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Capacity of Reinforced Rectangular Columns Subject to Biaxial Bending

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Comprehensive design charts complying to Section 1905(a) of the ACI Building Code (318-63) relating the biaxial bending capacity of rectangular columns to the uniaxial bending capacity by a single parameter are presented. Differences in the behavior of columns due to bar arrangement and steel strengths are noted. An approximate procedure which facilitates the determination of the required size for columns subject to biaxial bending is suggested and evaluated.

Key words: analysis; biaxial bending, column, design; rectangular column; reinforced concrete; ultimate strength.

■ JUST AS THE DIFFICULTY ASSOCIATED with the determination of the ultimate capacity of reinforced columns subject to combined axial load and uniaxial bending is primarily an arithmetical one, so is that of an axially loaded column subject to biaxial bending, with the added complication arising from the introduction of another variable. With the exception of a few cases, the bending resistance of a column with specified reinforcement subject to a given axial load, is determined through iteration of simple but lengthy computations. These extensive calculations are compounded when optimization of the reinforcement or cross section is sought. Consequently, in practice, it is customary to depend on design aids in the form of interaction curves or tables for the design of eccentrically loaded columns. Fortunately, for uniaxial bending, an abundance of design aids well suited for this purpose are available.[1-3]

In contrast to this, although several noteworthy articles[4-7] on biaxial bending which contributed greatly to the understanding of this subject have appeared in recent years, significant gaps in the area of design aids for biaxial bending still exist. To lessen these gaps, a number of comprehensive design aids are presented, implemented by a discussion of the importance of the parameters. It is of course obvious that the presentation of interaction curves for various ratios of the bending moments about each axis is impractical because of the voluminous nature of the data and the difficulty of two or threefold interpolation. Hence, to cover adequately and comprehensively biaxial bending within a manageable compilation, the response of columns to biaxial bending must be related to its uniaxial resistance. Several approaches based on acceptable approximations have been suggested to achieve this. Of these, the one that appears to have the most merit from the viewpoint of accuracy, condensation of design charts, and simplification potential is that in which the relative bending moments about each axis are related by a single though variable exponent.

The biaxial bending resistance of an axially loaded column can be represented graphically (Fig. 1a) as a surface formed by a series of interaction curves drawn radially from the P_u axis. When the bending resistance is plotted in terms of the dimensionless parameters P_u/P_o, M_y/M_{uy}, and M_x/M_{ux} with the latter two terms designated as the relative moments, the ultimate capacity surface generated assumes the typical shape shown in Fig. 1b. The advantage of expressing the behavior in relative terms is that the contours of the surface (Fig. 1b), i.e., the intersection formed by planes of constant P_u/P_o and the surface,

