Geometric Nonlinearities in Unbraced Multistory Frames

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Abstract: The geometric nonlinearities in elastic sway frames are examined. Various approximate methods of second-order analysis for sway frames are reviewed and the conditions limiting the use of these procedures are stated. A rational method for combining the nonsway and sway moments is developed. A practical method of including the effects of sway deflections due to gravity load moments and out-of-plumb construction in the approximate second-order analysis is derived.

Introduction

In the analysis and design of multistory structures the "geometric nonlinearities" caused by the influence of displacements on the equilibrium of the structure must be considered. The modification of first-order analyses to include this geometric nonlinearity is the primary objective of this paper and a companion paper (8). According to the principle of superposition, a frame can be analyzed separately as a nonsway frame and a sway frame with the final force resultants obtained by superposition, provided that the axial forces in the members of both frames are equal to those of the original frame under the actual state of loading. A nonsway frame, which is completely braced against sidesway at floor levels and subjected to gravity load moments, was included in an earlier study (8). This paper deals with the sway frame subjected to lateral loads and axial forces in the columns.

This paper consists of four parts. First the effects of geometric nonlinearities are studied. Second, a number of approximate methods of second-order elastic analysis are examined. Next a method of combining the nonsway and sway moments is developed, and finally, procedures for considering sway deflections due to gravity load moments and out-of-plumb construction are presented.

Geometric Effects

Fig. 1(*a*) shows a column which can be any column in a sway frame, subjected to forces at the ends. A straight line joining the ends of the column forms an angle of a/L with respect to the vertical. The axial load N may be replaced by inclined and horizontal components as shown in Fig. 1(*b*). The first of these acts parallel to the line joining the ends of the column and, assuming small deformations, is equal to N. The horizontal component is equal to Na/L. Consequently, the total shear acting

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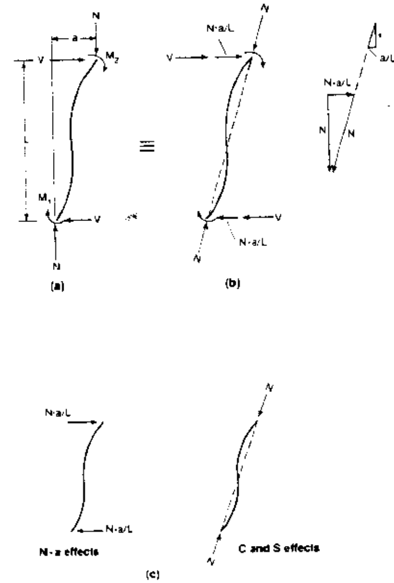


FIG. 1.—Geometric Effects Due to Axial Loads

at the end of the column is the sum of the original end shear V resisting the external lateral loads, and the N-a shears (Na/L) resulting from the moments induced by N acting through the deflection a.

Accordingly, the geometric effects due to the axial load can be decomposed into two types as shown in Fig. 1(*c*). First are effects due to the N-a shear which are termed the "N-a effects" (also called $P\Delta$ effects). The N-a shear produces an overturning moment which in turn tends to increase the lateral displacement and the overturning moment. The second type of geometric effect occurs due to secondary moments produced by N times the displacements from the chord line of the column. These will be termed the "C and S effects" because the axial load acting in this way changes the values of the stability functions C and S in the slope-deflection equation

$$M_2 = \frac{EI}{L}\left[C\theta_2 + S\theta_1 - \frac{a}{L}(C + S)\right] \quad (1)$$

in which M and θ = the end moment and rotation of a given column, respectively. Subscripts 1 and 2 refer to the ends of the column. The C and S terms in Eq. 1 take into account the additional bending moments contributed by the axial load acting on the column. If this effect is neglected, as in a first-order analysis, C and S are taken as 4 and 2, respectively. The C and S effects in a nonsway frame have been discussed in a companion paper (8). The N-a effects take into account the additional moments at the ends of the column contributed by the vertical axial load. In a first-order analysis, both of these effects are neglected. An analysis including these two types of effects is considered a complete second-order analysis.

An Example Frame.—The mechanics of these two types of geometric effects can be illustrated using an elastic single-story frame subjected to lateral and vertical loads as shown in Fig. 2(*a*). The vertical loads are expressed as a function of the Euler load, $N_e = \pi^2 EI/L^2$. The column axial load is assumed equal to the vertical load above it, the beam is rigid, and EI is constant for all the columns. In the absence of vertical loads a lateral load $H = 4V_0$ produces a shear of V_0 in each of the columns, and the frame undergoes a lateral deflection a_0 as shown in Fig. 2(*b*). The end moment M_0 is equal in all the columns.

To study the *N-a* effects only, the C and S will be taken equal to 4 and 2, respectively. The vertical loads are replaced by a horizontal force equal to the sum of *N-a* shears from all the columns as shown in Fig. 2(*c*). With this additional force, the sway of the frame is increased to $\bar{a} = \bar{f}_s a_0$. Since the lateral stiffness of each column remains unaffected, the shear resisting the total horizontal forces in each column becomes $\bar{f}_s V_0$, and the end moment in each column is also equal to $\bar{f}_s M_0$. The shear resisting the lateral load H in each column, as presented in Fig. 2(*b*), can be obtained by subtracting the *N-a* shear for a particular column from the total shear $\bar{f}_s V_0$. As the *N-a* shear is different for each column, the lateral load shears have been redistributed compared to the first-order shears in Fig. 2(*b*). Thus, the capacity of a column to resist the lateral load diminishes as the axial load is increased.

