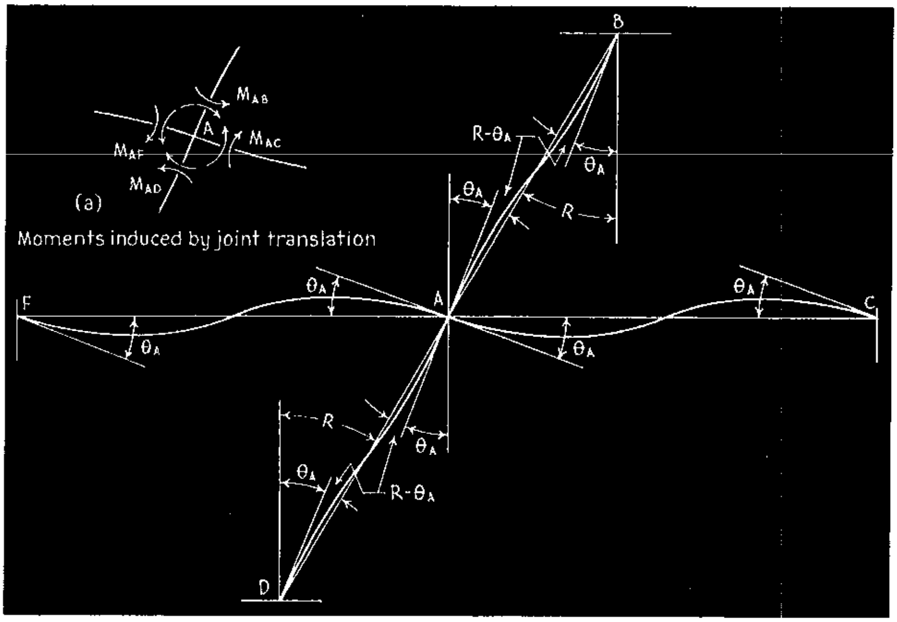


Fig. 26 — Frame deformed by wind pressure.

4. The angle change, θ_A, is the same at B, A and D, θ_A being measured from a vertical line.

The part of the bent included in Fig. 26 is distorted under wind pressure as shown diagrammatically, and angle R represents the translation of joints. Combined angle change at ends of columns is $R - \theta_A$, while the angle change at ends of beams is θ_A.

It can be shown by application of the formulas derived in Section 4, "Stiffness and Carry-over Factor," that:

$$M_{AC} = 2EK_{AC}(2\theta_A + \theta_A) = 6EK_{AC}\theta_A.$$
$$M_{AF} = 2EK_{AF}(2\theta_A + \theta_A) = 6EK_{AF}\theta_A.$$

As indicated in Fig. 26 (a), the moments in the beams tend to rotate joint A in one direction and the moments in the columns tend to rotate A in the opposite direction. Changing sign and substituting $R - \theta_A$ for θ_A give:

$$M_{AB} = -6EK_{AB}(R - \theta_A) = 6EK_{AB}\theta_A - 6EK_{AB}R.$$
$$M_{AD} = -6EK_{AD}(R - \theta_A) = 6EK_{AD}\theta_A - 6EK_{AD}R.$$

Since joint A is in equilibrium, the sum of the four moments must equal zero, or:

$$\Sigma M_{AX} = 6E\theta_A \Sigma K_{AX} - 6ER(K_{AB} + K_{AD}) = 0,$$

from which

$$\theta_A = R\left(\frac{K_{AB} + K_{AD}}{\Sigma K_{AX}}\right).$$

Inserting this expression for θ_A in the formula for M_{AB} gives:

$$M_{AB} = 6ERK_{AB}\left(\frac{K_{AB} + K_{AD}}{\Sigma K_{AX}}\right) - 6ERK_{AB} = 6ERK_{AB}\left(\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}\right).$$

If the shear in column AB is denoted as V_{AB},

$$V_{AB} = \frac{2}{h} \times M_{AB} = \left(\frac{12ER}{h}\right) K_{AB}\left(\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}\right);$$

and when $\frac{R}{h}$ is considered constant for all columns in a story, the relative value of shear in a column AB is

$$v_{AB} = K_{AB}\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}.$$

K_{AC} and K_{AF} are $\frac{I}{L}$-values for the beams adjacent to A; ΣK_{AX} is the sum of $\frac{I}{L}$-values for all members adjacent to A. For column AD below A, substitute K_{AD} for K_{AB}.

When relative values of shear in columns and the total wind shear are known, shears and subsequently moments may be calculated in the columns. Shears and moments may then be determined in the beams as illustrated in Fig. 23.

appendix

vertical load analysis

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Printed in U.S.A.

EB033.04D