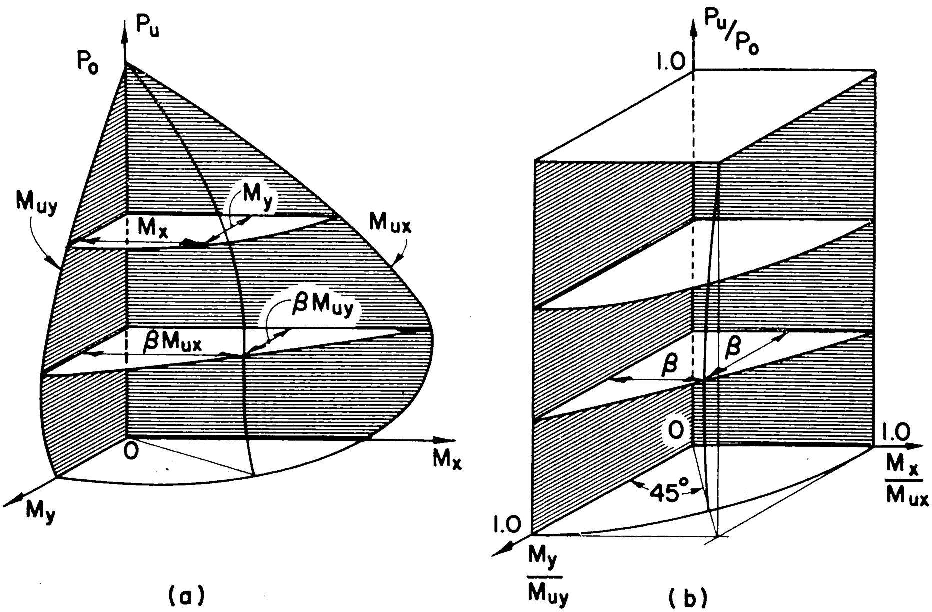

Fig. 1—Ultimate capacity surfaces

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can be considered for design purposes to be symmetrical about the vertical plane bisecting the two coordinate planes. Even for sections that are rectangular or have unequal reinforcement in the two adjacent faces, this approximation yields values sufficiently accurate for design. As suggested by Bresler, the contours can be approximated by the expression:

$$(M_x/M_{ux})^n + (M_y/M_{uy})^n = 1 \qquad \text{(1a)}$$

which can be restated in the less convenient but more meaningful form:

$$(M_x/M_{ux})^{\log 0.5/\log \beta} + (M_y/M_{uy})^{\log 0.5/\log \beta} = 1 \qquad \text{(1b)}$$

in which β equals the ordinate of the contours at the point at which the relative moments are equal. When $\beta = 0.5$, its lower limit, Eq. (1b) describes a straight line joining the points at which the relative moments equal one at the coordinate planes. When $\beta = 1.0$, its upper limit, Eq. (1b) describes two lines each of which is parallel to one of the coordinate planes. For intermediate values of β, Eq. (1b) describes curves which have been sometimes called sub- and superellipses. For design convenience, a plot of the curves generated by nine values of β are given in Fig. 2. With β known, the bending resistance in any direction of an eccentrically loaded column can be readily obtained.

The excellent agreement between this assumed variation and that obtained in the conventional manner by solving the equations of equilibrium, is shown in Fig. 3 for a wide range of conditions. The difference between the assumed and theoretical curves in the direction of the resultant does not exceed 5 percent, even for the case where the reinforcement on one face is 2.5 times that on the adjacent face. The close proximity of the curves in Fig. 3 signifies that the problem of biaxial bending reduces to merely the determination of β's and the uniaxial bending resistance about each axis of an axially loaded column. As previously stated, considerable data on this latter item already exist.

The value of β is a function of the amount, distribution and location of the reinforcement, the dimensions of the column and of the strength and elastic properties of the steel and concrete. To obtain values of β, a computer program was prepared on the basis of Section 1503 of the 1963 ACI Code using a rectangular stress block. In the equation of equilibrium, the effect of the concrete displaced by the steel was neglected since it had no significant effect on β. Data were accumulated for many values of g, b/t, q, f_c', f_y, and bar arrangements. It was found that the parameters g, b/t, and f_c' had minor effect. The maximum difference in β amounted to about 5 percent for a given value of P_u/P_o ranging from 0.1 to 0.9. The bulk of the values, especially those in the