When the C and S effects are also included in the analysis (i.e., a complete second-order analysis) the sway of the frame is increased further to $a = f_s a_0$ because the lateral stiffness of those columns subjected to axial loads are reduced to the values shown in Fig. 2(*d*). Note that the reduced lateral stiffness shown in the figure corresponds to the total

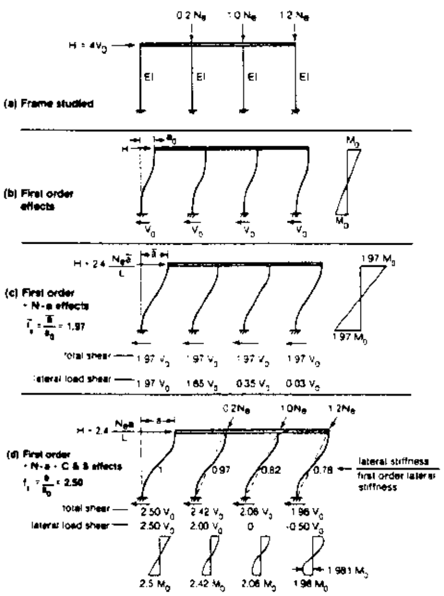


FIG. 2.—Single-Story Frame with Rigid Beams

shears including the *N-a* shears. The values would be different if only the lateral load shears are considered. This will be discussed later.

The reduction in column stiffness increases with higher axial loads. Since the total horizontal forces are resisted by the columns in proportion to their relative lateral stiffnesses, the total shear in each column becomes different. The shear in the column without any axial load is equal to $f_s V_0$, whereas the total shear in each of the axially loaded columns is less than $f_s V_0$. The most highly loaded column resists the least amount of shear. Similarly, the end moment is equal to $f_s M_0$ for the column not axially loaded, and smaller for the others. The moment diagrams for the axially loaded columns are nonlinear due to the C and S effects, compared to the linear moment distribution which results if only the *N-a* effects are considered (Fig. 2(c)). In the most highly loaded column, the maximum moment occurs away from the end.

The lateral load shear in each column is also presented in Fig. 2(*d*). Because the reduced stiffnesses of the axially loaded columns reduce their ability to resist the lateral loads, more lateral load shear is added to the less heavily loaded columns. For the column with a vertical load of N_e, the column does not offer any resistance to the lateral load, because N_e is equal to the free-to-sway critical load of that column. For the column with a vertical load greater than N_e, the lateral load shear has reversed direction, indicating that a negative shear is required to brace it from failing laterally.

To summarize, the *N-a* ($P\Delta$) effects cause an increase in the lateral deflections and overturning moments in a structure, while the C and S effects reduce the lateral stiffness of a structure and cause a redistribution of internal end moments and total lateral load.

Approximate Second-Order Analysis of Sway Frames

Iterative Method.—A number of authors (11,19,20) have described an iterative calculation of the *N-a* ($P\Delta$) effects in which the lateral deflections caused by the Na_0/L shears (a_0 = first order deflections) give rise to a new set of Na/L shears which in turn give rise to a new set of deflections. After several iterations a good estimate of the *N-a* effects is obtained. This procedure assumes that C and S effects are negligible and hence corresponds to the case illustrated in Fig. 2(*c*).

Modified Iterative Method.—Based on the assumption that the deflected shape produced by lateral and vertical loads is equal to that produced by lateral loads plus sway forces applied at the floor levels, a "modified iterative method" can be derived using the principle of minimum potential energy. In this method, a sway frame is analyzed according to the first-order theory for lateral loads plus sway forces H_s given by

$$H_{si} = \bar{V}_{s,i} - \bar{V}_{s,i+1} \quad (2)$$

in which
$$\bar{V}_s = \left(\sum \frac{\gamma N}{L}\right) a \quad (3)$$

the subscript i refers to floor level at which H_s acts and the story below this floor, where $\bar{V}_{si}$ acts. The summation sign Σ in Eq. 3 denotes summation for all columns and inclined bracing elements in a given story. The term γ is the ratio of the downward displacement of the top of the bent column (Fig. 3(b)) to that of a straight column (Fig. 3(a)) and is defined in Fig. 3. This factor accounts for the effects of bending between the ends of the column and is referred to as the "flexibility factor." The evaluation of this factor will be discussed later. The term N denotes the axial compressive force in a column in a given story when the frame is subjected to the original state of loading (i.e. both gravity and lateral loads), and L is the corresponding column height. In the case of a pin-ended inclined bracing member N = the vertical force component of the axial force in the member (positive for compression), L = the vertical projection of the member length, and $\gamma = 1.0$. The term a is the horizontal deflection of the top of a given story relative to its bottom. The analysis is iterative as the deflection a is the result of the analysis.