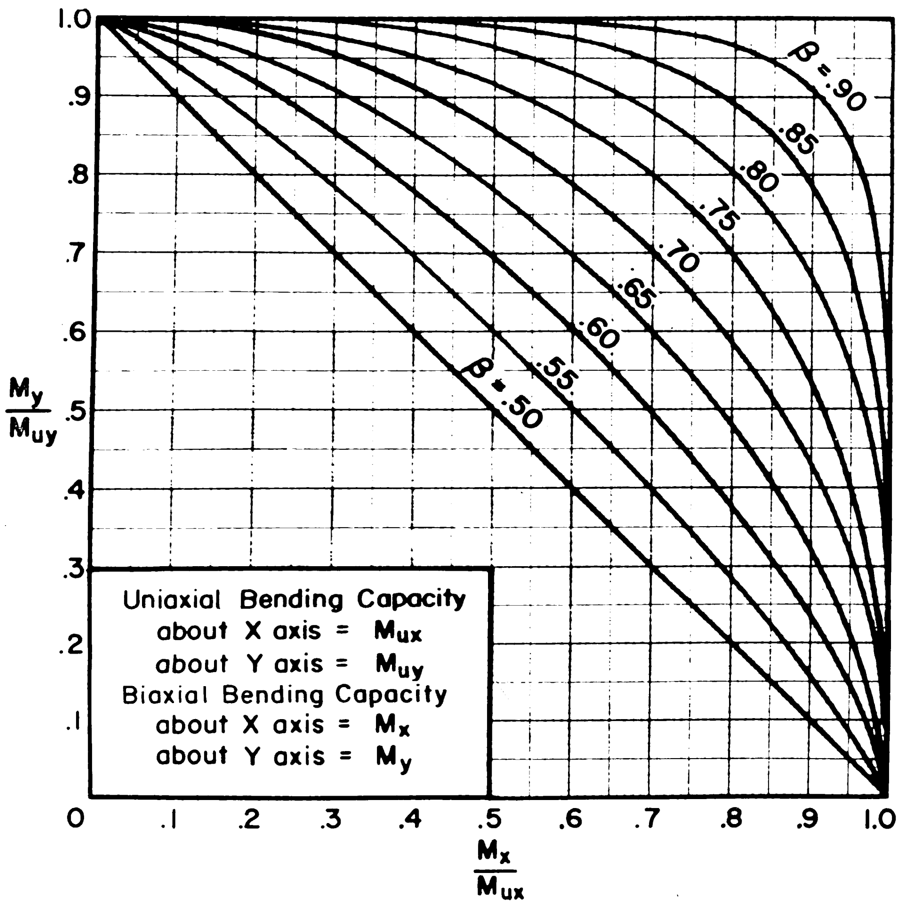

Fig. 2—Biaxial moment relationship

most frequently used range of P_u/P_o did not differ by more than 3 percent. For $P_u/P_o = 0$, differences as much as 8 percent were obtained. In view of these small differences, only envelopes of lowest β values were plotted for two values of f_y and different bar arrangements. These are presented in Fig. 5 to 8.

As can be seen from an inspection of these four figures, β is dependent primarily on the ratio P_u/P_o and to a lesser though still significant extent on the bar arrangement, the reinforcement index q and the strength of the reinforcement. To check the validity of interpolating between the 40,000 and 60,000-psi curves, a few cases with $f_y = 50,000$ psi were computed. The interpolated and computed values agreed reasonably well.

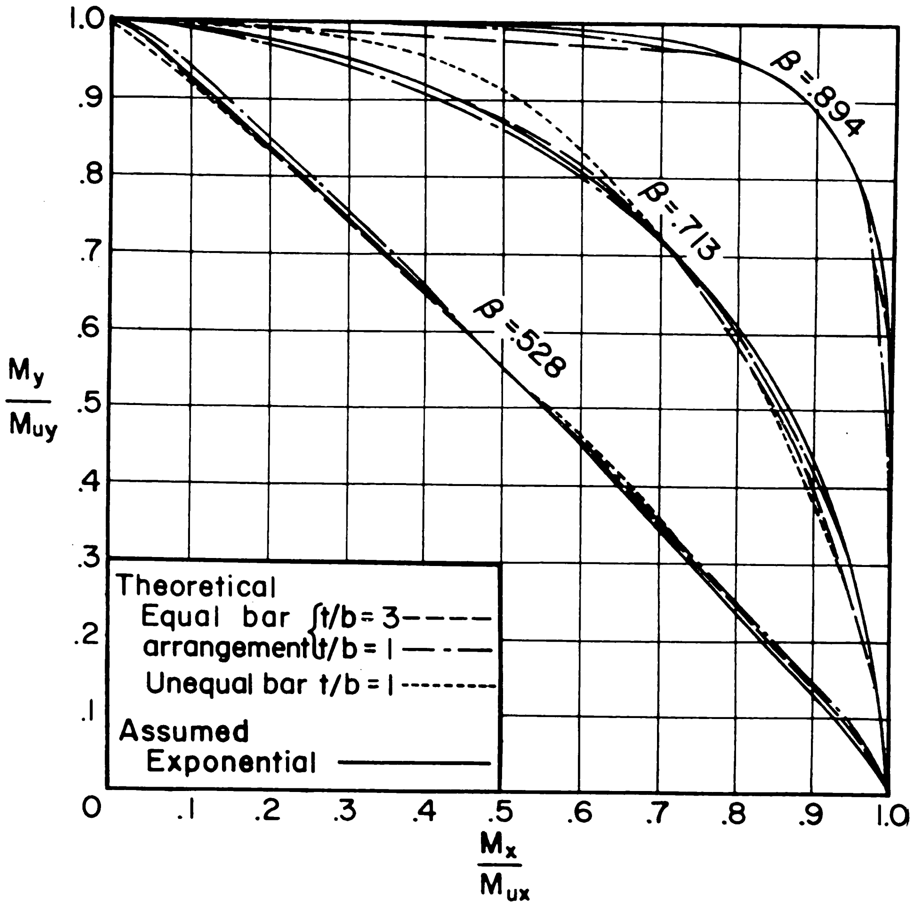

Fig. 3—Assumed versus theoretical moment

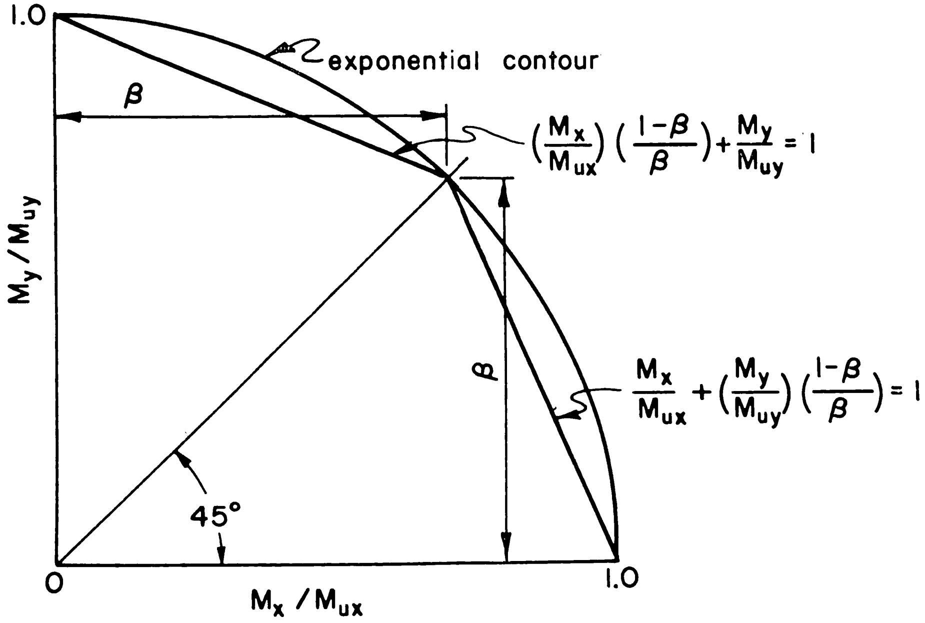

Fig. 4—Interaction curves

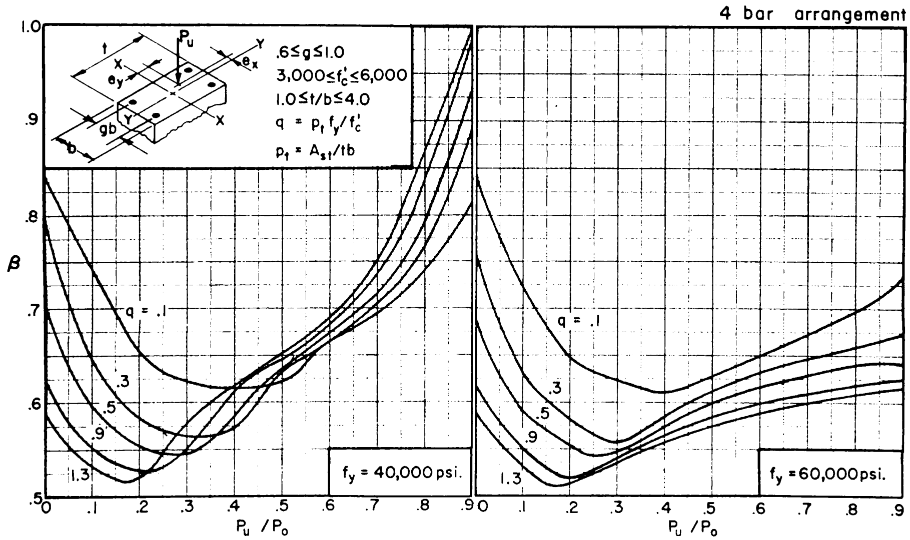

Fig. 5—Biaxial bending design constants (4 bar arrangement)