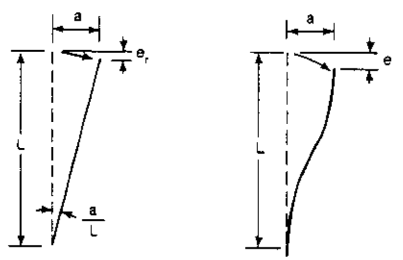
The axial load N in Eq. 3 can be assumed equal to its first-order value to simplify the calculation. This will not introduce any significant errors since the summation sign offsets the errors introduced by this assumption unless L varies widely from column to column. It should be noted that the term L in Eq. 3 is placed within the summation operator, and therefore the equation can be applied to the case of unequal column heights in the bottom story. For the other stories, L is the same as the story height.

The flexibility factor γ for a given column can be determined by

$$\gamma = 1 + 0.22\left\{\frac{4(G_1 - G_2)^2 + (G_1 + 3)(G_2 + 3)}{[(G_1 + 2)(G_2 + 2) - 1]^2}\right\} \quad \text{(4)}$$

in which $$G_1 = \frac{\left(\sum \dfrac{EI}{L}\right)_{col}}{\left(\sum \dfrac{EI_B}{L_B}\right)_{beam}} \quad \text{(5)}$$

The sign Σ denotes summation for columns or beams rigidly connected to one end of a given column, and similarly for G_2 at the other end. The



$e_r = 0.5\,\dfrac{a^2}{L}$ (a)

$e = \gamma e_r$ (b)

FIG. 3.—Vertical Displacement of Column Distortion

lower limit of γ given by Eq. 4 is 1.0 when $G_1 = G_2 = \infty$ (hinged at both ends). The upper limit is 1.22 when $G_1 = G_2 = 0$ (rigid beams at both ends) or $G_1 = \infty$ and $G_2 = 0$.

Eq. 4 is derived on the basis of three major assumptions (7). First, a first-order deflected shape is assumed for the column. Second, the rotational stiffness of the beams is equal to $6EI_B/L_B$. Third, the column end moments at the joints are distributed between the column above and the column below in the ratio of the EI/L values of the two columns. Similar relationships for the flexibility factor γ have been developed by a number of investigators (5,9,15,16,18).

The first assumption is compatible with the basic assumption in the modified iterative method that the deflected shape of a sway frame under vertical and lateral loads can be represented by the deflected shape produced by horizontal loads only. The second assumption is reasonable for beams with both ends rigidly connected to columns which undergo similar deformations. In the case that the far end of a beam rigidly connected to the column under consideration is hinged or fixed, the beam length should be multiplied by 2 or 1.5, respectively, when calculating the corresponding value of G.

The third assumption is reasonable for frames with stiff beams where the relative story deflections are roughly the same in the upper and lower stories such as, for example, a multistory frame with stiff beams where the lateral stiffness and loading of a given story do not differ significantly from the story above or below. In other words, this assumption is most valid for the value of the flexibility factor close to the upper limit. For a frame with flexible beams, this assumption becomes less accurate, but its effect is proportionally smaller because the values of the flexibility factor become smaller. The offsetting effect is best reflected by the extreme case of a shear wall. For a shear wall this assumption breaks down, but for G approaching zero Eq. 4 gives $\gamma = 1.0$. This is a reasonable answer since a shear wall within a story deflects in a very similar manner to a rigid column (Fig. 3). Therefore, this assumption leads to reasonable values of γ although it may not represent the actual behavior in the case of flexible beams.

As is apparent from Eq. 3, the flexibility factor needs to be evaluated for each column. In recognition of the small range of the values of the flexibility factor (from 1.0 to 1.22), it may be preferable to use a single value of γ for the entire story of frame, when a precise calculation is deemed unnecessary. Table 1 shows the range of the values of the flexibility factor corresponding to given values of G_s, the smaller value of G for a column. The range of G_1 and G_2 from 0.1 to 10 used in the table includes most practical cases. An average flexibility factor $\bar{\gamma}$ which tends to be on the conservative side is suggested for a given range of the values of G_s. Since G_s will rarely be less than 0.1 in practical frames, a conservative value of $\bar{\gamma} = 1.15$ can be used for any frame and the calculation will be simplified. Stevens (18) proposed the use of $\bar{\gamma} = 10/9 = 1.11$.

The modified iterative method differs from the conventional iterative analysis discussed earlier (6,19,20) by introducing the flexibility factor γ into the expression for the N-a shears (Eq. 3). This was first suggested by Rosenblueth (15) and later by Rubin (16).

TABLE 1.—Suggested Values for the Average Flexibility Factor

G_s (1)	Stiff restraints 0.1–0.4 (2)	0.4–1.0 (3)	Flexible restraints 1.0–10 (4)
γ	1.09–1.18	1.05–1.12	1.0–1.07
$\bar{\gamma}$	1.15	1.10	1.05

Note: G_s is the smaller value of G for a given column.