Fig. 2, in combination with Fig. 5 to 8, furnishes a convenient and direct means of determining the biaxial bending capacity of a given cross section subject to an axial load, since the values of P_o, M_{ux}, and M_{uy} can be readily obtained from available design aids. While analysis is easy, the determination of a section which will satisfy the strength requirements imposed by a load eccentric about both axes can only be achieved by successive analyses of assumed sections. Rapid and easy convergence to a satisfactory section can be secured by approximating in the initial steps the curves in Fig. 2 by straight lines intersecting where the relative moments are equal, as shown in Fig. 4. For a value of $\beta = 0.65$, the maximum difference between the resultants as given by the curved and straight line contours amounts to 6.5 percent. This difference increases slightly for values of β between 0.65 and 0.80 and decreases when β is less than 0.65.

By simple geometry, it can be shown that the equation of the lines are:
when

$$M_y/M_{uy} > M_x/M_{ux}$$

$$\frac{M_y}{M_{uy}} + \frac{M_x}{M_{ux}} \frac{(1-\beta)}{\beta} = 1 \qquad \text{(2a)}$$

which can be restated for design convenience:

$$M_y + M_x \frac{M_{uy}}{M_{ux}} \frac{(1-\beta)}{\beta} = M_{uy} \qquad \text{(2b)}$$

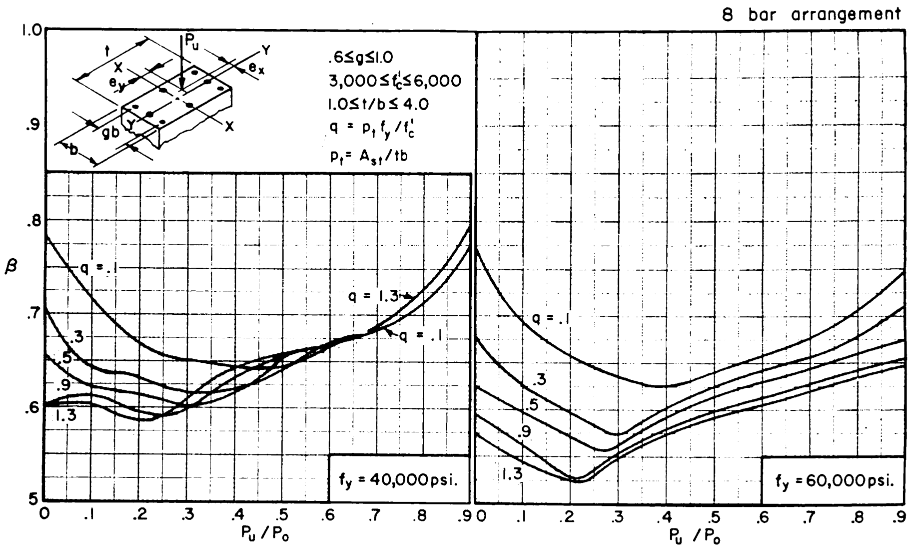

Fig. 6—Biaxial bending design constants (8 bar arrangement)

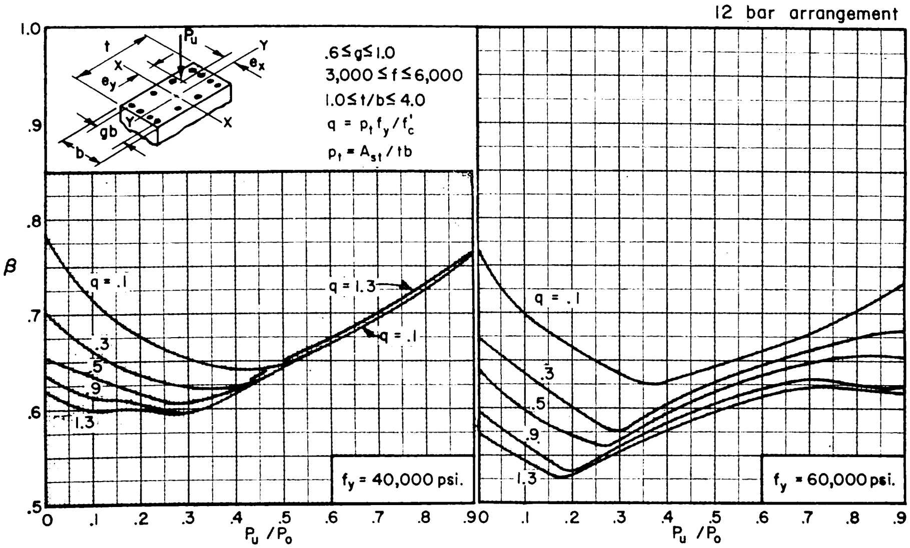

Fig. 7 — Biaxial bending design constants (12 bar arrangement)

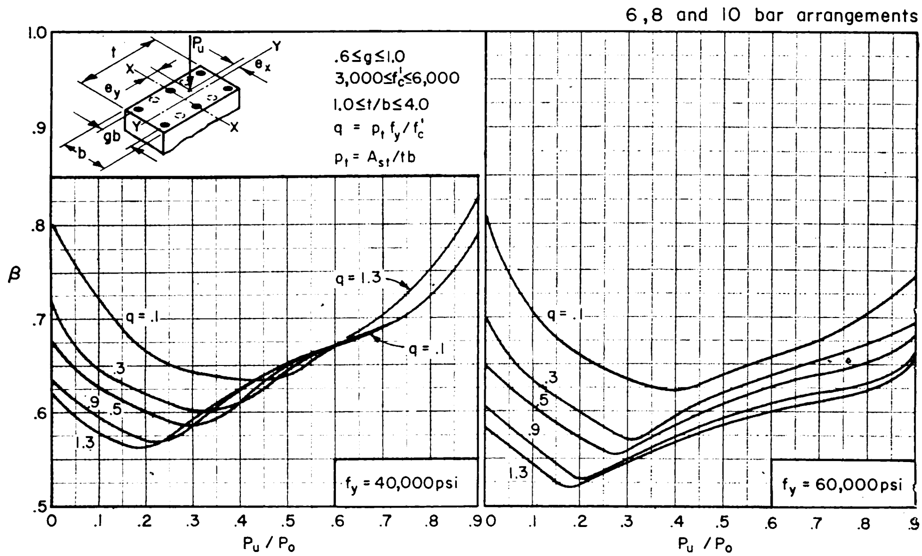

Fig. 8 — Biaxial bending design constants (6, 8, and 10 bar arrangements)

For rectangular sections with reinforcement equally distributed on all faces, Eq. (2b) can be approximated by:

$$M_y + M_x \left(\frac{b}{t}\right) \frac{(1-\beta)}{\beta} \approx M_{uy} \qquad (2c)$$

and similarly:
when

$$M_y/M_{uy} < M_x/M_{ux}$$

$$M_y \frac{M_{ux}}{M_{uy}} \frac{(1-\beta)}{\beta} + M_x = M_{ux} \qquad (2d)$$

and

$$M_y \left(\frac{t}{b}\right) \frac{(1-\beta)}{\beta} + M_x \approx M_{ux} \qquad (2e)$$

In Design Eq. (2c) and (2e), the ratio t/b must be chosen and the value of β must be assumed. For lightly loaded columns, β will generally vary from 0.55 to about 0.70. Hence, 0.65 represents a good initial choice. With these quantities known, M_{ux} or M_{uy} is computed by Eq. (2c) or (2e). From uniaxial design aids, the section and reinforcement satisfying P_u, and M_{ux} or M_{uy} is determined. This section is then used to calculate β. If the assumed and calculated β's are in close agreement, Fig. 2 is used as a final check. If they differ greatly, a second uniaxial bending moment is calculated by the design equation with the improved β. For rectangular sections with unequal number of bars on adjacent faces, Design Eq. (2c) and (2e) are satisfactory for initial trial, but subsequent trials should be made with Eq. (2b) or (2d) using the calculated β and M_{ux}/M_{uy}.

To illustrate the use of the charts, three design examples are included. Because of the uniaxial design aid employed, the loads and moments given represent the effect of design loads times appropriate load factors divided by the ϕ factor.

EXAMPLES

Example 1

Find the required square column size with the load and moments given, assuming reinforcement is equally distributed on all faces.