The physical significance of the flexibility factor can be explained by recalling the discussion of the C and S effects. (Note that $\gamma = 1.0$ if C and S effects were neglected.) As stated previously, inclusion of the C and S effects produce two consequences: (1) Further increasing the lateral deflections; and (2) redistributing the total shears or the story moments. The former is artificially looked after by the introduction of the flexibility factor. The latter, however, is not directly taken into account and is a source of possible errors in the moments. This, however, can be rectified by using the moment-correction factors developed by Hellesland (5).

Based on the assumption that the lateral deflections and second order end moments M_1 and M_2 are accurately determined in the analysis, the maximum moment, M_{max}, along the length of a column is given by Eq. 6.

$$M_{max} = \delta_{ns} M_{m2} \quad (6)$$

in which δ_{ns} = the moment magnifier for a restrained nonsway column end moments M_{m1} and M_{m2} from the modified iterative analysis. For simplicity, the approximate formula for δ_{ns} suggested in the companion paper (8) is used here. The column end moments, which are also needed to determine the moments required in the attached beams, can be conservatively assumed equal to M_{m1} and M_{m2} (6).

Modified Negative Brace Method.—This method gives essentially the same results as the modified iterative method but allows a direct calculation without iteration. A fictitious pin-ended diagonal bracing member with a negative value of AE given by Eq. 7 is inserted in every story of the structure.

$$\mathrm{AE} = -\frac{\left(\sum \frac{\gamma N}{L}\right) L_b}{\cos^2 \alpha} \quad (7)$$

in which L_b = the length of the negative brace and α = its angle of inclination. The structure, including the negative braces, is then analyzed for the lateral loads acting alone (i.e., a first-order analysis). The column end moments obtained from this analysis are equivalent to M_m in Eq. 6.

The concept of negative braces originated with Nixon et al. (12) who derived an expression similar to Eq. 7 without considering the flexibility factor. A complete derivation of this method is given by Lai (7).

Story Magnifier Method.—In this method the modified iterative method is further simplified by making the additional assumption that each story can behave independently of other stories. As a result, any story in a frame subjected to lateral loads plus sway forces can be treated like a single-story frame subjected to lateral load shears, ΣV, plus the modified N-a shears, $(\Sigma\ \gamma N/L)a$, of that story. Because the deflection of a single-story sway frame under horizontal forces only is directly proportional to the applied horizontal force, the following relation is obtained:

$$\frac{a}{a_0} = \frac{\Sigma V + \left(\sum \frac{\gamma N}{L}\right) a}{\Sigma V} \quad (8)$$

in which a_0 = the first-order deflection of the story. After rearranging the terms, the deflection magnifier f_s defined as a/a_0 is equal to

$$f_s = \frac{1}{1 - \frac{\left(\sum \frac{\gamma N}{L}\right) a_0}{\Sigma V}} \quad (9)$$

in which Σ denotes the summation for all columns and inclined braces (if any) in a given story. Because the moments in a single story frame subjected to horizontal forces are directly proportional to the deflection, the column end moments are equal to $f_s M_{0s}$ which is equivalent to M_m in Eq. 6. The term M_{0s} is the first-order end moment in a given column in a sway frame. Expressions similar to Eq. 9 have been developed by many investigators (2,5,9,11,13,15,18).

The simplifying assumption of the behavior of a story being independent of other stories is reasonable for a frame with stiff beams. Based on extensive evaluation of this method (7), this assumption was shown to be valid subject to two specific conditions. First, the maximum value of f_s in the structure is less than about 1.5. Second, an inflection point should occur at or between the ends of each column in every story of the structure when it is subjected to lateral loads.

Frame Magnifier Method.—This method is also a simplification of the modified iterative method. Here the additional assumption is made that the deflection ratio a/a_0 is equal for all the stories of the frame subjected to lateral loads H plus sway forces. In other words, the total lateral deflections of the frame are those produced by the lateral loads $f_s H$, where $f_s = a/a_0$. As a result, the energy stored in the structure due to the lateral loads plus the sway forces is identical with the energy resulting from the lateral loads $f_s H$. Based on this, the deflection magnifier f_s becomes equal to

$$f_s = \frac{1}{1 - \left\{ \frac{\sum_{i=1}^{n} \left(\sum \frac{\gamma N}{L}\right)_i a_{0i}^2}{\sum_{i=1}^{n} (\Sigma V)_i a_{0i}} \right\}} \quad (10)$$

in which i = the story level; and n = the number of stories in the structure. Since all the deflections are increased by the same ratio, the column end moments are equal to $f_s M_{0s}$ which is equivalent to M_m in Eq. 6.

The critical load factor implied by Eq. 10 (i.e., when $f_s = \infty$) is similar to the one given by Stevens (18) although derived in a different manner.

The simplifying assumption of equal a/a_0 in all the stories was found (7) to be valid provided the structure includes a distinct shear wall extending from the base to the top of the structure and f_s is less than about 1.5. A "distinct shear wall" is defined here as a stiff vertical element which has less than two points of contraflexure. This assumption was first suggested by Perrez-V. (14) but the limitation of the method was not developed.