Given: $P_u = 220$ kips, $M_x = 180$ ft-kips, $M_y = 80$ ft-kips, $f_y = 40{,}000$ psi, and $f_c' = 3000$ psi.

Assuming $\beta = 0.65$, then by Eq. (2e) we have:

$$M_{ux} = 80(1 - 0.65)/0.65 + 180 = 43 + 180 = 223 \text{ ft-kips}$$

From Reference 2, the uniaxial bending capacity of a 16 in. square column with eight #9 bars ($p_t = 0.0313$) is 225 ft-kips for $P_u = 220$ kips. The concentric load capacity P_o is 973 kips.

Hence

$$P_u/P_o = 220/973 = 0.23$$

and

$$q = 0.0313 \text{ x } 40/3 = 0.42$$

From Fig. 6, $\beta = 0.62$.

From Fig. 2 with:

$$M_x/M_{ux} = 180/225 = 0.80 \text{ and } \beta = 0.62$$

$$M_y/M_{uy} = 0.39$$

Hence

$$M_y = 0.39 \text{ x } 225 = 88 \text{ ft-kips}$$

The section is satisfactory.

Example 2

Find the required square column size, with the same loading and strength as previously given, but steel placed on only two faces. Assuming $\beta = 0.65$, then by Eq. (2e) assuming temporarily $M_{ux}/M_{uy} = 1.0$, as previously obtained, $M_{ux} = 223$ ft-kips.

From Reference 2, the uniaxial bending capacity about the x and y axes of a 16 in. square column with six #10 bars ($p_t = 0.0298$) is respectively 235 and 211 ft-kips for $P_u = 220$ kips. The concentric load capacity P_o is 958 kips.

Hence

$$P_u/P_o = 220/958 = 0.23$$

and

$$q = 0.0298 \text{ x } 40/3 = 0.40$$

From Fig. 8, $\beta = 0.60$.

From Fig. 2 with:

$$M_x/M_{ux} = 180/235 = 0.77 \text{ and } \beta = 0.60$$

$$M_y/M_{uy} = 0.40$$

Hence

$$M_y = 0.40 \text{ x } 211 = 84 \text{ ft-kips}$$

The section is satisfactory. It is worthwhile to mention that the conventional linear biaxial relationship would have increased the amount of reinforcement by 20 percent, since with $\beta = 0.5$, the section would have to be proportioned for $M_{ux} = 260$ ft-kips.

Example 3

Find rectangular section with width equal to one-half the depth and reinforcement on only the longer faces.

Given: $P_u = 300$ kips, $M_x = 200$ ft-kips, $M_y = 80$ ft-kips, $f_y = 40{,}000$ psi, and $f_c' = 3000$ psi.

Assuming $\beta = 0.65$, and $M_x/M_{ux} > M_y/M_{uy}$, then by Eq. (2e) with $t/b = 2.0$ we have:

$$M_{ux} = 80 \times 2 \times (1 - 0.65)/0.65 + 200 = 286 \text{ ft-kips}$$

From the column load tables, the uniaxial bending capacity about the X and Y axes of a 12 x 24-in. column reinforced with eight #7 bars ($p_t = 0.167$) is respectively 283 and 146 ft-kips for $P_u = 300$ kips. The concentric load capacity is 926 kips.

Hence

$$P_u/P_o = 300/926 = 0.32$$

and

$$q = 0.167 \times 40/3 = 0.22$$

From Fig. 8, $\beta = 0.62$.

From Fig. 2 with:

$$M_x/M_{ux} = 200/283 = 0.71 \text{ and } \beta = 0.62$$
$$M_y/M_{uy} = 0.52$$

Hence

$$M_y = 0.52 \times 146 = 76 \text{ ft-kips}$$

The section is satisfactory.

CONCLUSION

The biaxial bending charts will always yield greater capacity for a given cross section than the conventional and most commonly used linear biaxial bending relationship. The increased capacity is of sufficient magnitude for columns with bending in the direction of the diagonal, that reduction in the size of the member or of the amount of reinforcement, of as much as 30 percent can be achieved for low steel percentages and when the axial load is less than 20 percent of the concentric load capacity. For columns subject primarily to axial load, the increase in bending capacity possible by the use of the charts does not lead to any substantial savings.

The approximation of the contours of the thrust moment surface by a bilinear relationship leads to considerable simplification and permits direct design of columns from uniaxial column tables with easily corrected estimates of β.

REFERENCES

1. ACI Committee 340, *Ultimate Strength Design of Reinforced Concrete Columns,* SP-7, American Concrete Institute, Detroit, 1964, 182 pp.

2. "Ultimate Load Tables for Tied Columns," *Concrete Information*, Portland Cement Association, Chicago, 1961, 29 pp.

3. Fenlon, H. F., *Reinforced Concrete Column Tables Ultimate Strength Design,* Hugh F. Fenlon, Atlanta, 1960, 316 pp.

4. Bresler, Boris, "Design Criteria for Reinforced Columns under Axial Load and Biaxial Bending," ACI JOURNAL, *Proceedings* V. 57, No. 5, Nov. 1960, pp. 481-490.

5. Fleming, J. F., and Werner, S. D., "Design of Columns Subjected to Biaxial Bending," ACI JOURNAL, *Proceedings* V. 62, No. 3, Mar. 1965, pp. 327-342.

6. Pannell, F. N., "Failure Surfaces for Members in Compression and Biaxial Bending," ACI JOURNAL, *Proceedings* V. 60, No. 1, Jan. 1963, pp. 129-140.

7. Furlong, R. W., "Ultimate Strength of Square Columns Under Biaxially Eccentric Loads," ACI JOURNAL, *Proceedings* V. 57, No. 9, Mar. 1961, pp. 1129-1140.

APPENDIX

NOTATION

The notation used conforms to that given in the ACI Building Code with the following exceptions:

M_{ux} = uniaxial ultimate moment capacity about the X-axis with axial load = P_u

M_{uy} = uniaxial ultimate moment capacity about the Y-axis with axial load = P_u

M_x = component about the X-axis of the biaxial bending capacity with axial load = P_u

M_y = component about the Y-axis of the biaxial bending capacity with axial load = P_u

n = an exponent dependent on column characteristics

P_u = ultimate eccentrically applied load

β = ratio M_x/M_{ux} and M_y/M_{uy} where they are equal

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Sinopsis—Résumé—Zusammenfassung

Capacidad de Columnas Rectangulares Reforzadas Sometidas a Flexión Biaxial

Se presentan gráficas de di seño comprensivas de acuerdo con la sección 1905(a) del Código de Construcción ACI relacionando la capacidad a flexión biaxial de columnas rectangulares con la capacidad a flexión axial por un solo parámetro. Se destaca las diferencias en el comportamiento de las columnas debido a la distribución de barras y resistencias del acero. Se sugiere y evalúa un procidimiento approximado el cual facilita la determinación de las dimençiones requeridas para columnas sometidas a flexión biaxial.

Capacité de Colonnes Rectangulaires Armées, Soumises à un Fléchissement Biaxial

Des tableaux de calculs simplifiés se référant à l'article 1905(a) du code de construction ACI traitant la capacité de fléchissement biaxial de colonnes, rectangulaires comparée à la capacité de fléchissement uniaxial par un simple paramètre sont présentés. Des différences dans le comportement des colonnes, dues à l'arrangement des barres d'armature et à la résistance de l'acier sont notées. Un procédé approximatif qui facilite la détermination des dimensions nécessaires pour des colonnes soumises á un fléchissement biaxial est suggéré et évalué.

Tragfähigkeit rechteckiger Stahlbetonsäulen unter zweiachsiger Biegung

Es werden vollständige Bemessungstabellen gegeben, welche die Tragfähigkeit rechteckiger Säulen unter zweiachsiger Biegung zur Festigkeit unter einachsiger Biegung in Beziehung setzen. Nur ein Parameter ist erforderlich, und die Anforderungen des Abschnittes 1905(a) des ACI Building Code werden erfüllt. Unterschiede im Verhalten der Säulen aufgrund der Bewehrungsanordnung und der Stahlfestigkeit werden hervorgehoben. Eine Näherungsmethode wird vorgeschlagen und ausgewertet, welche die Bestimmung des erforderlichen Querschnittes von Säulen unter zweiachsiger Biegung erleichtert.