Effective Length Method.—This method can also be considered a further simplification of the modified iterative method, but it has the restriction that the frame to be considered cannot include any distinct bracing elements such as shear walls or inclined bracing members. Two additional simplifying assumptions are required in the calculation of the effective lengths as implemented in the ACI Code (1). First, the rotational stiffness of the beams is equal to $6EI_B/L_B$ corresponding to inflection points at mid-span of the beams. Second, the column end moments at the joints are distributed between the column above and the column below in the ratio of the EI/L values of the two columns. These two assumptions permit a story to be isolated from the frame with the column end rotational restraints expressed as a function of G_1 and G_2. For an isolated single-story frame, the modified iterative method is simplified to Eq. 9. Based on Eq. 9 and the idealized end restraints, the deflection magnifier f_s can be expressed as

$$f_s = \frac{1}{1 - \dfrac{\sum \dfrac{N}{L}}{\sum \dfrac{N_{fs}}{L}}} \qquad (11)$$

in which N_{fs} is the free-to-sway critical load of a column. The free-to-sway effective length factor, which is a function of G_1 and G_2 only, is available from an effective length factor alinement chart or other standard method. The sign Σ denotes summation for all columns in a given story. The above equation was given by Hellesland (5). For a story with equal column heights, the terms L in Eq. 11 cancel out, and the equation becomes the same as given in the American Concrete Institute (ACI) Code (1) for sway frames.

The assumption of mid-span inflection points is reasonable for beams rigidly connected to columns. In fact, the summation of N_{fs} for all columns in a story offsets some of the error resulting from the inflection points not occurring exactly in mid-span of the beams. If the far end of a beam framed into the column under consideration is hinged or fixed, the beam length should be multiplied by 2 or 1.5, respectively when calculating the corresponding value of G, in order to obtain the correct value of k_{fs} from the alinement chart.

The second assumption, which permits a story to be separated from the frame, is reasonable for a regular multistory frame with stiff beams and regular loading as discussed earlier. The conditions suggested for the story magnifier are also suggested here to limit the application of the effective length method. That is, an inflection point should occur at or between the ends of each column in every story of the structure when subjected to lateral loads, and f_s should be less than about 1.5. Subject to these conditions, the effective length method was found (7) to give quite accurate results except at discontinuities such as stories where the column stiffness changes abruptly. Because of its inability to adequately account for discontinuities, the effective length method was generally less accurate than the story magnifier method.

Accuracy of Approximate Second-Order Analyses.—In Ref. 7 the approximate analyses are compared to "exact" second-order, elastic slope deflection analyses of a series of short and tall frames. Three of these are summarized in Fig. 4 to Fig. 6. In each case the building studied is a 24-story frame with a single concentrated lateral load applied at the top and concentrated vertical loads applied at each joint in the frame and at the top of the wall in the third frame. The stepped lines in each figure give the ratio M_2/M_{02} from the exact analysis. For the frame with the wall this is given as the deflection magnifier a/a_0. In all calculations γ was taken as 1.05 except in the bottom story where 1.20 was used due to the fixed base.

Fig. 4 shows a frame with constant beam stiffness and column stiffnesses varying in 3 steps from 1.04 times the beam stiffness at the top to 3.7 times at the bottom. The magnifiers computed by the story magnifier method give excellent agreement. The ACI procedure is also excellent except at changes in column stiffness. The frame magnifier method gives a single value which approaches the average magnifier.

Fig. 5 shows a frame with very stiff columns in the lower stories (18.8 times beam stiffness). The first order bending moment diagram for this building shows single curvature bending in the columns in the bottom 4 stories. The story magnifier method shows the correct trend while the

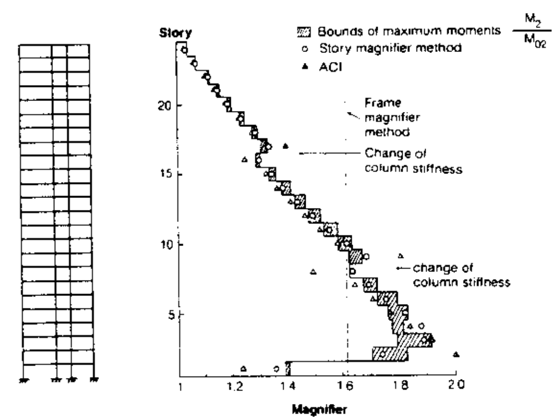


FIG. 4.—Comparison of Approximate and Exact Solutions—Frame

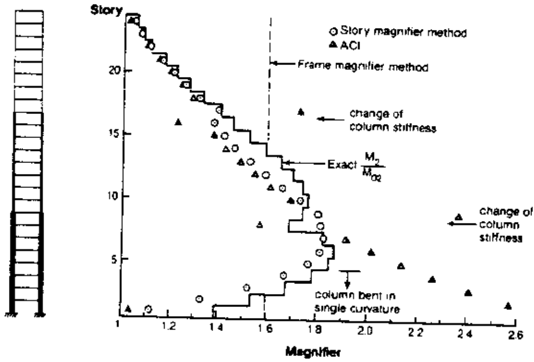


FIG. 5.—Comparison of Approximate and Exact Solutions—Frame with very Stiff Columns

ACI procedure is less accurate throughout, particularly at changes of stiffness and in the lower floors.

Fig. 6 shows a frame with a discontinuous shear wall. Here the frame magnifier method applies in the region with the shear wall and the story magnifier method applies in the region above the wall. The ACI procedure has not been compared since the bottom 16 stories are braced by the ACI definition.

Superposition of Nonsway and Sway Moments

Once nonsway and sway moments have been computed in a frame, they must be combined before the columns can be designed. The major difficulty arises from the fact that the column end moments in the nonsway frame are not known (8). Presently, there are three basic approaches to deal with this problem. The first approach is the one recommended in the 1977 ACI Code (1) or the 1978 American Institute of Steel Construction (AISC) Specification (16). The maximum moment,

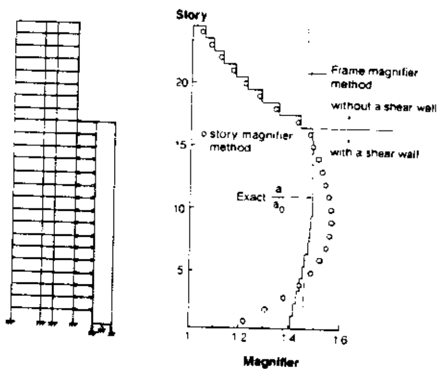


FIG. 6.—Comparison of Approximate and Exact Solutions—Frame with Discontinuous Wall

$M_{\max}$, in a column of an unbraced frame is given by

$$M_{\max} = f_s \cdot (M_{0ns} + M_{0s})_2 \quad \text{(12)}$$

in which M_{0ns} = the first-order end-moment of the column from the nonsway analysis; and M_{0s} = the first-order end moment from the sway analysis. The summation of the two end moments is performed at each end of the column and the numerically larger sum, denoted by the subscript 2, is used. This, obviously, is not a rational approach because the sway magnifier f_s is only applicable for M_{0s}.

The second approach (3,4) is to combine directly the nonsway maximum moment, $M_{ns,\max}$, and the sway maximum moment, $M_{s,\max}$, of a given column:

$$M_{\max} = M_{ns,\max} + M_{s,\max} \quad \text{(13)}$$

This is a conservative approach since the summation of the two maximum values must be greater than or equal to the actual maximum moment. It may be overconservative when the two maximum moments occur in different sections and the two values are comparable in magnitude.

The third approach (6,11) is based on the assumption that the C and S effects can be neglected. As a result of this assumption, the nonsway end moments are equal to the first-order values. Therefore, the end-moments of the nonsway and sway frame can be superimposed at each end of the column. Once this is done, the maximum moment in the column with the known end-moments can be determined as in a pin-ended column (11).

According to the principle of superposition in which load effects are superimposed at the same section, the third approach is more rational than the second one. Nevertheless, the second approach can take into account the C and S effects, which the third approach cannot. The method to be proposed is similar to the third approach in that moments are summed at the two ends of the columns, thus satisfying the principle of superposition. However, an attempt will be made to include the C and S effects.

A method of combining the nonsway and sway moments is developed schematically in Fig. 7. A frame subjected to external moments, lateral loads and column axial forces is shown in its equilibrium position in Fig. 7(*a*). In Fig. 7(*b*) the same frame, subjected to external moments only, is braced against sway with lateral deformations, a, equal to those in the original state of loading (Fig. 7(*a*)). This frame can be decomposed into a nonsway frame subjected to external moments and a laterally deformed nonsway frame with forced deformations a, as shown in Fig. 7(*c*). Therefore, it can be seen that the column end moment, M_e, in the frame in Fig. 7(*b*) is equal to the sum of M_{0ns} and M_m. The term M_m is the column end moment in the laterally deformed nonsway frame or the result obtained from the modified iterative analysis or the modified negative brace analysis. In the story magnifier method or other simplified methods, the general term M_m will be replaced by $f_s M_{0s}$ which corresponds to the definition of the moments from the modified iterative method or negative brace method. When the frame in Fig. 7(*b*) is subjected to the column axial forces (Fig. 7(*d*)), the resulting load effects are identical with those in the original state of loading. Therefore, it can be

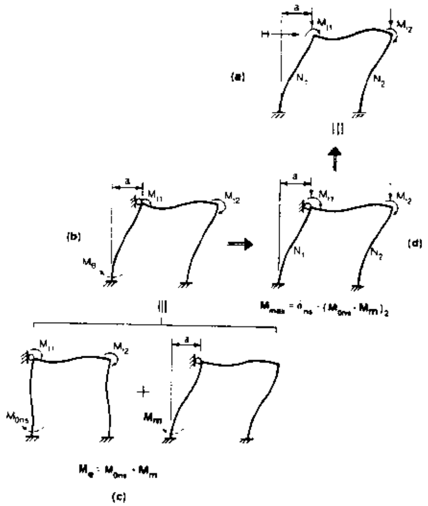


FIG. 7.—Schematic Development of the Proposed Approach for Combining the Nonsway and Sway Moments

seen that the moment at the end of the column is

$$M_e = M_{0ns} + M_m$$

and the maximum moment along the length of the column, M_{max}, can be given by

$$M_{max} = \delta_{ns} \cdot (M_{0ns} + f_s M_{0s})_2 \quad \text{(14)}$$

in which δ_{ns} = the moment magnifier for the restrained nonsway column with given first-order end moments. The approximate formula for δ_{ns} suggested in the companion paper (8) is used here with the end moments taken equal to $(M_{0ns} + f_s M_{0s})$ in the computation of M_{max}. Note that the summation is performed separately at each end of the column and the numerically larger value from the two ends (denoted by the subscript 2 in Eq. 14) is multiplied by δ_{ns}. Note that Eq. 14 is the same as Eq. 6 when M_{0ns} is equal to zero.

Theoretically, the approach developed above entails no assumptions other than those required in the approximate methods for the analysis of nonsway and sway frames. In other words, if δ_{ns} and $f_s M_{0s}$ were exact in the nonsway analysis and sway analysis, respectively, the value of M_{max} given by Eq. 14 would also be exact.

Deflections Due to Gravity Load Moments.—When the nonsway frame is prevented from swaying laterally under gravity loads, holding forces are developed in the lateral bracing elements. These holding forces can be added to the actual lateral loads when the sway frame is analyzed to include the sidesway effects due to gravity loads. Generally it is sufficient to use the holding forces from a first-order analysis of the nonsway frame, thereby neglecting the *C* and *S* effects. This assumption is based on the normal condition that the internal moments resulting from the holding forces are small compared to the lateral load moments or the

gravity load moments. As a result, the errors resulting from this assumption are relatively insignificant in design.

Frequently, it is more convenient to perform a first-order analysis for gravity loads without bracing the structure against sidesway. The results of such an analysis can be used in the load superposition by following the procedure described below. For simplicity, this procedure is derived using a single-story frame, but it can be generalized to apply to multistory frames.

A single-story frame, subjected to gravity loads only, displaces a distance a_{0g}, the first-order displacement due to gravity load moments, M_{t1} and M_{t2}. It is then braced against further sway while the column axial forces are applied, as shown in Fig. 8(*a*). The first-order moments in the resulting nonsway frame are those obtained from a first-order gravity load analysis of the frame without bracing the structure against sway. This frame can be decomposed into the two frames shown in Figs. 8(*b*) and 8(*c*). The frame in Fig. 8(*b*) is a nonsway frame, subjected to the external moments and column axial forces, with a zero displacement at the joint. The holding force in this frame is denoted by V_g. The frame in Fig. 8(*c*) is a nonsway frame, subjected to column axial forces, with an imposed displacement of a_{0g} at the joint. The holding force is denoted by $\bar{V}$. In this way, the holding force V_a in the original frame (Fig. 8(*a*)) is equal to

$$V_a = V_g - \bar{V} \quad \text{(15)}$$

For the frame shown in Fig. 8(*c*), the column axial forces are assumed to be replaced by a horizontal load equal to $(\Sigma\gamma N/L)\, a_{0g}$ (Fig. 8(*d*)) where Σ denotes summation for all columns in the story. This can be derived based on the same assumption and method as in the modified iterative method except that "a" is replaced by "a_{0g}." Consequently, the holding force $\bar{V}$ (Fig. 8(*c*)) is assumed equal to

$$\bar{V} = V_{0g} - \left(\sum \frac{\gamma N}{L}\right) a_{0g} \quad \text{(16)}$$

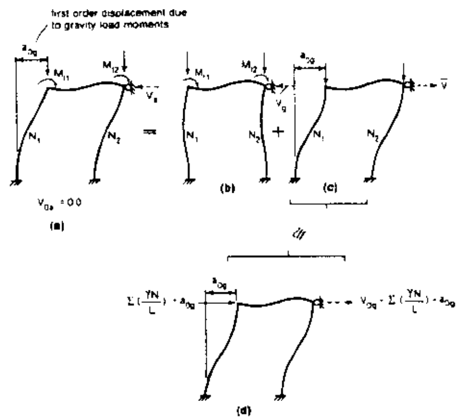


FIG. 8.—Holding Shears in a Nonsway Frame with Imposed Displacements

in which V_{0g} = the first-order value of V_g (Fig. 8(*b*)). Substituting Eq. 16 into Eq. 15 and making the assumption that $V_g = V_{0g}$ as discussed earlier, V_a is equal to

$$V_a = \left(\sum \frac{\gamma N}{L}\right) a_{0g} \quad \text{(17)}$$

which is the horizontal force that should be added to the actual lateral loads in the sway analysis. To apply Eq. 17 to a multistory frame, the holding force V_a for each story represents a holding shear for that story which is added to the lateral load shear in the sway analysis. In other words, the total lateral load at a given floor level is equal to the actual load plus the algebraic sum of the holding shears V_a (Eq. 17) from the story above and below the floor in the manner implied by Eq. 2. Thus, this is similar to the calculation of the sway force in the modified iterative method.

In the modified iterative method, the holding shears can be included more conveniently by writing the modified *N-a* shear $\Sigma \bar{V}_s$ (Eq. 3) as

$$\Sigma \bar{V}_s = \left(\sum \frac{\gamma N}{L}\right)(a + a_{0g}) \quad \text{(18)}$$

in which term "*a*" still represents the final value of the second-order deflection of the structure due to lateral loads.

In the story magnifier method, the procedure can be simplified. For a single-story frame subjected to the lateral load shear plus the modified *N-a* shear as assumed in the story magnifier method, the deflection is directly proportional to the horizontal load applied at the joint. Noting this condition, the following relation, based on Eq. 9, can be derived:

$$\frac{a}{a_{0H}} = f_{sH} + (f_{sH} - 1)\frac{a_{0g}}{a_{0H}} \quad \text{(19)}$$

where the sway magnifier for lateral load effects is

$$f_{sH} = \frac{1}{1 - \dfrac{\left(\sum \dfrac{\gamma N}{L}\right) a_{0H}}{\Sigma V_H}} \quad \text{(20)}$$

The terms a_{0H} and ΣV_H represent the first-order story displacement and the total story shear, respectively, produced by the actual lateral loads *H* only. Similarly, the value of $f_s M_{0s}$ in the analysis for the sway frame subjected to lateral loads and holding forces can be assumed equal to

$$f_s M_{0s} = \left[f_{sH} + (f_{sH} - 1)\frac{a_{0g}}{a_{0H}}\right] M_{0H} \quad \text{(21)}$$

in which M_{0H} = the column end moment from a first-order analysis of the frame subjected to the actual lateral loads *H* only. Eq. 21 is also applicable for the effective length method except that f_{sH} is taken equal to f_s given by Eq. 11.

Out-of-Plumbs.—In real structures, the centroid of the top of a column often is not directly over the centroid at the other end due to construction errors. In other words, columns are frequently "out of plumb." As a result, the gravity loads acting through the initially inclined columns generate additional forces within the structures. For design purposes, all columns in the same story are generally assumed to lean in the same direction with the same amount of out-of-plumb (10). Based on this assumption, the inclusion of the effect of out-of-plumbs in the second-order analysis is derived below.

Assume that fictitious external moments acting at the joints of a strucure can produce the prescribed initial out-of-plumbs. It is understood that the first-order moments caused by these fictitious external moments do not exist, and therefore only those caused by the column axial forces acting through the out-of-plumbs are considered. The methodology follows that for the gravity load deflections. In Fig. 8, the term a_{0g} is replaced by a_{0p} which denotes the initial out-of-plumb for any story. Consequently, the holding shear, which should be added to the lateral load shear in the sway analysis of an initially undeformed structure, is equal to $(\Sigma\ \gamma N/L)\ a_{0p}$ for a story. Note that the real first-order moments in the nonsway frame (Fig. 8(*a*)) with the lateral displacement equal to a_{0p} are equal to zero. The additional moments caused by the column axial forces in the nonsway frame are assumed negligible.

Thus, the out-of-plumbs are included in the analysis in the same way as the gravity load deflections and the term a_{0g} in the preceding section can be replaced by $(a_{0g} + a_{0p})$ in the sway analysis.

Summary

The geometric effects in a sway frame have been separated into two types: the "*N-a* effects" and the "*C* and *S* effects." A single-story frame was used to illustrate these two types of effects.

Various approximate methods of second-order analysis for sway frames have been reviewed and the assumptions in each discussed. The conditions limiting the use of these procedures are stated.

A rational method for combining the nonsway and sway moments is proposed. The proposed method entails no assumptions other than those required in the approximate methods for the second-order analysis of nonsway and sway frames. A practical method of including the effects of sway deflections due to gravity load moments and out-of-plumb columns in the approximate second-order analysis is presented.

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Appendix II.—Notation

The following symbols are used in this paper:

A = area;
a = lateral deflection of the top of a column relative to the bottom;
C = stability function in the slope-deflection equation (Eq. 1);
EI = flexural stiffness of a column;
EI_B = flexural stiffness of a beam;
f_s = sway deflection magnifier defined by a/a_0;

G = relative stiffness parameter given by Eq. 5;
H = horizontal load;
L = column height;
L_B = beam length;
M = end moment of a column;
M_m = end moment of a column obtained from the modified iterative analysis;
N = axial compression in a column;
N_{fs} = free-to-sway critical load of an elastic column;
n = total number of stories in a frame;
S = stability function in the slope-deflection equation (Eq. 1);
V = end shear in a column due to lateral loads;
$\bar{V}_s$ = sum of modified N-a shears in a story (Eq. 3);
α = angle of inclination of brace;
δ_{ns} = moment magnifier for a restrained nonsway column with given first-order end moments;
γ = flexibility factor;
$\bar{\gamma}$ = average flexibility factor; and
θ = end rotation of a column.

Subscripts

0 = first-order effects (implying $N = 0$);
$1,2$ = indicate ends of a column;
g = effects due to gravity load moments;
H = effects due to horizontal loads;
i = story or floor level;
ns = nonsway column; and
s = sway column.

Shears, V, moments, M, and displacements, a, without subscripts refer to the final second-order values. The subscript "0" as in V_{01} or a_0 refers to the first-order value. The subscript "2" as in M_2 refers to the larger end moment in a column.